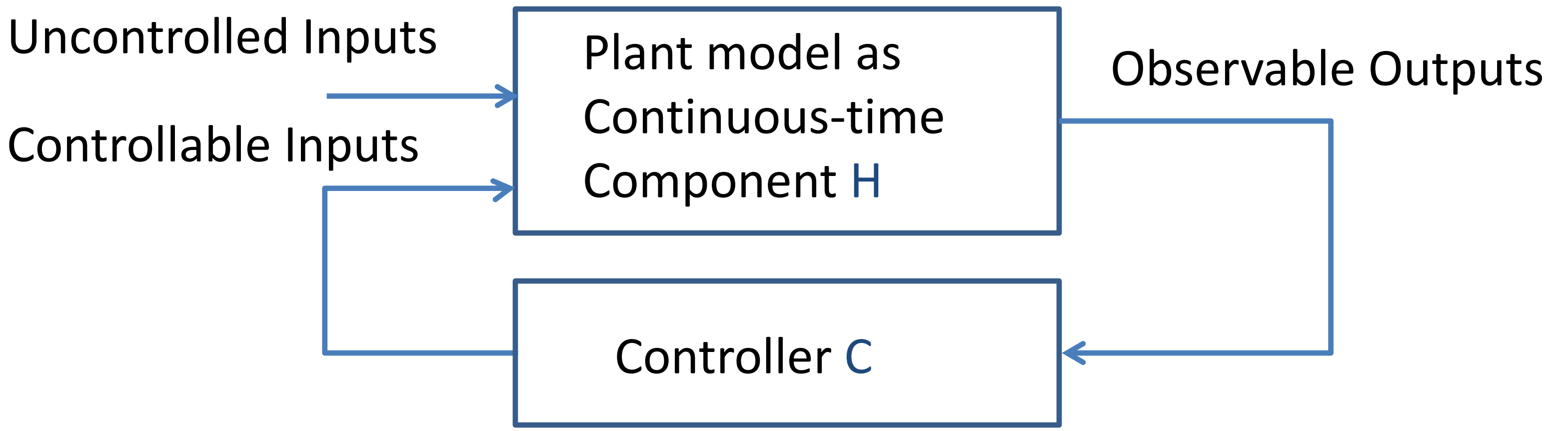
CS:4980
Foundations of Embedded Systems

Dynamical Systems

Part III

Control Design Problem



- We want to design a controller C so that C || H is stable
- Is there a mathematical way to check when a system is stable?
- Is there in fact a systematic way to design C so that C || H is stable?
- Yes, if the plant model is linear

Linear Component

A *linear* expression over real variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$$

where $a_1, a_2, \ldots$ are rational constants

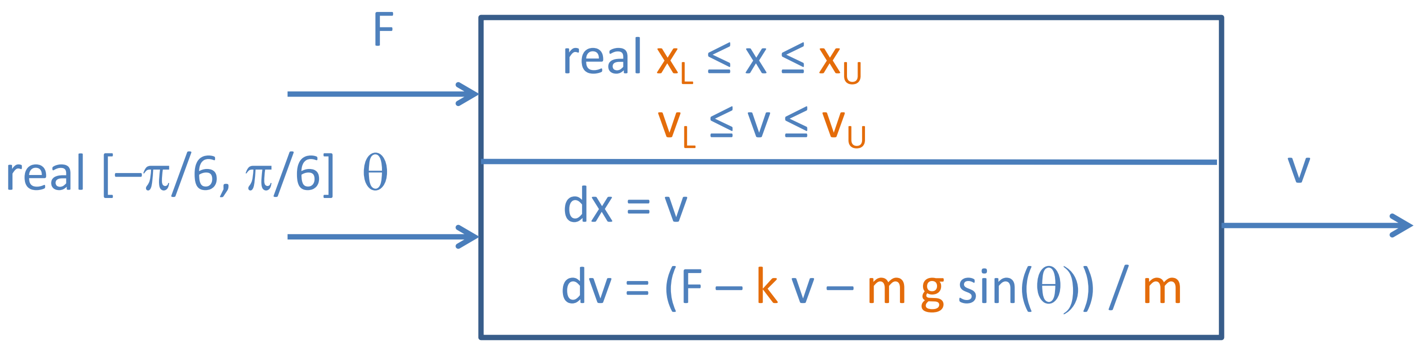
A continuous-time component H with state variables S, input variables I, and output variables O is *linear* if

