SMT-based Model Checking

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Modeling Computational Systems

Software or hardware systems can be often represented as a *state transition system* $\mathcal{M} = (\mathcal{S}, \mathcal{I}, \mathcal{T}, \mathcal{L})$ where

- $\mathcal{S}$ is a set of *states*, the state space
- $\mathcal{I} \subseteq \mathcal{S}$ is a set of *initial states*
- $\mathcal{T} \subseteq \mathcal{S} \times \mathcal{S}$ is a (right-total) *transition relation*
- $\mathcal{L} : \mathcal{S} \to 2^{\mathcal{P}}$ is a *labeling function* where $\mathcal{P}$ is a set of *state predicates*

Typically, the state predicates denote variable-value pairs $x = v$

Model Checking

Software or hardware systems can be often represented as a state transition system $\mathcal{M} = (\mathcal{S}, \mathcal{I}, \mathcal{T}, \mathcal{L})$

$\mathcal{M}$ can be seen as a *model* both

1. in an engineering sense:

   an abstraction of the real system

   and

2. in a mathematical logic sense:

   a Kripke structure in some modal logic

Model Checking

The functional properties of a computational system can be expressed as *temporal* properties

- for a suitable model $\mathcal{M} = (\mathcal{S}, \mathcal{I}, \mathcal{T}, \mathcal{L})$ of the system
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Two main classes of properties:

- *Safety properties*: nothing bad ever happens
- *Liveness properties*: something good eventually happens

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I will focus on checking safety in this talk

Talk Roadmap

- Checking safety properties
- Logic-based model checking
- Satisfiability Modulo Theories
  - theories
  - solvers
- SMT-based model checking
  - main approaches
  - k-induction
    - basic method
    - enhancements
  - interpolation

Basic Terminology

Let $\mathcal{M} = (\mathcal{S}, \mathcal{I}, \mathcal{T}, \mathcal{L})$ be a transition system

The set $\mathcal{R}_\mathcal{I}$ of *reachable states (of $\mathcal{M}$)* is the smallest subset of $\mathcal{S}$ such that

1. $\mathcal{I} \subseteq \mathcal{R}_\mathcal{I}$ (initial states are reachable)
2. $\mathcal{R}_\mathcal{I} \bowtie \mathcal{T} \subseteq \mathcal{R}_\mathcal{I}$ ($\mathcal{T}$-successors of reachable states are reachable)

Let $\mathcal{E} \subseteq \mathcal{S}$ (a *state property*)

The set $\mathcal{B}_\mathcal{E}$ of *bad states wrt $\mathcal{E}$* is the smallest subset of $\mathcal{S}$ such that

1. $\mathcal{E} \subseteq \mathcal{B}_\mathcal{E}$ (the states of $\mathcal{E}$ are bad)
2. $\mathcal{T} \bowtie \mathcal{B}_\mathcal{E} \subseteq \mathcal{B}_\mathcal{E}$ ($\mathcal{T}$-predecessors of bad states are bad)

Safety and Invariance

$\mathcal{M}$ is *safe* wrt a state property $\mathcal{E}$ if $\mathcal{R}_{\mathcal{I}} \cap \mathcal{E} = \emptyset$

iff $\mathcal{I} \cap \mathcal{B}_{\mathcal{E}} = \emptyset$

A state property $\mathcal{P}$ is *invariant (for $\mathcal{M}$)* iff $\mathcal{R}_{\mathcal{I}} \subseteq \mathcal{P}$

Note:

$\mathcal{M}$ is safe wrt $\mathcal{E}$ iff $\mathcal{S} \setminus \mathcal{E}$ is invariant for $\mathcal{M}$

Checking Safety

In principle, to check that $\mathcal{M}$ is safe wrt $\mathcal{E}$ it suffices to

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This can be done explicitly only if $\mathcal{S}$ is finite, and relatively small ($< 10M$ states)

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- BDD-based methods, if $\mathcal{S}$ is finite,
- automata-based methods,
- logic-based methods, or
- abstract interpretation methods

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Logic-based Symbolic Model Checking

Applicable if we can encode $\mathcal{M} = (\mathcal{S}, \mathcal{I}, \mathcal{T}, \mathcal{L})$ in some (classical) logic $\mathbb{L}$ with decidable entailment $\models_{\mathbb{L}}$

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Examples of $\mathbb{L}$:

- Propositional logic
- Quantified Boolean Formulas
- Bernay-Schönfinkel logic
- Quantifier-free real (or linear integer) arithmetic with arrays and uninterpreted functions
- ...

