# Online Accelerated Gradient Descent and Variational Regret Bounds

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#### Abstract

In this note, we study Nesterov's accelerated gradient descent method in an online setting and establish both variational *static* and *dynamic* regret bounds using the functional variation, which "match" previous regret bounds in terms of gradient variation. To the best of our knowledge, this is the first work to study Nesterov's accelerated gradient method in an online setting and our regret bounds are better than previous variational regret bounds in terms of functional variation.

## 1 Introduction

Nesterov's accelerated gradient method and its various variants have been utilized to solve smooth convex optimization problems in optimization and mathematical programming community. However, it is under explored for online optimization, where the goal is to minimize the regret. A natural question arising is whether we can extend the update of accelerated gradient methods to the online setting and establish improved regret bounds. On the other hand, for online convex optimization and online strongly convex optimization, online gradient descent has been shown to achieve optimal (static) <sup>1</sup> regret bounds. Therefore, one potential for achieving improved regret bounds is to consider online smooth optimization, where the cost functions received at all rounds are smooth and convex.

Recently, there emerge several studies on developing improved regret bounds for online smooth optimization using the variation of the cost functions [3, 7, 2, 9, 4, 5, 10]. In particular, these studies have proposed well-designed online optimization algorithms that maintain and update auxiliary sequence of solutions besides the sequence of actions. Nonetheless, no work has been done to study the regret bounds of Nesterov's accelerated gradient methods applied in the online setting, though they also maintain auxiliary sequences of solutions. This is the main motivation of the present work given that Nesterov's accelerated gradient methods have been very successful for smooth optimization. In this paper, we study a variant of Nesterov's accelerated gradient methods, which can be explained as linear coupling of gradient update and mirror descent update [8, 6, 1]. By combining the basic convergence analysis of gradient descent method and mirror descent method,

<sup>&</sup>lt;sup>1</sup>To differentiate with the dynamic regret bound introduced later.

we establish improved variational static regret bounds and dynamic regret bounds for online smooth optimization. In particular, the variational regret bounds are in terms of the functional variation instead of the gradient variation used in previous studies. To the best of our knowledge, this is the first work that establishes variational regret bounds for Nesterov's accelerated gradient method in an online setting.

## 2 Preliminaries and Notations

In this section, we present some preliminaries of online convex optimization and some notations. Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded convex set and  $\|\cdot\|$  be a norm and  $\|\cdot\|_*$  be its conjugate dual norm.

In online convex optimization, at the t-th round the learner needs to play an action  $\mathbf{x}_t \in \Omega \subseteq \mathbb{R}^d$ and then suffers a loss  $f_t(\mathbf{x}_t)$  measured by a convex cost function  $f_t(\cdot) : \Omega \to \mathbb{R}$  chosen by an adversary. In order to compute the action in the next round, the learner usually needs to query for some feedback about the cost function. We consider the most commonly used feedback model, where the leaner is allowed to query for the gradient of a point  $\mathbf{y}_t \in \Omega$ , i.e.,  $\nabla f_t(\mathbf{y}_t)$ , where  $\mathbf{y}_t$ is not necessarily equal to the played action  $\mathbf{x}_t$ . The goal of online convex optimization is to minimize the total cost suffered in a total of T rounds, i.e.,  $\sum_{t=1}^T f_t(\mathbf{x}_t)$ . In order to measure the performance of different online optimization algorithms, a regret can be defined that compares the performance of the learner to that of a competitor who is assumed to know all the cost functions in advance. Depending on whether a single comparator or a sequence of comparators is used in the definition of regret, we can have two notions of regret, i.e., static regret and dynamic regret. Let  $\mathbf{x}_* = \arg \min_{\mathbf{x} \in \Omega} \sum_{t=1}^T f_t(\mathbf{x})$  and  $\mathbf{x}_t^* = \arg \min_{\mathbf{x} \in \Omega} f_t(\mathbf{x})$ . The static regret is defined as

$$\mathcal{R}_T^s = \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_*) = \sum_{t=1}^T f_t(\mathbf{x}_t) - \min_{\mathbf{x} \in \Omega} \sum_{t=1}^T f_t(\mathbf{x})$$
(1)