1. for every state variable s, the dynamics is given by $ds = f_s(S, I)$, where f_s is a linear expression
2. every output variable o is defined by $o = h_o(S, I)$, where h_o is a linear expression

Examples

- **linear:** heatflow, car, helicopter
- **nonlinear:** pendulum

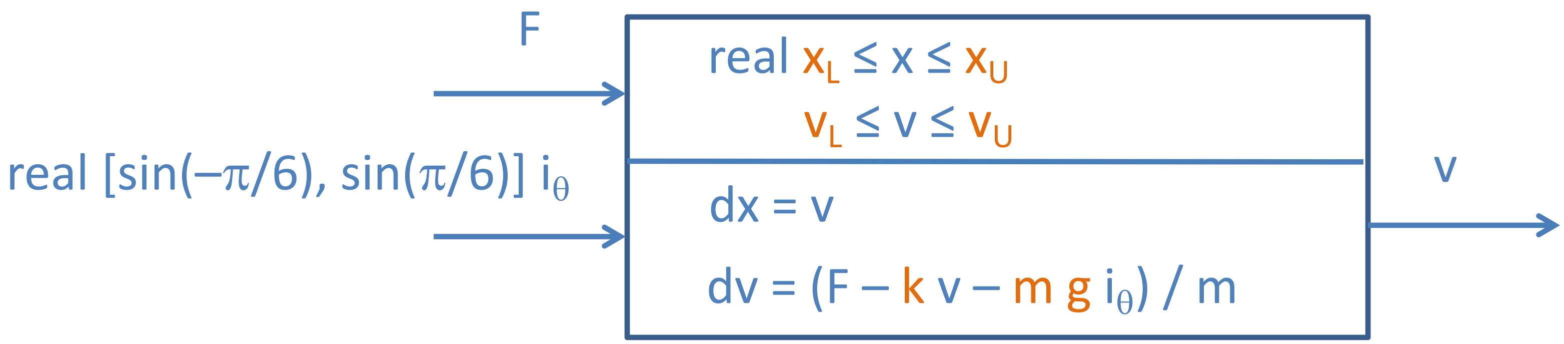
Continuous-time Component Car2



Problem: right-hand side of dv equation is not linear

Easy fix: replace disturbance θ by another variable $i_\theta = \sin \theta$

Continuous-time Component Car2



Rewriting to normal form:

$$dx = 0x + 1v + 0F + 0i_\theta$$

$$dv = 0x + (-k/m)v + (1/m)F + (-g)i_\theta$$

$$v = 0x + 1v + 0F + 0i_\theta$$

Matrix-based representation:

$S = (x \;\; v)^T \qquad I = (F \;\; i_\theta)^T \qquad O = (v)$

$$dS = A\,S + B\,I$$

$$O = C\,S + D\,I$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -k/m \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1/m & -g \end{pmatrix}$$

$$C = (0 \;\; 1) \qquad D = (0 \;\; 0)$$

(A,B,C,D) Representation of Linear Components

Suppose a linear continuous-time component has

- n state variables $S = \{x_1, x_2, \ldots, x_n\}$
- m input variables $I = \{u_1, u_2, \ldots, u_m\}$
- k output variables $O = \{y_1, y_2, \ldots, y_k\}$

Then the dynamics is given by

$$dS = A\,S + B\,I \quad \text{and} \quad O = C\,S + D\,I$$

where

A is an $n \times n$ matrix   C is a $k \times n$ matrix

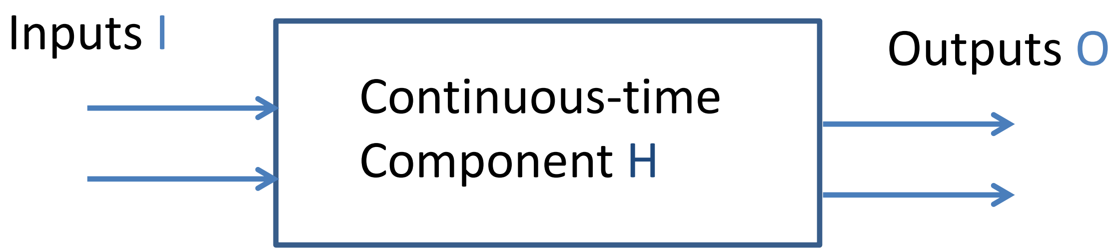
B is an $n \times m$ matrix   D is a $k \times m$ matrix