Logical encodings of transitions systems

$\mathcal{M} = (\mathcal{S}, \mathcal{I}, \mathcal{T}, \mathcal{L})$ X: set of *variables* V: set of *values* in $\mathbb{L}$

Not.: if $\mathbf{x} = (x_1, \ldots, x_n)$ and $\sigma = (v_1, \ldots, v_n)$, $\phi[\sigma] := \phi[v_1/x_1, \ldots, v_n/x_n]$

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- State properties encoded as formulas $P[\mathbf{x}]$

Strongest Inductive Invariant

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Logic-based model checking is about approximating R
as efficiently as possible and as precisely as needed

Main Logic-based Approaches

- Bounded model checking [CBRZ01, AMP06, BHvMW09]
- Interpolation-based model checking [McM03, McM05a]
- Property Directed Reachability [BM07, Bra10, EMB11]
- Temporal induction [SSS00, dMRS03, HT08]
- Backward reachability [ACJT96, GR10]
- . . .

Past accomplishments: mostly based on propositional logic, with SAT solvers as reasoning engines

New frontier: based on logics decided by solvers for Satisfiability Modulo Theories [Seb07, BSST09]

Model Checking Modulo Theories

We invariably reason about transition systems in the context of some theory $\mathcal{T}$ of their data types

Examples

- Pipelined microprocessors: theory of equality, atoms like $f(g(a,b),c) = g(c,a)$
- Timed automata: theory of integers/reals, atoms like $x - y < 2$
- General software: combination of theories, atoms like $a[2*j+1] + x \geq car(l) - f(x)$

Such reasoning can be reduced to checking the satisfiability of certain formulas in (or *modulo*) the theory $\mathcal{T}$.

Satisfiability Modulo Theories

Let $\mathcal{T}$ be a first-order theory of signature Σ

> The $\mathcal{T}$-satisfiability problem for a class $\mathcal{C}$ of Σ-formulas:
> decide for $\varphi[\mathbf{x}] \in \mathcal{C}$ whether $\mathcal{T} \cup \{\exists \mathbf{x}.\, \varphi\}$ is satisfiable

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- Equality with "Uninterpreted Function Symbols"
- Linear Arithmetic (Real and Integer)
- Arrays (i.e., updatable maps)
- Finite sets and multisets
- Inductive data types (enumerations, lists, trees, . . .)
- . . .

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Thanks to advances in SAT and in decision procedures, this can be done very efficiently in practice by current SMT solvers

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Provide additional functionalities besides satisfiability checking

- compute satisfying assignments
- evaluate terms
- identify unsatisfiable cores
- generate interpolants
- construct proof objects
- . . .

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Are now the backend of a variety of FM tools :

model checkers, equivalence checkers, extended static checkers, type checkers, program verifiers, symbolic simulators, malware detectors, test case generators, invariant generators, ...

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Increasingly conform to a standard I/O language: the SMT-LIB format [BST10]

Modern SMT Solvers

Such as Alt-Ergo, CVC3, MathSat, OpenSMT, VeriT, Yices, Z3, . . . ,

- are based on many-sorted first-order logic
- support a combination of several built-in theories
- allow user-defined function and predicate symbols
- follow a stack-based, assert-and-query execution model
- provide a rich API

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  declare: symbol → type → unit

  define: symbol → λ-term → unit

  assert: formula → unit

  push: unit → unit

  pop: unit → unit

  check_sat: unit → unit

  eval: term → value

  next_model: unit → unit

  . . .

Model Checking: SAT or SMT?

SMT encodings in model checking provide several advantages over SAT encodings

- more powerful language

  (unquantified) first-order formulas instead of Boolean formulas

- satisfiability still efficiently decidable
- similar high level of automation
- more natural and compact encodings
- greater scalability
- not limited to finite state systems

Model Checking: SAT or SMT?

