Foundations of Lazy SMT and DPLL(T)

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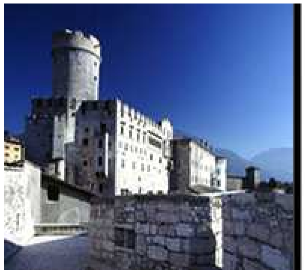

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Acknowledgments: Many thanks to Albert Oliveras for contributing some of the material used in these slides.

Disclamer: The literature on SMT and its applications is already vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

Introduction

Historically, automated reasoning $\equiv$ uniform proof-search procedures for First Order Logic

Limited success: is FOL the best compromise between expressivity and efficiency?

More recent trend [Sha02] focuses on:

- addressing mostly (expressive enough) decidable fragments of a certain logic
- incorporating domain-specific reasoning, e.g on:
  - arithmetic reasoning
  - equality
  - data structures (arrays, lists, stacks, ...)

Introduction

Examples of this trend:

SAT: propositional formalization, Boolean reasoning

- $+$ high degree of efficiency
- $-$ expressive (all NP-complete problems) but involved encodings

SMT: first-order formalization, Boolean $+$ domain-specific reasoning

- $+$ improves expressivity and scalability
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This lecture: overview of SMT formal foundations

The SMT Problem

Some problems are more naturally expressed in logics other than propositional logic, e.g:

- Software verification needs reasoning about equality, arithmetic, data structures, ...

SMT is about deciding the satisfiability of a (usually quantifier-free) FOL formula with respect to some *background theory*

- Example (Equality with Uninterpreted Functions):

$$g(a) = c \quad \wedge \quad (\ f(g(a)) \neq f(c) \vee g(a) = d\ ) \quad \wedge \quad c \neq d$$

Wide range of applications: Extended Static Checking [FLL$^+$02], Predicate abstraction [LNO06], Model checking [AMP06, HT08], Scheduling [BNO$^+$08b], Test generation [TdH08], ...

Theories of Interest: EUF

Equality ($=$) with Uninterpreted Functions [NO80, BD94, NO07]

Typically used to abstract unsupported constructs, e.g.:

- non-linear multiplication in arithmetic
- ALUs in circuits

Example: The formula

$$a * (|b| + c) = d \;\; \wedge \;\; b * (|a| + c) \neq d \;\; \wedge \;\; a = b$$

is unsatisfiable, but no arithmetic reasoning is needed

If we abstract it to

$$mul(a, add(abs(b), c)) = d \;\; \wedge \;\; mul(b, add(abs(a), c)) \neq d \;\; \wedge \;\; a = b$$

it is still unsatisfiable

Theories of Interest: Arithmetic(s)

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$ [BBC+05a]
- Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
- Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06]
- Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM+09, ZM10]

Theories of Interest: Arrays

Used in software verification and hardware verification (for memories) [SBDL01, BNO+08a, dMB09]

Two interpreted function symbols read and write

Axiomatized by:

- $\forall a\, \forall i\, \forall v\ \mathrm{read}(\mathrm{write}(a,i,v),i) = v$
- $\forall a\, \forall i\, \forall j\, \forall v\ i \neq j \to \mathrm{read}(\mathrm{write}(a,i,v),j) = \mathrm{read}(a,j)$

Sometimes also with *extensionality*:

- $\forall a\, \forall b\ (\forall i\, \mathrm{read}(a,i) = \mathrm{read}(b,i)\ \to a = b)$

Is the following set of literals satisfiable in this theory?

$$\mathrm{write}(a,i,x) \neq b,\ \mathrm{read}(b,i) = y,\ \mathrm{read}(\mathrm{write}(b,i,x),j) = y,\ a = b,\ i = j$$

Theories of Interest: Bit vectors

Useful both in hardware and software verification [BCF+07, BB09]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- *String-like*: concat, extract, ...
- *Logical*: bit-wise not, or, and, ...
- *Arithmetic*: add, subtract, multiply, ...
- *Comparison*: $<, >$, ...

Is this formula satisfiable over bit vectors of size 3?

$$a[0:1] \neq b[0:1] \ \wedge \ (a \mid b) = c \ \wedge \ c[0] = 0 \ \wedge \ a[1] + b[1] = 0$$

Combinations of Theories

In practice, theories are seldom used in isolation

E.g., software verifications may need a combination of arrays, arithmetic, bit vectors, data types, ...