The rate of change of i-th state variable and the value of j-th output are

$$dx_i = A_{i,1}\,x_1 + A_{i,2}\,x_2 + \ldots + A_{i,n}\,x_n + B_{i,1}\,u_1 + B_{i,2}\,u_2 + \ldots + B_{i,m}\,u_m$$

$$y_j = C_{j,1}\,x_1 + C_{j,2}\,x_2 + \ldots + C_{j,n}\,x_n + D_{j,1}\,u_1 + D_{j,2}\,u_2 + \ldots + D_{j,m}\,u_m$$

Input-Output Linearity



With a fixed initial state, a continuous-time component H maps input signals **I**(t) to output signals **O**(t)

Theorem: If H is linear, then both of the following hold.

1. *Scaling*: If the output response of H to the input signal **I**(t) is **O**(t), then for every constant α, the output response of H to the input signal α**I**(t) is α**O**(t)
2. *Additivity*: If the output responses of H to the input signals $\mathbf{I}_1(t)$ and $\mathbf{I}_2(t)$ are $\mathbf{O}_1(t)$ and $\mathbf{O}_2(t)$, then the output response of H to the input signal $(\mathbf{I}_1 + \mathbf{I}_2)(t)$ is $(\mathbf{O}_1 + \mathbf{O}_2)(t)$

Response of Linear Systems

Consider a one-dimensional linear system with no inputs:

$$dx = ax \quad ; \text{initial state } x_0$$

Its execution is given by the signal

$$\mathbf{x}(t) = x_0 \, e^{at}$$

- Recall that $e^a = 1 + \sum_{n>0} a^n/n!$
- Verify that solution $\mathbf{x}(t)$ satisfies the differential equation
- See textbook on how solution is found

Response of Linear Systems

General Case with no inputs

- State set S
- Dynamics is given by

  $$dS = A\ S$$

  initial state s_0

- Execution is described by the signal

  $$\mathbf{S}(t) = e^{At}\, s_0$$

  - At = scalar product of A and t
  - $e^M = \mathrm{I} + M + M^2/2 + M^3/3! + M^4/4! + \ldots = \mathrm{I} + \sum_{n>0} M^n/n!$
  - I = identity matrix ($\mathrm{I}_{i,j}$ = if ($i = j$) then 1 else 0)

Response of Linear Systems

General Case with inputs input signal $\mathbf{I}(t)$

- State set S, input set I
- Dynamics is given by

  $dS = A\,S + B\,I$

  initial state s_0

- Execution is described by the signal

  $\mathbf{S}(t) = e^{At} s_0 + \int_0^t (e^{A(t-\tau)} B\, \mathbf{I}(\tau)\, d\tau)$

Matrix Exponential

- Matrix exponential $e^A = I + A + A^2/2 + A^3/3! + A^4/4! + \ldots$
- Each term in the sum is an $n \times n$ matrix
- How do we compute e^A?
  - If $A^k = 0$ for some k, the sum is finite and can be computed directly
  - If A is a diagonal matrix $\mathbf{D}(a_1, a_2, \ldots, a_n)$ (A_{ij} = if ($i = j$) then a_i else 0), then $e^A = \mathbf{D}(e^{a1}, e^{a2}, \ldots, e^{an})$
  - In general, the sum of the first k terms will give an approximation (whose quality is proportional to k)
  - Otherwise, we can use analytical methods based on eigenvalues and similarity transformations

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix, λ a scalar value and x an n-dimensional non-zero vector.

If $A x = \lambda x$, then x is an *eigenvector* of A, and

λ is the corresponding *eigenvalue*

- ❑ How to compute eigenvalues and eigenvectors?