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> SMT-based model checking techniques are blurring the line between traditional model checking and deductive verification

Talk Roadmap

- √ Checking safety properties
- √ Logic-based model checking
- √ Satisfiability Modulo Theories
  - √ theories
  - √ solvers
- SMT-based model checking
  - main approaches
  - k-induction
    - basic method
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SMT-based Model Checking

A few approaches:

- Predicate abstraction + finite model checking
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Will focus more on temporal induction

Technical Preliminaries

Let's fix

- $\mathbb{L}$, a logic decided by an SMT solver
  (e.g., quantifier-free linear arithmetic and EUF)
- $M = (I[\mathbf{x}], T[\mathbf{x}, \mathbf{x}'])$, an encoding in $\mathbb{L}$ of a system $\mathcal{M}$
- $P[\mathbf{x}]$, a state property to be proven invariant for S

Example: Parametric Resettable Counter

Model

Vars

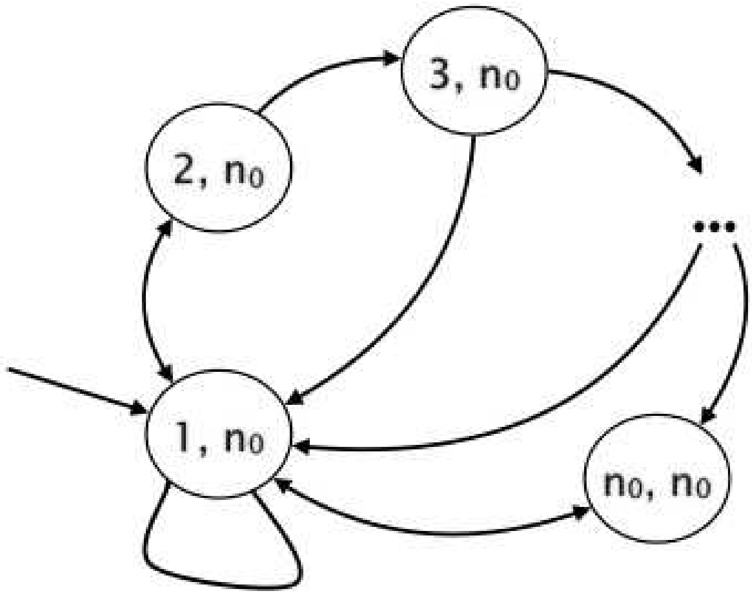
input pos int n_0
input bool r
int c, n

Initialization

c := 1
n := n_0

Transitions

n' := n
c' := if (r' or c = n)
then 1
else c + 1



The transition relation contains infinitely many instances of the schema above, one for each $n_0 > 0$

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Encoding in $\mathbb{L}$

$$\mathbf{x} := (c, n, r, n_0)$$

$$I[\mathbf{x}] := (c = 1) \wedge (n = n_0)$$

$$\begin{aligned} T[\mathbf{x}, \mathbf{x}'] &:= (n' = n) \\ &\wedge \ (r' \vee (c = n) \rightarrow (c' = 1)) \\ &\wedge \ (\neg r' \wedge (c \neq n) \rightarrow (c' = c + 1)) \end{aligned}$$

$$P[\mathbf{x}] := c < n + 1$$

Inductive Reasoning

Let $S = (I[\mathbf{x}], T[\mathbf{x}, \mathbf{x}'])$

To prove $P[x]$ invariant for S it suffices to show that it is *inductive* for S, i.e.,

1. $I[\mathbf{x}] \models_{\mathbb{L}} P[\mathbf{x}]$ (base case)

   and

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Problem: Not all invariants are inductive

Example: In the parametric resettable counter, $P := c \leq n + 1$ is invariant but (2) above is falsifiable, e.g., by $(c, n, r) = (4, 3, \mathit{false})$ and $(c, n, r)' = (5, 3, \mathit{false})$

Improving Induction's Precision

1. $I[\mathbf{x}] \models_{\mathbb{L}} P[\mathbf{x}]$ 2. $P[\mathbf{x}] \wedge T[\mathbf{x}, \mathbf{x}'] \models_{\mathbb{L}} P[\mathbf{x}']$

A few options:

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  Easy to automate (but fairly weak in its basic form)

Basic k-Induction (Naive Algorithm)

Notation: $I_i := I[\mathbf{x}_i],\ P_i := P[\mathbf{x}_i],\ T_i := T[\mathbf{x}_{i-1}, \mathbf{x}_i]$

```
for i = 0 to ∞ do
    if not (I_0 ∧ T_1 ∧ ··· ∧ T_i ⊨_𝕃 P_i) then
        return fail
    if (P_0 ∧ ··· ∧ P_i ∧ T_1 ∧ ··· ∧ T_{i+1} ⊨_𝕃 P_{i+1}) then
        return success
```

P is *k-inductive* for some $k \geq 0$, if the first entailment holds for all $i = 0, \ldots, k$ and the second entailment holds for $i = k$

Example: In the parametric resettable counter, $P := c \leq n + 1$ is 1-inductive, but not 0-inductive

Basic k-Induction (Naive Algorithm)