Formulas of the following form usually arise:

$$i = j + 2 \;\; \wedge \;\; a = \mathrm{write}(b, i+1, 4) \;\; \wedge$$
$$(\mathrm{read}(a, j+3) = 2 \;\vee\; f(i-1) \neq f(j+1))$$

Often decision procedures for each theory combine modularly [NO79, TH96, BBC+05b]

Solving SMT Problems

Fact: Many theories of interest have (efficient) decision procedures for the satisfiability of sets (or conjunctions) of literals.

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3. formulas with quantifiers

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This lecture focuses more on general methods to address (1), mostly, and (2)

More details on (2) and (3) will be given in later lectures today

Structure of this Lecture

Introduction

Part I

From sets of literals to arbitrary quantifier-free formulas

Part II

From a single theory T to multiple theories $T_1, \ldots, T_n$

Part I

From sets of literals to arbitrary quantifier-free formulas

Satisfiability Modulo a Theory T

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Current solution: Exploit propositional satisfiability technology

Lifting SAT Technology to SMT

Two main approaches:

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1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]
    - translate into an equisatisfiable propositional formula
    - feed it to any SAT solver

Notable systems: UCLID

Lifting SAT Technology to SMT

Two main approaches:

2. "Lazy" [ACG00, dMR02, BDS02, ABC$^+$02]
   - abstract the input formula to a propositional one
   - feed it to a (DPLL-based) SAT solver
   - use a theory decision procedure to refine the formula and guide the SAT solver

Notable systems: Barcelogic, Boolector, CVC3, MathSAT, Yices, Z3, ...

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This talk will focus on the lazy approach

(Very) Lazy Approach for SMT – Example

$$g(a) = c \quad \wedge \quad f(g(a)) \neq f(c) \; \vee \; g(a) = d \quad \wedge \quad c \neq d$$

Theory T: Equality with Uninterpreted Functions

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Simplest setting:

- Off-line SAT solver
- Non-incremental *theory solver* for conjunctions of equalities and disequalities
- Theory atoms (e.g., $g(a) = c$) abstracted to propositional atoms (e.g., 1)

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- SAT solver finds $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4\}$ unsat.
  Done: the original formula is unsatisfiable in EUF.

Lazy Approach – Enhancements

Several enhancements are possible to increase efficiency:

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- If M is T-unsatisfiable, bactrack to some point where the assignment was still T-satisfiable

Lazy Approach – Main Benefits

- Every tool does what it is good at:
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  - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
  - SAT and theory solvers communicate via a simple API [GHN+04]
  - SMT for a new theory only requires new theory solver
  - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)

An Abstract Framework for Lazy SMT

Several variants and enhancements of lazy SMT solvers exist

They can be modeled abstractly and declaratively as *transition systems*

A transition system is a binary relation over states, induced by a set of conditional transition rules

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07]

Advantages of Abstract Framework

An abstract framework helps one:

- skip over implementation details and unimportant control aspects
- reason formally about solvers for SAT and SMT
- model advanced features such as non-chronological bactracking, lemma learning, theory propagation, ...
- describe different strategies and prove their correctness
- compare different systems at a higher level
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The one described next is a re-elaboration of those in [NOT06, KG07]

The Original DPLL Procedure

- Modern SAT solvers are based on the DPLL procedure [DP60, DLL62]
- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F
- M is grown by
  - deducing the truth value of a literal from M and F, or
  - guessing a truth value
- If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

An Abstract Framework for DPLL

States:

$$\text{fail} \quad \text{or} \quad \langle M, F \rangle$$

where

- M is a sequence of literals and *decision points* $\bullet$ denoting a partial truth *assignment*
- F is a set of clauses denoting a CNF *formula*

Def. If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level* i of M
- $M^{[i]} \stackrel{\text{def}}{=} M_0 \bullet \cdots \bullet M_i$

An Abstract Framework for DPLL

States:

fail or $\langle M, F \rangle$

Initial state:

- $\langle (), F_0 \rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- fail if F_0 is unsatisfiable
- $\langle M, G \rangle$ otherwise, where
    - G is equivalent to F_0 and
    - M satisfies G

Transition Rules: Notation

States treated like records:

- M denotes the truth assignment component of current state
- F denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

$$\frac{p_1 \quad \cdots \quad p_n}{[\mathsf{M} := e_1] \quad [\mathsf{F} := e_2]}$$

updating M, F or both when premises $p_1, \ldots, p_n$ all hold

NB: When convenient, will treat M as the set of its literals

Transition Rules for the Original DPLL

Extending the assignment

Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

Not. Clauses are treated modulo ACI of $\vee$

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Decide $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

Not. $\mathrm{Lit}(F) \stackrel{\text{def}}{=} \{l \mid l \text{ literal of } F\} \cup \{\bar{l} \mid l \text{ literal of } F\}$

Transition Rules for the Original DPLL

Repairing the assignment

Fail $$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

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Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\ N \quad \bullet \notin N}{\mathsf{M} := M\ \bar{l}}$$

NB: Last premise of **Backtrack** enforces chronological backtracking

From DPLL to CDCL Solvers (1)

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

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Replace **Backtrack** with

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Explain $$\frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Backjump

$$\frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\ \bar{l}_1, \ldots, \mathsf{lev}\ \bar{l}_n \leq \ i < \mathsf{lev}\ \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\ l}$$

Not. $l \prec_{\mathsf{M}} l'$ if l occurs before l' in M

$\mathsf{lev}\ l = i$ iff l occurs in decision level i of M

From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

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Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Not. $\models_{\mathrm{p}}$ denotes propositional entailment

From DPLL to CDCL Solvers (3)

Modify **Fail** to

Fail $\dfrac{C \neq no \quad \bullet \notin M}{fail}$

Execution Example

$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$

M	F	C	rule
	F	no	//

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1	F	no	by **Propagate**

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$1\ 2$	F	no	by **Propagate**

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	F	no	//
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**

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	F	no	//
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	//
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**

Execution Example

$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$

M	F	C	rule
	F	no	//
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**

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$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	//
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**

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	F	no	$//$
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**

Execution Example

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$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$

Execution Example

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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**

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$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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1	F	no	by **Propagate**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

From DPLL to CDCL Solvers (4)

Also add

Learn $\dfrac{\mathsf{F} \models_{\mathrm{p}} C \quad C \notin \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{C\}}$

Forget $\dfrac{\mathsf{C} = \mathsf{no} \quad \mathsf{F} = G \cup \{C\} \quad G \models_{\mathrm{p}} C}{\mathsf{F} := G}$

Restart $\dfrac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad \mathsf{C} := \mathsf{no}}$

NB: Learn can be applied to any clause stored in C when $\mathsf{C} \neq \mathsf{no}$

Modeling Modern SAT Solvers

At the core, current CDCL SAT solvers are implementations of the transition system with rules

Propagate, **Decide**,

Conflict, **Explain**, **Backjump**,

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Basic DPLL $\stackrel{\text{def}}{=}$

{ **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump** }

DPLL $\stackrel{\text{def}}{=}$ Basic DPLL + { **Learn**, **Forget**, **Restart** }

The Basic DPLL System – Correctness

Some terminology:

Irreducible state: state to which no Basic DPLL rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = ()$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Note: This is not so immediate, because of **Backjump**.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Lemma Every exhausted execution ends with either $C = \text{no}$ or fail.

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Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, the clause set F_0 is satisfied by M.

The DPLL System – Strategies

- Applying
  - one Basic DPLL rule between each two **Learn** applications and
  - **Restart** less and less often

ensures termination

The DPLL System – Strategies

- A common basic strategy applies the rules with the following priorities:
  1. If $n > 0$ conflicts have been found so far, increase n and apply **Restart**

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  7. Apply **Decide**

The DPLL System – Correctness

Proposition (Termination) Every execution in which
(a) **Learn**/**Forget** are applied only finitely many times and
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is finite.

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(a) **Learn**/**Forget** are applied only finitely many times and
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is finite.

Proposition (Soundness) As before.

Proposition (Completeness) As before.

(For simplicity the statement of the termination result is not entirely accurate.
See [NOT06] for more details.)