where the performance of the learner is compared to a single best solution that minimizes the total loss. In contrast, the dynamic regret is defined as

$$\mathcal{R}_{T}^{d} = \sum_{t=1}^{T} f_{t}(\mathbf{x}_{t}) - \sum_{t=1}^{T} f_{t}(\mathbf{x}_{t}^{*}) = \sum_{t=1}^{T} f_{t}(\mathbf{x}_{t}) - \sum_{t=1}^{T} \min_{\mathbf{x} \in \Omega} f_{t}(\mathbf{x})$$
(2)

where the performance of the learner is compared to a sequence of minimizers for the received cost functions. Note that since  $\sum_{t=1}^{T} \min_{\mathbf{x} \in \Omega} f(\mathbf{x}) \leq \min_{\mathbf{x} \in \Omega} \sum_{t=1}^{T} f_t(\mathbf{x})$ , therefore the dynamic regret is always larger than the static regret. Moreover, it is known that in the worst case it is impossible to achieve a sublinear dynamic regret bound unless a regularity measure is imposed on the sequence of loss functions [2, 10]. In literature, three regularity measures have been used, namely functional variation, gradient variation and path variation. The functional variation is defined as

$$V_T^f = \sum_{t=1}^T \max_{\mathbf{x} \in \Omega} |f_t(\mathbf{x}) - f_{t+1}(\mathbf{x})|$$
(3)

The gradient variation is defined as

$$V_T^g = \sum_{t=1}^T \max_{\mathbf{x} \in \Omega} \|\nabla f_t(\mathbf{x}) - \nabla f_{t+1}(\mathbf{x})\|_*^2$$
(4)

Regret Type	Reference	Bound	Algorithm	Remark
	[3, 7]	$O\left(\sqrt{V_T^g}\right)$	Optimistic Mirror Descent	
static	Present work	$O\left(\sqrt{V_T^f}\right)$	Online Accelerated Gradient Descent	
	[4]	$O(V_T^p\sqrt{T})$	Online Mirror Descent	
	[2]	$O((V_T^f)^{1/3}T^{2/3})$	Restarted Online Gradient Descent	
	[10]	$O(\sqrt{V_T^p T})$	Online Gradient/Mirror Descent	
dynamic	[10]	$O(V_T^p)$	Online Gradient/Mirror Descent	vanishing gradient
	[5]	$O(\sqrt{V_T^p V_T^g})$	Optimistic Mirror Descent	
	Present work	$O(\sqrt{V_T^p V_T^f})$	Online Accelerated Gradient Descent	

Table 1: Summary of variational regret bounds for online smooth optimization under gradient feedback.

The path variation is given by

$$V_T^p = \sum_{t=1}^T \|\mathbf{x}_t^* - \mathbf{x}_{t+1}^*\|$$
(5)

A sublinear dynamic regret bound is achievable only when  $V_T^{f/g/p} \leq o(T)$  [2, 10]. Several variational regret bounds have been established. In Table 1, we summarize the variational static and dynamic regret bounds developed in previous works and the present work to facilitate the comparison. Note that the listed results focus on the similar setting as considered in this paper. Before ending this section, we introduce some notations and assumptions. Let  $\omega(\mathbf{x})$  be a 1-strongly convex function with respect to  $\|\cdot\|$ , i.e.,

$$\omega(\mathbf{x}) \ge \omega(\mathbf{y}) + \langle \nabla \omega(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2$$

Denote by  $V(\mathbf{x}, \mathbf{y})$  the Bregman divergence induced by  $\omega(\mathbf{x})$ , i.e.,

 $V(\mathbf{x}, \mathbf{y}) = \omega(\mathbf{x}) - \omega(\mathbf{y}) - \langle \nabla \omega(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle$ 

In the sequel, we will make the following assumptions.

**Assumption 1.** Suppose  $\Omega$  is bounded so that there exists R > 0 such that  $V(\mathbf{x}, \mathbf{y}) \leq R$  for all  $\mathbf{x}, \mathbf{y} \in \Omega$ .