Notation: $I_i := I[\mathbf{x}_i]$, $P_i := P[\mathbf{x}_i]$, $T_i := T[\mathbf{x}_{i-1}, \mathbf{x}_i]$

$\textbf{for } i = 0 \textbf{ to } \infty \textbf{ do}$
    $\textbf{if not } (I_0 \wedge T_1 \wedge \cdots \wedge T_i \models_{\mathbb{L}} P_i) \textbf{ then}$
        return fail
    $\textbf{if } (P_0 \wedge \cdots \wedge P_i \wedge T_1 \wedge \cdots \wedge T_{i+1} \models_{\mathbb{L}} P_{i+1}) \textbf{ then}$
        return success

P is *k-inductive* for some $k \geq 0$, if the first entailment holds for all $i = 0, \ldots, k$ and the second entailment holds for $i = k$

Note:

- inductive = 0-inductive
- k-inductive $\Rightarrow$ $(k+1)$-inductive $\Rightarrow$ invariant
- some invariants are not k-inductive for any k

Basic k-Induction with SMT Solvers

Solver maintains current set of *asserted* formulas

Two solver instances: b, i

(0) $\textsf{assert}_\mathrm{b}(I_0)$
(0) $k := 0$
(0) **loop**
(0) $\quad\textsf{assert}_\mathrm{b}(T_k)$ // $T_0 = \textit{true}$ by convention
(0) $\quad$**if not** $\textsf{entailed}_\mathrm{b}(P_k)$ **then return** $\textsf{cex}_\mathrm{b}()$
(0) $\quad\textsf{assert}_\mathrm{i}(P_k)$; $\textsf{assert}_\mathrm{i}(T_{k+1})$
(0) $\quad$**if** $\textsf{entailed}_\mathrm{i}(P_{k+1})$ **then return** success
(0) $\quad k := k + 1$

$\textsf{assert}_s(F)$: adds formula F to asserted formulas
$\textsf{entailed}_s(F)$: checks if F is entailed by asserted formulas
$\textsf{cex}_s()$: returns counterexample after failed entailment

Actual k-Induction with SMT Solvers

Solver maintains current set of *asserted* formulas

Two solver instances: b, i

(0) $\text{assert}_{\mathrm{b}}(I_0)$; $\text{assert}_{\mathrm{i}}(\neg P_1)$
(0) $k := 0$
(0) **loop**
(0) $\quad \text{assert}_{\mathrm{b}}(T_k)$ $\quad // \; T_0 = \mathit{true}$ by convention
(0) $\quad$ **if not** $\text{entailed}_{\mathrm{b}}(P_k)$ **then return** $\text{cex}_{\mathrm{b}}()$
(0) $\quad \text{assert}_{\mathrm{i}}(P_{-k})$; $\text{assert}_{\mathrm{i}}(T_{-k+1})$
(0) $\quad$ **if** $\text{unsat}_{\mathrm{i}}()$ **then return** success
(0) $\quad k := k + 1$

$\text{assert}_s(F)$: adds formula F to asserted formulas
$\text{entailed}_s(F)$: checks if F is entailed by asserted formulas
$\text{cex}_s()$: returns counterexample after failed entailment
$\text{unsat}_s()$: succeeds iff asserted formulas are jointly unsatisfiable

Definition of $\mathsf{entailed}_s$

(0) **proc** $\mathsf{entailed}_s(F)$
(0) $\quad$ $\mathsf{push}()$
(0) $\quad$ $\mathsf{assert}_s(\neg F)$
(0) $\quad$ $r := \mathsf{unsat}()$
(0) $\quad$ $\mathsf{pop}()$
(0) $\quad$ **return** r

$\mathsf{unsat}_s()$: succeeds iff asserted formulas are jointly unsatisfiable

Enhancements to k-Induction

- Abstraction and refinement
- Path compression
- Termination checks
- Property strengthening
- Invariant generation
- Multiple property checking

Path Compression (simplified)

Let $E[\mathbf{x}, \mathbf{y}]$ be a formula s.t. $E[\mathbf{x}, \mathbf{y}] \models_{\mathbb{L}} \forall \mathbf{z} \, (T[\mathbf{x}, \mathbf{z}] \Leftrightarrow T[\mathbf{y}, \mathbf{z}])$

(**Ex:** $E[\mathbf{x}, \mathbf{y}] := \mathbf{x} = \mathbf{y}$)

Path Compression (simplified)