- ❑ We solve the *characteristic equation* of A:

$$\det(A - \lambda I) = 0$$

Recall: the *determinant* $\det(M)$ of a 2×2 matrix M is

$$M_{1,1}M_{2,2} - M_{1,2}M_{2,1}$$

Eigenvalues and Eigenvectors

The eigenvalues of an $n \times n$ matrix A are the roots of the *characteristic polynomial* $p = \det(A - \lambda I)$

Note:

- The multiplicity of an eigenvalue (as a root of p) can be > 1
- An eigenvalue can be a complex number
- If A is a diagonal matrix then its eigenvalues are exactly its diagonal elements
- For a given eigenvalue λ, we can compute the corresponding eigenvector(s) by solving the linear system $A x = \lambda x$, with vector x of unknowns
- If all eigenvalues of are A distinct, then the set of corresponding eigenvectors is linearly independent

Similarity Transformation

Where P is an invertible $n \times n$ matrix of reals, consider the systems

1. H w/o inputs and with dynamics $dS = A\,S$; s_0 (initial state)
2. H' w/o inputs and with dynamics $dS' = J\,S'$; s'_0

   where $S' = P^{-1} S$, $J = P^{-1} A\,P$, $s'_0 = P^{-1} s_0$

Then, $\mathbf{S'}(t) = e^{Jt}\, s'_0$ and $\mathbf{S}(t) = P\, e^{Jt}\, P^{-1} s_0$

Note:

- H' is called the *transformed system* (since $S' = P^{-1} S$)
- Matrix $J = P^{-1} A\,P$ is said to be *similar* to A
- $dS' = d(P^{-1} S) = P^{-1} dS = P^{-1} A\,S = P^{-1} A\,P\,P^{-1} S = P^{-1} A\,P\,S' = J\,S'$
- When is this useful? When can choose P so that J is diagonal

Similarity Transformation using Eigenvectors

Consider system H with dynamics given by:

$$dS = A\ S \text{ ; initial state } s_0$$

1. Calculate eigenvalues $\lambda_1, \ldots, \lambda_n$ and suppose they are all distinct
2. Calculate corresponding eigenvectors $x_1, \ldots, x_n$ (which must be linearly independent, vertical vectors of size n)
3. Consider the $n \times n$ matrix $P = (x_1\ \ x_2\ \ \ldots\ \ x_n)$
4. Find its inverse P^{-1} (which must exist in this case)

Claim: The matrix $J = P^{-1} A\ P$ is the diagonal matrix $\mathbf{D}(\lambda_1, \ldots, \lambda_n)$

Then, execution of H is given by:

$$\mathbf{S}(t) = P\ \mathbf{D}(e^{\lambda 1\ t}, \ldots, e^{\lambda n\ t})\ P^{-1} s_0$$

Example: Response of Linear Systems

Consider 2-dimensional system with dynamics given by

$ds_1 = 4\, s_1 + 6\, s_2$ initial state $(s_1, s_2) = (1, 1)^T$

$ds_2 = s_1 + 3\, s_2$

1. Compute eigenvalues λ_1 and λ_2 of $A = ((4\ \ 1)^T\ \ (6\ \ 3)^T)$
   - $\lambda_1 = 6$ and $\lambda_2 = 1$
2. Compute eigenvectors x_1 and x_2
   - $x_1 = (3\ \ 1)^T$ and $x_2 = (2\ \ -1)^T$
3. Choose the similarity transformation matrix $P = (x_1\ \ x_2) = ((3\ \ 1)^T\ \ (2\ \ -1)^T)$
4. Compute the inverse P^{-1} of P
   - $P^{-1} = ((-1\ \ -1)^T\ \ (-2\ \ 3)^T) / (-3-2) = ((1/5\ \ 1/5)^T\ \ (2/5\ \ -3/5)^T)$
5. Verify that $J = P^{-1} A P$ is diagonal matrix $\mathbf{D}(\lambda_1, \lambda_2) = ((6\ \ 0)^T\ \ (0\ \ 1)^T)$
   - $J = P^{-1} A P = ((6/5\ \ 1/5)^T\ \ (12/5\ \ -3/5)^T)\, ((3\ \ 1)^T\ \ (2\ \ -1)^T) = ((6\ \ 0)^T\ \ (0\ \ 1)^T)$
6. Desired solution is $\mathbf{S}(t) = P\, \mathbf{D}(e^{\lambda 1\, t}, e^{\lambda 2\, t})\, P^{-1} (1, 1)^T$
   - $\mathbf{S}(t) = ((3\ \ 1)^T\ \ (2\ \ -1)^T)\, ((e^{6t}\ \ 0)^T\ \ (0\ \ e^{t})^T)\, ((1/5\ \ 1/5)^T\ \ (2/5\ \ -3/5)^T)\, (1, 1)^T = \ldots$