From SAT to SMT

Same sort of states and transitions but

- F contains quantifier-free clauses in some theory T
- M is a sequence of theory literals and decision points
- the DPLL system augmented with rules

  T-**Conflict**, T-**Propagate**, T-**Explain**

- maintains invariant: F $\models_T$ C and M $\models_{\mathrm{p}}$ $\neg$C when C $\neq$ no

Def. $F \models_T G$ iff every model of T that satisfies F satisfies G as well

SMT-level Rules

Fix a theory T

T-Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

Not: $\bot$ = empty clause

NB: $\models_T$ decided by theory solver

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Fix a theory T

T-**Conflict** $$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-**Propagate** $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

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T-Explain $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Not: $\bot$ = empty clause

NB: $\models_T$ decided by theory solver

Modeling the Very Lazy Theory Approach

T-**Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier EUF example

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	//
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by T-**Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T-**Conflict**, **Learn**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **Conflict**
fail			by **Fail**

A Better Lazy Approach

The very lazy approach can be improved considerably with

- An *on-line* SAT engine,
  which can accept new input clauses on the fly

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The very lazy approach can be improved considerably with

- An *on-line* SAT engine, which can accept new input clauses on the fly
- an *incremental and explicating T*-solver, which can
  1. check the T-satisfiability of M as it is extended and
  2. identify a small T-unsatisfiable subset of M once M becomes T-unsatisfiable

A Better Lazy Approach

$$\underbrace{g(a) = c}_{1} \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \lor \underbrace{g(a) = d}_{3} \quad \land \quad \underbrace{c \neq d}_{\overline{4}}$$

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	//
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2$	by **T-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**
fail			by **Fail**

Lazy Approach – Strategies

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
2. If M is T-unsatisfiable, apply T-**Conflict**
3. Apply **Fail** or **Explain**+**Learn**+**Backjump** as appropriate
4. Apply **Propagate**
5. Apply **Decide**

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Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
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3. Apply **Fail** or **Explain**+**Learn**+**Backjump** as appropriate
4. Apply **Propagate**
5. Apply **Decide**

NB: Depending on the cost of checking the T-satisfiability of M, Step (2) can be applied with lower frequency or priority

Theory Propagation

With T-**Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

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With T-**Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

With T-**Propagate** and T-**Explain**, it can also be used to guide the engine's search [Tin02]

T-**Propagate** $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T-**Explain** $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Theory Propagation Example

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	//
1	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** $(1 \models_T 2)$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** $(1,\ \overline{4} \models_T \overline{3})$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{2} \vee 3$	by **Conflict**
fail			by **Fail**

NB: T-propagation eliminates search altogether in this case,
no applications of **Decide** are needed

Theory Propagation Example (2)

$$\underbrace{g(a) = e}_{0} \vee \underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	//
$\overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$\overline{4} \bullet 1$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** $(1 \models_T 2)$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** $(1,\ \overline{4} \models_T \overline{3})$
$\overline{4} \bullet 1\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{2} \vee 3$	by **Conflict**
$\overline{4} \bullet 1\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 3$	by T-**Explain**
$\overline{4} \bullet 1\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 4$	by T-**Explain**
$\overline{4}\ \overline{1}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$\cdots$			(exercise)

Theory Propagation Features

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- For some theories, e.g., difference logic, detecting T-entailed literals is cheap and so theory propagation is extremely effective.
- For others, e.g., the theory of equality, detecting all T-entailed literals is too expensive.
- If T-**Propagate** is not applied exhaustively, T-**Conflict** is needed to repair T-unsatisfiable assignments.

Modeling Modern Lazy SMT Solvers

At the core, current lazy SMT solvers are implementations of the transition system with rules

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) T-**Conflict**, T-**Propagate**, T-**Explain**

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Basic DPLL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2)

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Updated terminology:

Irreducible state: state to which no Basic DPLL MT rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = ()$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

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Exhausted execution: execution ending in an irreducible state

Proposition (Termination) Every execution in which
(a) **Learn**/**Forget** are applied only finitely many times and
(b) **Restart** is applied with increased periodicity
is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

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Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is T-unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, F_0 is T-satisfiable; specifically, M is T-satisfiable and $\mathsf{M} \models_{\mathrm{p}} F_0$.