Let $E[\mathbf{x},\mathbf{y}]$ be a formula s.t. $E[\mathbf{x},\mathbf{y}] \models_{\mathbb{L}} \forall \mathbf{z}\,(T[\mathbf{x},\mathbf{z}] \Leftrightarrow T[\mathbf{y},\mathbf{z}])$
(**Ex:** $E[\mathbf{x},\mathbf{y}] := \mathbf{x} = \mathbf{y}$)

Can strengthen the premise of the inductive step as follows

$$2. \quad P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \wedge C_k \models_{\mathbb{L}} P_{k+1}$$

where $C_k := \bigwedge_{0 \le i < j \le k} \neg E[\mathbf{x}_i, \mathbf{x}_j]$

Path Compression (simplified)

Let $E[\mathbf{x}, \mathbf{y}]$ be a formula s.t. $E[\mathbf{x}, \mathbf{y}] \models_{\mathbb{L}} \forall \mathbf{z}\, (T[\mathbf{x}, \mathbf{z}] \Leftrightarrow T[\mathbf{y}, \mathbf{z}])$
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where $C_k := \bigwedge_{0 \le i < j \le k} \neg E[\mathbf{x}_i, \mathbf{x}_j]$

Rationale: Consider a path that breaks original (2)

$$\pi := \sigma_0, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_j, \sigma_{j+1}, \ldots, \sigma_{k+1}$$

with $E[\sigma_i, \sigma_j]$ and $i < j$. If π is on an actual execution of $\mathcal{M}$, so is the shorter path $\sigma_0, \ldots, \sigma_i, \sigma_{j+1}, \ldots, \sigma_{k+1}$

Path Compression (simplified)

Let $E[\mathbf{x}, \mathbf{y}]$ be a formula s.t. $E[\mathbf{x}, \mathbf{y}] \models_{\mathbb{L}} \forall \mathbf{z}\, (T[\mathbf{x}, \mathbf{z}] \Leftrightarrow T[\mathbf{y}, \mathbf{z}])$
(**Ex:** $E[\mathbf{x}, \mathbf{y}] := \mathbf{x} = \mathbf{y}$)

Can further strengthen the premise of the inductive step with

2. $P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \wedge C_k \wedge N_k \models_{\mathbb{L}} P_{k+1}$

where $N_k := \bigwedge_{1 \leq i \leq k+1} \neg I[\mathbf{x}_i]$

Path Compression (simplified)

Let $E[\mathbf{x}, \mathbf{y}]$ be a formula s.t. $E[\mathbf{x}, \mathbf{y}] \models_{\mathbb{L}} \forall \mathbf{z}\, (T[\mathbf{x}, \mathbf{z}] \Leftrightarrow T[\mathbf{y}, \mathbf{z}])$
(**Ex:** $E[\mathbf{x}, \mathbf{y}] := \mathbf{x} = \mathbf{y}$)

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where $N_k := \bigwedge_{1 \leq i \leq k+1} \neg I[\mathbf{x}_i]$

Rationale: if
$\sigma_0, \ldots, \sigma_i, \ldots, \sigma_{k+1}$ breaks original (2) and $I[\sigma_i]$, then
$\sigma_i, \ldots, \sigma_{k+1}$ breaks the base case in the first place

Path Compression (simplified)

Let $E[\mathbf{x}, \mathbf{y}]$ be a formula s.t. $E[\mathbf{x}, \mathbf{y}] \models_{\mathbb{L}} \forall \mathbf{z}\, (T[\mathbf{x}, \mathbf{z}] \Leftrightarrow T[\mathbf{y}, \mathbf{z}])$
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Better E's than $\mathbf{x} = \mathbf{y}$ can be generated by an analysis of $\mathcal{M}$

More sophisticated notions of compressions have been proposed [dMRS03]

Termination check

$C_k := \bigwedge_{0 \le i < j \le k} \neg E[\mathbf{x}_i, \mathbf{x}_j]$

(0) **for** $k = 0$ **to** ∞ **do**
(0) **if not** $(I_0 \wedge T_1 \wedge \cdots \wedge T_k \models_{\mathbb{L}} P_k)$ **then**
(0) **return** fail
(0) **if** $(P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} P_{k+1})$ **then**
(0) **return** success
(0) **if** $(I_0 \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} \neg C_{k+1})$ **then**
(0) **return** success