**Assumption 2.** Suppose each cost function  $f_t(\mathbf{x}) : \Omega \to \mathbb{R}$  is bounded, i.e., there exists B > 0 such that  $0 \le f_t(\mathbf{x}) \le B$  for all t = 1, ..., T.

**Assumption 3.** Suppose all the cost functions  $f_t(\mathbf{x}) : \Omega \to \mathbb{R}$  are convex and L-smooth, i.e.,  $f_t(\mathbf{x})$  satisfies

$$f_t(\mathbf{x}) \ge f_t(\mathbf{y}) + \langle \nabla f_t(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \tag{6}$$

and

$$\|\nabla f_t(\mathbf{x}) - \nabla f_t(\mathbf{y})\|_* \le L \|\mathbf{x} - \mathbf{y}\|$$
(7)

Algorithm 1 Online Accelerated Gradient Decent Method

1: Initialization:  $\mathbf{x}_1 = \mathbf{z}_1 \in \Omega$ 

- 2: for t = 1, ..., T do
- 3: Play  $\mathbf{x}_t$
- 4: Compute  $\mathbf{y}_t = \tau \mathbf{z}_t + (1 \tau) \mathbf{x}_t$  and Query for the gradient of the loss function  $f_t(\mathbf{x})$  at  $\mathbf{y}_t$
- 5: Update  $\mathbf{x}_{t+1}$  by

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{y}\in\Omega} \frac{\gamma}{2} \|\mathbf{y} - \mathbf{y}_t\|^2 + \langle \nabla f_t(\mathbf{y}_t), \mathbf{y} - \mathbf{y}_t \rangle$$

6: Update  $\mathbf{z}_{t+1}$  by

$$\mathbf{z}_{t+1} = \arg\min_{\mathbf{y}\in\Omega} \langle \alpha \nabla f_t(\mathbf{y}_t), \mathbf{y} - \mathbf{z}_t \rangle + V(\mathbf{y}, \mathbf{z}_t)$$

7: end for

# 3 Online Accelerated Gradient Descent Method

In this section, we will first present a variant of Nesterov's accelerated gradient methods and then extend it to the online setting. Then we establish its variational regret bounds.

#### 3.1 Nesterov's Accelerated Gradient Descent Method

Nesterov's accelerated gradient descent methods have been used widely for solving smooth optimization problem. Consider the following optimization problem

$$\min_{\mathbf{x}\in\Omega} f(\mathbf{x}) \tag{8}$$

where  $f(\mathbf{x})$  is a convex and L-smooth function. There are several variants of Nesterov's accelerated gradient methods. Here, we consider a particular variant that can be explained as the linear coupling of the gradient descent method and the mirror descent method. The updates of this variant are given by  $t = 1, \ldots, T - 1$ :

$$\mathbf{y}_t = \tau \mathbf{z}_t + (1 - \tau) \mathbf{x}_t \tag{9}$$

$$\mathbf{x}_{t+1} = \min_{\mathbf{y}\in\Omega} \frac{L}{2} \|\mathbf{y} - \mathbf{y}_t\|^2 + \langle \nabla f(\mathbf{y}_t), \mathbf{y} - \mathbf{y}_t \rangle$$
(10)

$$\mathbf{z}_{t+1} = \min_{\mathbf{y} \in \Omega} \langle \alpha \nabla f(\mathbf{y}_t), \mathbf{y} - \mathbf{z}_t \rangle + V(\mathbf{y}, \mathbf{z}_t)$$
(11)

Note that the update in (10) is the gradient descent update and the update in (11) is the mirror descent update.

### 3.2 Online Accelerated Gradient Descent Method

The online Nesterov's accelerated gradient descent method is presented in Algorithm 1. The regret bounds of online accelerated gradient descent method are presented in the following two theorems.