DPLL(T) Architecture

The approach formalized so far can be implemented with a simple architecture named DPLL(T) [GHN+04, NOT06]

$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

DPLL(X):

- Very similar to a SAT solver, enumerates Boolean models
- Not allowed: pure literal, blocked literal detection, ...
- Required: incremental addition of clauses
- Desirable: partial model detection

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T-solver:

- Checks the T-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of T-unsatisfiability/propagation
- Must be incremental and backtrackable

Reasoning by Cases in Theory Solvers

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$$M = \{ \underbrace{r(w(a,i,x),j) \neq x}_{1},\ \underbrace{r(w(a,i,x),j) \neq r(a,j)}_{2} \}$$

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$i \neq j$) Then, $r(w(a,i,x),j) = r(a,j)$. Contradiction with 2.

Conclusion: M is T-unsatisfiable

Case Splitting

A *complete* T-solver reasons by cases via (internal) case splitting and backtracking mechanisms.

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A *complete* T-solver reasons by cases via (internal) case splitting and backtracking mechanisms.

An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine.

Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them.

Possible benefits:

- All case-splitting is coordinated by the SAT engine
- Only have to implement case-splitting infrastructure in one place
- Can learn a wider class of lemmas

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$$\mathsf{M} = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

- Main SMT module: “Is M T-unsatisfiable?”
- T-solver: “I do not know yet, but it will help me if you consider these *theory lemmas*:

$$s = s' \wedge i = j \rightarrow s = t, \quad s = s' \wedge i \neq j \rightarrow s = r(a,j) \text{ ”}$$

Modeling Splitting on Demand

To model the generation of theory lemmas for case splits, add the rule

T-**Learn**

$$\frac{\models_T \exists \mathbf{v}(l_1 \vee \cdots \vee l_n) \quad l_1, \ldots, l_n \in L_{\mathrm{S}} \quad \mathbf{v} \text{ vars not in } \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

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where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

NB: For many theories with a theory solver, there exists an appropriate finite L_{S} for every input F

The set L_{S} does not need to be computed explicitly

Modeling Splitting on Demand

Now we can relax the requirement on the theory solver:

When $M \models_T F$, it must either

- determine whether $M \models_T \perp$ or
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NB: In practice, to determine if $\mathsf{M} \models_T \bot$ the T-solver only needs a small subset of L_{S} to be defined in M

Example — Theory of Finite Sets

$$F: \quad x = y \cup z \;\;\wedge\;\; y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \;\bullet\; y = \emptyset$	F	by **Decide**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z$	F	by **Propagate**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by T-**Learn**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z \;\bullet\; e \in x$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Decide**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z \;\bullet\; e \in x\; e \notin z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Propagate**

T-solver can make the following deductions at this point:

$$e \in x \quad \cdots \quad \Rightarrow \quad e \in y \cup z \quad \cdots \quad \Rightarrow \quad e \in y \quad \cdots \quad \Rightarrow \quad e \in \emptyset \quad \Rightarrow \quad \bot$$

This enables an application of T-**Conflict** with clause

$$x \neq y \cup z \;\vee\; y \neq \emptyset \;\vee\; x = z \;\vee\; e \notin x \;\vee\; e \in z$$

Correctness Results

Correctness results can be extended to the new rule.

Soundness: The new T-**Learn** rule maintains satisfiability of the clause set.

Completeness: As long as the theory solver can decide $\mathsf{M} \models_T \bot$ when all literals in L_{S} are determined, the system is still complete.

Termination: The system terminates under the same conditions as before. Roughly:

- Any lemma is (re)learned only finitely many times
- **Restart** is applied with increased periodicity

Part II

From a single theory T to multiple theories $T_1, \ldots, T_n$

Need for Combining Theories and Solvers

Recall: Many applications give rise to formulas like:

$$a \approx b + 2 \wedge A \approx \mathrm{write}(B, a + 1, 4) \wedge$$
$$(\mathrm{read}(A, b + 3) \approx 2 \vee f(a - 1) \neq f(b + 1))$$

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Solving that formula requires reasoning over

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Question: Given solvers for each theory, can we combine them modularly into one for $T_{\mathrm{LA}} \cup T_{\mathrm{A}} \cup T_{\mathrm{EUF}}$?

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Question: Given solvers for each theory, can we combine them modularly into one for $T_{\mathrm{LA}} \cup T_{\mathrm{A}} \cup T_{\mathrm{EUF}}$?

Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]

Motivating Example (Convex Case)

Consider the following set of literals over $T_{\mathrm{LRA}} \cup T_{\mathrm{EUF}}$ (T_{LRA}, linear real arithmetic):

$$
\begin{array}{rcl}
f(f(x) - f(y)) & = & a \\
f(0) & > & a + 2 \\
x & = & y
\end{array}
$$

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f(f(x) - f(y)) = a \quad \Longrightarrow \quad \begin{aligned} f(e_1) &= a \\ e_1 &= f(x) - f(y) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} f(e_1) &= a \\ e_1 &= e_2 - e_3 \\ e_2 &= f(x) \\ e_3 &= f(y) \end{aligned}
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$$\begin{array}{ccccc} f(0) = a + 2 & \Longrightarrow & f(e_4) = a + 2 & \Longrightarrow & f(e_4) = e_5 \\ & & e_4 = 0 & & e_4 = 0 \\ & & & & e_5 > a + 2 \end{array}$$

Motivating Example (Convex Case)

Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

L_1	L_2
$f(e_1) = a$	$e_2 - e_3 = e_1$
$f(x) = e_2$	$e_4 = 0$
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$$L_2 \models_{\mathrm{LRA}} e_1 = e_4$$

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$$L_1 \models_{\mathrm{EUF}} a = e_5$$

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Third step: check for satisfiability locally

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Report unsatisfiable

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Consider the following unsatisfiable set of literals over $T_{\mathrm{LIA}} \cup T_{\mathrm{EUF}}$ (T_{LIA}, linear integer arithmetic) :

$$
\begin{array}{rcl}
1 \leq & x & \leq 2 \\
f(1) & = & a \\
f(x) & = & b \\
a & = & b + 2
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$$f(1) = a \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = 1 \end{array}$$

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$$f(2) = f(1) + 3 \implies \begin{array}{r} e_2 = 2 \\ f(e_2) = e_3 \\ f(e_1) = e_4 \\ e_3 = e_4 + 3 \end{array}$$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
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$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

No more entailed equalities, but $L_1 \models_{\text{LIA}} x = e_1 \vee x = e_2$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leq x$	$f(e_1) = a$
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$e_3 = e_4 + 3$	
$a = e_4$	

Consider each case of $x = e_1 \vee x = e_2$ separately

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Case 1) $x = e_1$

Motivating Example (Non-convex Case)

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$x \leq 2$	$f(x) = b$
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$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_1$
$e_3 = e_4 + 3$	
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$x = e_1$	

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$a = e_4$	
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$L_2 \models_{\mathrm{EUF}} a = b$, which entails $\perp$ when sent to L_1

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$e_2 = 2$	
$e_3 = e_4 + 3$	
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Case 2) $x = e_2$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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$x \leq 2$	$f(x) = b$
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$L_2 \models_{\mathrm{EUF}} e_3 = b$, which entails $\perp$ when sent to L_1

The Nelson-Oppen Method

- For $i = 1, 2$, let T_i be a first-order theory of *signature* Σ_i
  (set of function and predicate symbols in T_i other than $=$)
- Let $T = T_1 \cup T_2$
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NB: Because of purification, there is no loss of generality in considering only ground $(\Sigma_i \cup \mathcal{C})$-literals

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1. Guess an *arrangement* A, i.e., a set of equalities and disequalities over $\mathcal{C}$ such that

$$c = d \in A \quad \text{or} \quad c \neq d \in A \quad \text{for all } c, d \in \mathcal{C}$$

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2. If $L_i \cup A$ is T_i-unsatisfiable for $i = 1$ or $i = 2$, return **unsat**
3. Otherwise, return **sat**

Correctness of the NO Method

Proposition (Termination) The method is terminating.

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Proposition (Completeness) If $\Sigma_1 \cap \Sigma_2 = \emptyset$ and T_1 and T_2 are stably infinite, when the method returns **sat** for some arrangement, the input is $(T_1 \cup T_2)$-is satisfiable.