Termination check

$C_k := \bigwedge_{0 \le i < j \le k} \neg E[\mathbf{x}_i, \mathbf{x}_j]$

for $k = 0$ **to** ∞ **do**
  if not $(I_0 \wedge T_1 \wedge \cdots \wedge T_k \models_{\mathbb{L}} P_k)$ **then**
    return fail
  if $(P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} P_{k+1})$ **then**
    return success
  if $(I_0 \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} \neg C_{k+1})$ **then**
    return success

Rationale: If the last test succeeds, every execution of length $k+1$ is compressible to a shorter one.
Hence, the whole reachable state space has been covered without finding counterexamples for P

Termination check

$C_k := \bigwedge_{0 \leq i < j \leq k} \neg E[\mathbf{x}_i, \mathbf{x}_j]$

(0) **for** $k = 0$ **to** ∞ **do**
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(0) **return** success
(0) **if** $(I_0 \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} \neg C_{k+1})$ **then**
(0) **return** success

Note: The termination check may slow down the process but increases precision in some cases
It even makes k-induction complete whenever the quotient $\mathcal{S}/E$ is finite (e.g., timed automata)

Property Strengthening

Suppose in the k-induction loop the SMT solver finds a counterexample $\sigma_0, \ldots, \sigma_{k+1}$ for

$$2. \quad P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} P_{k+1}$$

Property Strengthening

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$$2. \quad P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \not\models_{\mathbb{L}} P_{k+1}$$

Then this property is satisfied by σ_0:

$$F[\mathbf{x}_0] := \exists \mathbf{x}_1, \ldots, \mathbf{x}_{k+1}(P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \wedge \neg P_{k+1})$$

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(Naive) Algorithm:

1. find a $E[\mathbf{x}]$ in $\mathbb{L}$ satisfied by σ_0 and s.t. $E[\mathbf{x}] \models_{\mathbb{L}} F[\mathbf{x}]$
2. restart the process with $P[\mathbf{x}] \wedge \neg E[\mathbf{x}]$ in place of $P[\mathbf{x}]$

Correctness of Property Strengthening

$$F[\mathbf{x}_0] := \exists \mathbf{x}_1, \ldots, \mathbf{x}_{k+1}\, (P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \wedge \neg P_{k+1})$$

When F is satisfied by some σ_0, we

1. find a $E[\mathbf{x}]$ in $\mathbb{L}$ satisfied by σ_0 and s.t. $E[\mathbf{x}] \models_{\mathbb{L}} F[\mathbf{x}]$
2. replace $P[\mathbf{x}]$ with $Q[\mathbf{x}] := P[\mathbf{x}] \wedge \neg E[\mathbf{x}]$
3. "restart" the k-induction process

- If all states satisfying E are unreachable, we can remove them from consideration in the inductive step
- Otherwise, P is not invariant and the base case is guaranteed to fail with Q

Viability of Property Strengthening

$$F[\mathbf{x}_0] := \exists \mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \, (P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \wedge \neg P_{k+1})$$

When F is satisfied by some σ_0, we

1. find a $E[\mathbf{x}]$ in $\mathbb{L}$ satisfied by σ_0 and s.t. $E[\mathbf{x}] \models_{\mathbb{L}} F[\mathbf{x}]$
2. replace $P[\mathbf{x}]$ with $Q[\mathbf{x}] := P[\mathbf{x}] \wedge \neg E[\mathbf{x}]$
3. "restart" the k-induction process

- Normally, computing a E equivalent to F requires QE, which may be impossible or very expensive
- Under-approximating F might be cheaper but less effective in pruning unreachable states.

(Undirected) Invariant Generation

1. Generate invariants for $\mathcal{M}$ independently from P, either before or in parallel with k-induction

2. For each invariant $J[\mathbf{x}]$, add $J_0 \wedge \cdots \wedge J_{k+1}$ to induction hypothesis in induction step

$$P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} P_{k+1}$$

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Correctness: states not satisfying J are definitely unreachable and so can be pruned

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Correctness: states not satisfying J are definitely unreachable and so can be pruned

Viability: can use any property-independent method for invariant generation (template-based [KGT11], abstract interpretation-based, . . .)