**Theorem 4.** (Variational Static Regret Bound) Suppose  $\gamma \ge \max\{L, \frac{B+V_T^f}{R}\}$ ,  $\alpha = \sqrt{\frac{R}{\gamma(B+V_T^f)}}$  and  $\tau = \frac{1}{\alpha\gamma} \le 1$ . We have

$$\mathcal{R}_T^s \le 2\sqrt{\gamma R(B + V_T^f)}$$

**Theorem 5.** (Variational Dynamic Regret Bound) Assume  $\omega(\mathbf{x})$  is Lipschitz continuous such that  $V(\mathbf{x}, \mathbf{z}) - V(\mathbf{y}, \mathbf{z}) \leq G \|\mathbf{x} - \mathbf{y}\|$  for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Omega$ . Suppose  $\gamma \geq \max\{L, \frac{B+V_T^f}{R+GV_T^p}\}, \ \alpha = \sqrt{\frac{R+GV_T^p}{\gamma(B+V_T^f)}}$  and  $\tau = \frac{1}{\alpha\gamma} \leq 1$ . We have

$$\mathcal{R}_T^d \le 2\sqrt{\gamma(B + V_T^f)(R + GV_T^p)}$$

To prove the above two theorems. We first prove a series of lemmas.

**Lemma 1.** Let  $\mathbf{z}_+ = \arg\min_{\mathbf{y}\in\Omega} \phi(\mathbf{y}) + V(\mathbf{y}, \mathbf{z})$ . Then for any  $\mathbf{y}\in\Omega$  we have

$$\phi(\mathbf{y}) + V(\mathbf{y}, \mathbf{z}) \ge \phi(\mathbf{z}_{+}) + V(\mathbf{z}_{+}, \mathbf{z}) + V(\mathbf{y}, \mathbf{z}_{+})$$

**Lemma 2.** Suppose  $\gamma \geq L$ . For any  $\mathbf{u} \in \Omega$ 

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{y}_t) \le \frac{\gamma}{2} \|\mathbf{x}_{t+1} - \mathbf{y}_t\|^2 + \langle \nabla f_t(\mathbf{y}_t), \mathbf{x}_{t+1} - \mathbf{y}_t \rangle \le \frac{\gamma}{2} \|\mathbf{u} - \mathbf{y}_t\|^2 + \langle \nabla f_t(\mathbf{y}_t), \mathbf{u} - \mathbf{y}_t \rangle$$

**Lemma 3.** Suppose  $\tau = \frac{1}{\alpha\gamma}$  For any  $\mathbf{x} \in \Omega$ , we have

$$\alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_t - \mathbf{x} \rangle \le \alpha^2 \gamma (f_t(\mathbf{y}_t) - f_t(\mathbf{x}_{t+1})) + V(\mathbf{x}, \mathbf{z}_t) - V(\mathbf{x}, \mathbf{z}_{t+1})$$

*Proof of Lemma 3.* Using Lemma 1 for the update of  $\mathbf{z}_{t+1}$  we have

$$\begin{aligned} \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{x} - \mathbf{z}_t \rangle + V(\mathbf{x}, \mathbf{z}_t) &\geq \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_{t+1} - \mathbf{z}_t \rangle + V(\mathbf{z}_{t+1}, \mathbf{z}_t) + V(\mathbf{x}, \mathbf{z}_{t+1}) \\ &\geq \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_{t+1} - \mathbf{z}_t \rangle + V(\mathbf{x}, \mathbf{z}_{t+1}) + \frac{1}{2} \|\mathbf{z}_t - \mathbf{z}_{t+1}\|^2 \end{aligned}$$

Therefore

$$\alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_t - \mathbf{x} \rangle \le V(\mathbf{x}, \mathbf{z}_t) - V(\mathbf{x}, \mathbf{z}_{t+1}) + \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_t - \mathbf{z}_{t+1} \rangle - \frac{1}{2} \|\mathbf{z}_t - \mathbf{z}_{t+1}\|^2$$