(Only non-immediate aspect)

Stably Infinite Theories

Def. A theory T is *stably infinite* iff every quantifier-free T-satisfiable formula is satisfiable in an infinite model of T

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Many interesting theories are stably infinite:

- Theories of an infinite structure (e.g., integer arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
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Def. A theory T is *convex* iff, for any set L of literals
$L \models_T s_1 = t_1 \vee \cdots \vee s_n = t_n \implies L \models_T s_i = t_i$ for some i

NB: With convex theories, arrangements do not need to be guessed—they can be computed by (theory) propagation

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Other interesting theories are not stably infinite:

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- Theories with models of bounded cardinality (e.g., theory of strings of bounded length)
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The Nelson-Oppen method has been extended to some classes of non-stably infinite theories [TZ05, RRZ05, JB10]

SMT Solving with Multiple Theories

Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$

How can we integrate all of them cooperatively into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

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Quick Solution:

1. Combine $S_1, \ldots, S_n$ with Nelson-Oppen into a theory solver for T
2. Build a DPLL(T) solver as usual

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Better Solution [Bar02, BBC+05b, BNOT06]:

1. Extend DPLL(T) to DPLL($T_1, \ldots, T_n$)
2. Lift Nelson-Oppen to the DPLL($X_1, \ldots, X_n$) level
3. Build a DPLL($T_1, \ldots, T_n$) solver

Modeling DPLL($T_1, \ldots, T_n$) Abstractly

- Let $n = 2$, for simplicity
- Let T_i be of signature Σ_i for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$
- Let $\mathcal{C}$ be a set of free constants
- Assume wlog that each input literal has signature $(\Sigma_1 \cup \mathcal{C})$ or $(\Sigma_2 \cup \mathcal{C})$ (no *mixed* literals)
- Let $\mathsf{M}|_i \stackrel{\text{def}}{=} \{(\Sigma_i \cup \mathcal{C})\text{-literals of M and their complement}\}$
- Let $\mathrm{I}(\mathsf{M}) \stackrel{\text{def}}{=} \{c = d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\} \cup$
  $\{c \neq d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\}$
  (*interface literals*)

Abstract DPLL Modulo Multiple Theories

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

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Decide $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$

Only change: decide on interface equalities as well

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T-**Propagate**

$$\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\} \quad \mathsf{M} \models_{T_i} l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

Only change: propagate interface equalities as well, but reason locally in each T_i

Abstract DPLL Modulo Multiple Theories

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \perp \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

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Abstract DPLL Modulo Multiple Theories

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1,2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1,2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each T_i

I-Learn

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1,2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

Example — Convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$

$$\underbrace{e_2 - e_3 = e_1}_{5} \wedge \underbrace{e_4 = 0}_{6} \wedge \underbrace{e_5 > a + 2}_{7}$$

$$\underbrace{e_2 = e_3}_{8} \quad \underbrace{e_1 = e_4}_{9} \quad \underbrace{a = e_5}_{10}$$

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$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$
$$\underbrace{e_2 - e_3 = e_1}_{5} \wedge \underbrace{e_4 = 0}_{6} \wedge \underbrace{e_5 > a + 2}_{7}$$
$$\underbrace{e_2 = e_3}_{8} \quad \underbrace{e_1 = e_4}_{9} \quad \underbrace{a = e_5}_{10}$$

M	F	C	rule
	F	no	//
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** $(1, 2, 4 \models_{\mathrm{EUF}} 8)$
0 1 2 3 4 5 6 7 8 9	F	no	by T-**Propagate** $(5, 6, 8 \models_{\mathrm{LRA}} 9)$
0 1 2 3 4 5 6 7 8 9 10	F	no	by T-**Propagate** $(0, 3, 9 \models_{\mathrm{EUF}} 10)$
0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \vee \overline{10}$	by T-**Conflict** $(7, 10 \models_{\mathrm{LRA}} \perp)$
fail			by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$

$$\underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = b}^{1} \land \overbrace{f(e_2) = e_3}^{2} \land \overbrace{f(e_1) = e_4}^{3} \land$$
$$\underbrace{1 \le x}_{4} \land \underbrace{x \le 2}_{5} \land \underbrace{e_1 = 1}_{6} \land \underbrace{a = b + 2}_{7} \land \underbrace{e_2 = 2}_{8} \land \underbrace{e_3 = e_4 + 3}_{9}$$
$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	//
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0, 3 \models_{\mathrm{EUF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \lor \overline{5} \lor 11 \lor 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by T-**Propagate** $(0, 1, 11 \models_{\mathrm{EUF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	$\overline{7} \lor \overline{13}$	by T-**Conflict** $(7, 13 \models_{\mathrm{EUF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by T-**Propagate** $(0, 1, \overline{13} \models_{\mathrm{EUF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Suggested Readings

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