(Undirected) Invariant Generation

1. Generate invariants for $\mathcal{M}$ independently from P, either before or in parallel with k-induction

2. For each invariant $J[\mathbf{x}]$, add $J_0 \wedge \cdots \wedge J_{k+1}$ to induction hypothesis in induction step

$$P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} P_{k+1}$$

Effectiveness: when P is invariant, can substantially improve

- speed, by making P k-inductive for a smaller k, and
- precision, by turning P from k-inductive for no k to k-inductive for some k

(Undirected) Invariant Generation

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$$P_0 \wedge \cdots \wedge P_k \wedge T_1 \wedge \cdots \wedge T_{k+1} \models_{\mathbb{L}} P_{k+1}$$

Shortcomings:

- Computed invariants may not prune the *right* unreachable states

- Adding too many invariants may swamp the SMT solver

Approximating R with Interpolation

Recall: If $R[\mathbf{x}]$ is the strongest inductive invariant for $\mathcal{M}$ in $\mathbb{L}$,

$\mathcal{M}$ is safe wrt some $E[\mathbf{x}]$ iff $R[\mathbf{x}] \land E[\mathbf{x}] \models_{\mathbb{L}} \bot$ ($\bot$ = false)

Problem: Such invariant may be very expensive or impossible to compute, or not even representable in $\mathbb{L}$

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Problem: Such invariant may be very expensive or impossible to compute, or not even representable in $\mathbb{L}$

Observation: It suffices to compute an $\widehat{R}[\mathbf{x}]$ such that

- $R[\mathbf{x}] \models_{\mathbb{L}} \widehat{R}[\mathbf{x}]$ ($\widehat{R}$ over-approximates R)
- $\widehat{R}[\mathbf{x}] \wedge B[\mathbf{x}] \models_{\mathbb{L}} \bot$ ($\widehat{R}$ is *disjoint* with E)

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- $\widehat{R}[\mathbf{x}] \wedge B[\mathbf{x}] \models_{\mathbb{L}} \bot$ ($\widehat{R}$ is *disjoint* with E)

A solution: Use theory interpolants to compute $\widehat{R}[\mathbf{x}]$

Logical Interpolation (simplified)

A logic $\mathbb{L}$ *has the interpolation property* if

for all $A[\mathbf{y}, \mathbf{x}]$ and $B[\mathbf{x}, \mathbf{z}]$ in $\mathbb{L}$ with $A[\mathbf{y}, \mathbf{x}] \wedge B[\mathbf{x}, \mathbf{z}] \models_{\mathbb{L}} \bot$

there is a $P[\mathbf{x}]$ in $\mathbb{L}$ such that

$$A[\mathbf{y}, \mathbf{x}] \models_{\mathbb{L}} P[\mathbf{x}] \quad \text{and} \quad P[\mathbf{x}] \wedge B[\mathbf{x}, \mathbf{z}] \models_{\mathbb{L}} \bot$$

P is *an interpolant* of A and B

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P is *an interpolant* of A and B

Intuitively, P

- is an abstraction of A from the viewpoint of B
- summarizes and explains in terms of the shared variables $\mathbf{x}$ why A is inconsistent with B

Logical Interpolation (simplified)

A logic $\mathbb{L}$ *has the interpolation property* if

for all $A[\mathbf{y},\mathbf{x}]$ and $B[\mathbf{x},\mathbf{z}]$ in $\mathbb{L}$ with $A[\mathbf{y},\mathbf{x}] \wedge B[\mathbf{x},\mathbf{z}] \models_{\mathbb{L}} \bot$

there is a $P[\mathbf{x}]$ in $\mathbb{L}$ such that

$$A[\mathbf{y},\mathbf{x}] \models_{\mathbb{L}} P[\mathbf{x}] \quad \text{and} \quad P[\mathbf{x}] \wedge B[\mathbf{x},\mathbf{z}] \models_{\mathbb{L}} \bot$$

P is *an interpolant* of A and B

Note: If $\mathbb{L}$ has quantifier elimination, the strongest interpolant (wrt $\models_{\mathbb{L}}$) is one equivalent to $\exists \mathbf{y}. A[\mathbf{y},\mathbf{x}]$

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P is *an interpolant* of A and B

Note: If $\mathbb{L}$ has quantifier elimination, the strongest interpolant (wrt $\models_{\mathbb{L}}$) is one equivalent to $\exists \mathbf{y}.A[\mathbf{y},\mathbf{x}]$

Interpolation is an over-approximation of quantifier elimination

Logics with Interpolation

The quantifier-free fragment of several theories used in SMT has the interpolation properties and computable interpolants:

- EUF [McM05b, FGG^{+}09]
- linear integer arithmetic with div_n [JCG09]
- real arithmetic [McM05b]
- arrays with diff [BGR11]
- combinations of any of the above [YM05, GKT09]
- ...