Letting  $\mathbf{u} = \tau \mathbf{z}_{t+1} + (1 - \tau) \mathbf{x}_t$ . Then  $\mathbf{y}_t - \mathbf{u} = \tau (\mathbf{z}_t - \mathbf{z}_{t+1})$  and

$$\begin{aligned} \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_t - \mathbf{z}_{t+1} \rangle &- \frac{1}{2} \| \mathbf{z}_t - \mathbf{z}_{t+1} \|^2 = \frac{\alpha}{\tau} \langle \nabla f_t(\mathbf{y}_t), \mathbf{y}_t - \mathbf{u} \rangle - \frac{1}{2\tau^2} \| \mathbf{y}_t - \mathbf{u} \|^2 \\ &= \alpha^2 \gamma \left( \langle \nabla f_t(\mathbf{y}_t), \mathbf{y}_t - \mathbf{u} \rangle - \frac{\gamma}{2} \| \mathbf{y}_t - \mathbf{u} \|^2 \right) \\ &\leq \alpha^2 \gamma (f_t(\mathbf{y}_t) - f_t(\mathbf{x}_{t+1})) \end{aligned}$$

Thus,

$$\alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_t - \mathbf{x} \rangle \le V(\mathbf{x}, \mathbf{z}_t) - V(\mathbf{x}, \mathbf{z}_{t+1}) + \alpha^2 \gamma (f_t(\mathbf{y}_t) - f_t(\mathbf{x}_{t+1}))$$

**Lemma 4.** Suppose  $\gamma \geq L$  and  $\tau = \frac{1}{\alpha \gamma}$  For any  $\mathbf{x} \in \Omega$ , we have

$$\alpha(f_t(\mathbf{x}_t) - f_t(\mathbf{x})) \le \alpha^2 \gamma(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_{t+1})) + V(\mathbf{x}, \mathbf{z}_t) - V(\mathbf{x}, \mathbf{z}_{t+1})$$

Proof of Lemma 4.

$$\begin{aligned} \alpha(f_t(\mathbf{y}_t) - f_t(\mathbf{x})) &\leq \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{y}_t - \mathbf{x} \rangle = \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{y}_t - \mathbf{z}_t \rangle + \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_t - \mathbf{x} \rangle \\ &\leq \frac{\alpha(1 - \tau)}{\tau} \langle \nabla f_t(\mathbf{y}_t), \mathbf{x}_t - \mathbf{y}_t \rangle + \alpha \langle \nabla f_t(\mathbf{y}_t), \mathbf{z}_t - \mathbf{x} \rangle \\ &\leq (\alpha^2 \gamma - \alpha) (f_t(\mathbf{x}_t) - f_t(\mathbf{y}_t)) + \alpha^2 \gamma (f_t(\mathbf{y}_t) - f_t(\mathbf{x}_{t+1})) + V(\mathbf{x}, \mathbf{z}_t) - V(\mathbf{x}, \mathbf{z}_{t+1}) \\ &= \alpha f_t(\mathbf{y}_t) + (\alpha^2 \gamma - \alpha) f_t(\mathbf{x}_t) - \alpha^2 \gamma f_t(\mathbf{x}_{t+1}) + V(\mathbf{x}, \mathbf{z}_t) - V(\mathbf{x}, \mathbf{z}_{t+1}) \end{aligned}$$

Then

$$\alpha(f_t(\mathbf{x}_t) - f(\mathbf{x})) \le \alpha^2 \gamma(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_{t+1})) + V(\mathbf{x}, \mathbf{z}_t) - V(\mathbf{x}, \mathbf{z}_{t+1})$$

Proof of Theorem 4. Let  $\mathbf{x} = \mathbf{x}_* = \arg \min_{\mathbf{x} \in \Omega} \sum_{t=1}^T f_t(\mathbf{x})$  in Lemma 4, we sum the above inequality over  $t = 1, \ldots, T$ , we have