Interpolation-based Model Checking

Let $(I[\mathbf{x}], T[\mathbf{x}, \mathbf{x}'])$ be an encoding in $\mathbb{L}$ of a system $\mathcal{M}$

Consider the *bounded reachability* formulas $(R^i[\mathbf{x}])_i$ where

- $R^0[\mathbf{x}] := I[\mathbf{x}]$
- $R^{i+1}[\mathbf{x}] := R^i[\mathbf{x}] \vee \exists \mathbf{y}(R^i[\mathbf{y}] \wedge T[\mathbf{y}, \mathbf{x}])$

Interpolation-based Model Checking

Let $(I[\mathbf{x}], T[\mathbf{x}, \mathbf{x}'])$ be an encoding in $\mathbb{L}$ of a system $\mathcal{M}$

Consider the *bounded reachability* formulas $(R^i[\mathbf{x}])_i$ where

- $R^0[\mathbf{x}] := I[\mathbf{x}]$
- $R^{i+1}[\mathbf{x}] := R^i[\mathbf{x}] \lor \exists \mathbf{y}(R^i[\mathbf{y}] \land T[\mathbf{y}, \mathbf{x}])$

We prove safety wrt a state property E by using interpolation [McM05a] to compute a sequence $(\widehat{R}^i)_{i \geq 0}$ such that

- each $\widehat{R}^i$ overapproximates R^i and is disjoint with E
- the sequence is increasing wrt $\models_{\mathbb{L}}$
- the sequence has a fixpoint $\widehat{R}$ (modulo equivalence in $\mathbb{L}$)

Constructing $(\widehat{R}^i)_{i\geq 0}$

Fix some $k > 0$, $\quad \widehat{R}^0 := I[\mathbf{x}]$

Base Case.

$A := \widehat{R}^0[\mathbf{x}_0] \wedge T[\mathbf{x}_0, \mathbf{x}_1]$

$B := T[\mathbf{x}_1, \mathbf{x}_2] \wedge \cdots \wedge T[\mathbf{x}_{k-1}, \mathbf{x}_k] \wedge (E[\mathbf{x}_1] \vee \cdots \vee E[\mathbf{x}_k])$

if $A \wedge B$ is satisfiable in $\mathbb{L}$ **then**

fail (M is not safe wrt E)

else

compute an interpolant $P[\mathbf{x}_1]$ of A and B

$\widehat{R}^1 := \widehat{R}^0[\mathbf{x}] \vee P[\mathbf{x}]$

Constructing $(\widehat{R}^i)_{i \geq 0}$

Step Case.

for $i = 1$ **to** ∞

$A := \widehat{R}^i[\mathbf{x}_0] \wedge T[\mathbf{x}_0, \mathbf{x}_1]$

$B := T[\mathbf{x}_1, \mathbf{x}_2] \wedge \cdots \wedge T[\mathbf{x}_{k-1}, \mathbf{x}_k] \wedge (E[\mathbf{x}_1] \vee \cdots \vee E[\mathbf{x}_k])$

if $A \wedge B$ is satisfiable in $\mathbb{L}$ **then**

restart the whole process with a larger k

else

compute an interpolant $P[\mathbf{x}_1]$ of A and B

$\widehat{R}^{i+1} := \widehat{R}^i[\mathbf{x}] \vee P[\mathbf{x}]$

if $\widehat{R}^{i+1} \models_{\mathbb{L}} \widehat{R}^i[\mathbf{x}]$ **then** succeed (fixpoint found)

Notes on the Interpolation Method

- It needs an interpolating SMT solver
- It is not incremental: a counter-example in the step case requires a real restart
- It can be made terminating when $\mathcal{M}$ has finite bisimulation quotient
- In the terminating cases, it converges more quickly than basic k-induction
  (k is bounded by $\mathcal{M}$'s radius, not just the reoccurence radius as in k-induction)

Conclusions

- SMT-based Model Checking is the new frontier in safety checking thanks to powerful and versatile SMT solvers
- Several SAT-based methods can be lifted to the SMT case
- SMT encodings of transitions systems are basically 1-to-1
- Reasoning is at the same level of abstraction as in the original system
- Scalability and scope are higher than approaches based on propositional logic
- Several approaches and enhancements are being tried, capitalizing on different features of SMT solvers
- Lots of anecdotal evidence of successful applications

Future Directions

- Quantifiers are often needed to encode
  - parametrized model checking problems
    (coming, e.g., from multi-process systems)
  - problems with arrays
- New SMT techniques are needed to generate/work with quantified transition relations, interpolants, invariants, ...
- Synergistic combinations with traditional abstract interpretation tools seem possible
- We are starting to see some promising work in these directions, but much is left to do

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