$$\begin{aligned} \alpha \sum_{t=1}^{T} \left( f_t(\mathbf{x}_t) - f_t(\mathbf{x}_*) \right) &\leq \alpha^2 L \sum_{t=1}^{T} \left( f_t(\mathbf{x}_t) - f_t(\mathbf{x}_{t+1}) \right) + V(\mathbf{x}_*, \mathbf{z}_0) - V(\mathbf{x}_*, \mathbf{z}_{T+1}) \\ &\leq \alpha^2 \gamma \sum_{t=1}^{T} \left( f_t(\mathbf{x}_t) - f_{t+1}(\mathbf{x}_{t+1}) + f_{t+1}(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_{t+1}) \right) + V(\mathbf{x}_*, \mathbf{z}_0) \\ &= \alpha^2 \gamma (f_1(\mathbf{x}_1) - f_{T+1}(\mathbf{x}_{T+1})) + \alpha^2 \gamma \sum_{t=1}^{T} (f_{t+1}(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_{t+1})) + V(\mathbf{x}_*, \mathbf{z}_0) \\ &\leq \alpha^2 \gamma B + \alpha^2 \gamma V_T^f + R \end{aligned}$$

Then we can bound the static regret by

$$\sum_{t=1}^{T} (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_*)) \le \alpha \gamma (B + V_T^f) + \frac{R}{\alpha}$$

By optimizing over  $\alpha$ , i.e., letting  $\alpha = \sqrt{\frac{R}{\gamma(B+V_T^f)}}$ , we obtain the following static regret bound

$$\sum_{t=1}^{T} (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_*)) \le 2\sqrt{\gamma R(B + V_T^f)}$$
(12)

Finally, we note that  $\alpha \gamma = \sqrt{\frac{R\gamma}{B+V_T^f}} \ge 1$ , therefore  $\tau = 1/(\alpha \gamma) \le 1$ .

Proof of Theorem 5. Let  $\mathbf{x} = \mathbf{x}_t^* = \arg\min_{\mathbf{x}\in\Omega} f_t(\mathbf{x})$  in Lemma 4, we have

$$f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t^*) \le \alpha \gamma (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_{t+1})) + \frac{1}{\alpha} V(\mathbf{x}_t^*, \mathbf{z}_t) - \frac{1}{\alpha} V(\mathbf{x}_t^*, \mathbf{z}_{t+1})$$

Summing over  $t = 1, \ldots, T$ , we have

$$\begin{split} \sum_{t=1}^{T} (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t^*)) &\leq \alpha \gamma (B + V_T^f) + \sum_{t=1}^{T} \left( \frac{1}{\alpha} V(\mathbf{x}_t^*, \mathbf{z}_t) - \frac{1}{\alpha} V(\mathbf{x}_t^*, \mathbf{z}_{t+1}) \right) \\ &\leq \alpha \gamma (B + V_T^f) + \frac{R}{\alpha} + \frac{1}{\alpha} \sum_{t=2}^{T} (V(\mathbf{x}_t^*, \mathbf{z}_t) - V(\mathbf{x}_{t-1}^*, \mathbf{z}_t)) \\ &\leq \alpha \gamma (B + V_T^f) + \frac{R}{\alpha} + \frac{G}{\alpha} \sum_{t=2}^{T} \|\mathbf{x}_t^* - \mathbf{x}_{t-1}^*\| \\ &\leq \alpha \gamma (B + V_T^f) + \frac{R + GV_T^p}{\alpha} \end{split}$$

wher the third equality uses the Lipschitz property  $V(\mathbf{x}, \mathbf{z}) - V(\mathbf{y}, \mathbf{z}) \leq G \|\mathbf{x} - \mathbf{y}\|$ . By optimizing over  $\alpha$ , i.e., letting  $\alpha = \sqrt{\frac{R+GV_T^p}{\gamma(B+V_T^p)}}$ , we get

$$\sum_{t=1}^{T} (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t^*)) \le 2\sqrt{\gamma(B + V_T^f)(R + GV_T^p)}$$

### References

- [1] Zeyuan Allen-Zhu and Lorenzo Orecchia. Linear coupling: An ultimate unification of gradient and mirror descent. ArXiv e-prints, abs/1407.1537, July 2014.
- [2] Omar Besbes, Yonatan Gur, and Assaf J. Zeevi. Non-stationary stochastic optimization. CoRR, abs/1307.5449, 2013.
- [3] Chao-Kai Chiang, Tianbao Yang, Chia-Jung Lee1, Mehrdad Mahdavi, Chi-Jen Lu, Rong Jin, and Shenghuo Zhu. Online optimization with gradual variations. In *Proceedings of the 25th* Annual Conference on Learning Theory (COLT), 2012.
- [4] Eric C. Hall and Rebecca M. Willett. Online optimization in dynamic environments. CoRR, abs/1307.5944, 2013.
- [5] Ali Jadbabaie, Alexander Rakhlin, Shahin Shahrampour, and Karthik Sridharan. Online optimization : Competing with dynamic comparators. In Proceedings of the Eighteenth International Conference on Artificial Intelligence and Statistics (AISTATS), 2015.
- [6] Guanghui Lan, Zhaosong Lu, and Renato D. C. Monteiro. Primal-dual first-order methods with O(1/e) iteration-complexity for cone programming. *Math. Program.*, 126(1):1–29, 2011.

- [7] Alexander Rakhlin and Karthik Sridharan. Optimization, learning, and games with predictable sequences. In Advances in Neural Information Processing Systems 26 (NIPS), pages 3066–3074, 2013.
- [8] P. Tseng. On accelerated proximal gradient methods for convex-concave optimization. *submitted to SIAM Journal on Optimization*, 2008.
- [9] Tianbao Yang, Mehrdad Mahdavi, Rong Jin, and Shenghuo Zhu. Regret bounded by gradual variation for online convex optimization. *Machine Learning*, 95(2):183–223, 2014.
- [10] Tianbao Yang, Lijun Zhang, Rong Jin, and Jinfeng Yi. Tracking slowly moving clairvoyant: Optimal dynamic regret of online learning with true and noisy gradient. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2016.

# A Proof of Lemma 1

By optimality condition of  $\mathbf{z}_+$ , for any  $\mathbf{y} \in \Omega$  we have

$$\langle \nabla \phi(\mathbf{z}_{+}) + \nabla V(\mathbf{z}_{+}, \mathbf{z}), \mathbf{y} - \mathbf{z}_{+} \rangle \ge 0$$

By the convexity of  $\phi(\mathbf{y})$ ,

$$\phi(\mathbf{y}) - \phi(\mathbf{z}_{+}) \ge \langle \nabla \phi(\mathbf{z}_{+}), \mathbf{y} - \mathbf{z}_{+} \rangle$$

Then

$$\phi(\mathbf{y}) - \phi(\mathbf{z}_+) \ge \langle \nabla V(\mathbf{z}_+, \mathbf{z}), \mathbf{z}_+ - \mathbf{y} \rangle$$

By the definition of  $V(\mathbf{y}, \mathbf{z})$ , we have  $\nabla V(\mathbf{z}_+, \mathbf{z}) = \nabla \omega(\mathbf{z}_+) - \nabla \omega(\mathbf{z})$ . Then

$$\phi(\mathbf{y}) - \langle \nabla \omega(\mathbf{z}), \mathbf{y} - \mathbf{z} \rangle \ge \phi(\mathbf{z}_{+}) - \langle \nabla \omega(\mathbf{z}), \mathbf{z}_{+} - \mathbf{z} \rangle - \langle \nabla \omega(\mathbf{z}_{+}), \mathbf{y} - \mathbf{z}_{+} \rangle$$

Adding  $\omega(\mathbf{y}) - \omega(\mathbf{z})$  to both sides and by noting that

$$V(\mathbf{y}, \mathbf{z}) = \omega(\mathbf{y}) - \omega(\mathbf{z}) - \langle \nabla \omega(\mathbf{z}), \mathbf{y} - \mathbf{z} \rangle$$
$$V(\mathbf{z}_{+}, \mathbf{z}) + V(\mathbf{y}, \mathbf{z}_{+}) = \omega(\mathbf{z}_{+}) - \omega(\mathbf{z}) - \langle \nabla \omega(\mathbf{z}), \mathbf{z}_{+} - \mathbf{z} \rangle + \omega(\mathbf{y}) - \omega(\mathbf{z}_{+}) - \langle \nabla \omega(\mathbf{z}_{+}), \mathbf{y} - \mathbf{z}_{+} \rangle$$

we can finish the proof.