Back to Equilibria and Stability

Consider a closed linear system H with dynamics given by:

$$dS = A\,S$$

Recall: a state s is an equilibrium state of H if $A\,s = 0$

How to compute equilibria: solve system of linear equations

Prop. 1: State 0 is an equilibrium

Prop. 2: If A is invertible, then 0 is the sole equilibrium

If state s is a non-zero equilibrium of H, consider the transformed system H' with state $S' = S - s$

The equilibria 0 of H' and s of H have the same properties

Back to Equilibria and Stability

Henceforth, we will focus on closed linear systems H and their equilibrium state 0

Definition:

1. H is *stable* if state 0 is stable
2. H is *asymptotically stable* if state 0 is asymptotically stable

Stability: One-Dimensional System

Consider a one-dimensional linear system H with dynamics given by: $dx = a\,x$; s_0

Recall: H is asymptotically stable iff 0 is asymptotically stable iff

1. (Stable) For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all initial states s with $||s|| < \delta$ and for all times t, $||e^{at}\, s|| < \varepsilon$
2. (Asymptotically) There is a $\delta > 0$ such that for all initial states s with $||s|| < \delta$, $||e^{at}\, s||$ goes to 0 as t goes to ∞

A. **Case a < 0**: $e^{at}\, s$ converges exponentially to 0 as t goes to ∞, regardless of s. Asymptotically stable

B. **Case a = 0**: dynamics is $dx = 0$. The state stays equal to initial state s_0. Stable but not asymptotically stable (unless $s_0 = 0$)

C. **Case a > 0**: $e^{at}\, s$ grows exponentially as t increases, and thus, state diverges away from 0. Unstable!

Stability: Diagonal State Dynamics

Consider n-dimensional linear system H with dynamics given by: $dS = A\,S;\ s$ with $A = \mathbf{D}(a_1, \ldots, a_n)$

Each dimension evolves independently: the i-th component of $\mathbf{S}(t)$ is $e^{a_i t}\, s_i$

A. **All coefficients $a_i < 0$**: State converges to the equilibrium 0 regardless of s. Asymptotically stable

B. **All coefficients $a_i \leq 0$**: Stable but not asymptotically stable if some coefficient $a_j = 0$ (j-th state component stays unchanged)

C. **Some coefficient $a_i > 0$**: Some state component grows unboundedly away from equilibrium 0. Unstable!

Similarity Transformations and Stability

Consider system H with dynamics given by: $dS = A\,S\,;\,s_0$

Let P be an invertible $n \times n$ matrix, and $J = P^{-1} A\, P$

Consider system H' with state $S' = P^{-1} S$ (and note that $S = P\, S'$)

Response signal of transformed system H': $\mathbf{S'}(t) = e^{Jt} P^{-1} s_0$

Response signal of original system H: $\mathbf{S}(t) = P\, e^{Jt} P^{-1} s_0$

Note: Response **S'**(t) is a linear transformation of **S**(t) and vice versa. Hence:

- If **S**(t) is bounded iff **S'**(t) is bounded
- If **S**(t) converges to 0 iff **S'**(t) converges to 0

Prop. 1: H is stable iff H' is stable

Prop. 2: H is asymptotically stable iff H' is asymptotically stable

Eigenvalues and Stability

Consider system H with dynamics given by: $dS = A\ S$

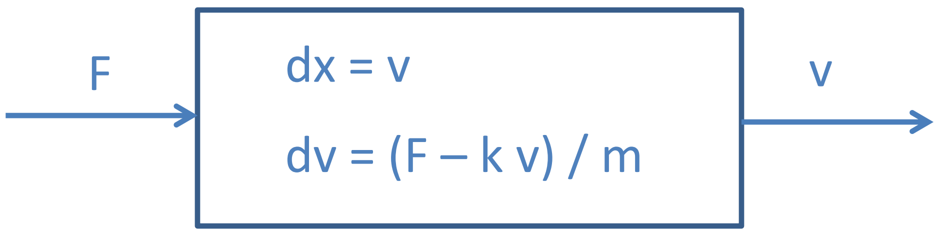
Suppose all eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are real and distinct

- Then the set of eigenvectors, $x_1, \ldots, x_n$ is guaranteed to be linearly independent
- Choose $n \times n$ matrix $P = (x_1\ \ x_2\ \ \ldots\ \ x_n)$ for similarity transformation
- The matrix $J = P^{-1} A\ P$ is the diagonal matrix $\mathbf{D}(\lambda_1,\ldots, \lambda_n)$

- If all eigenvalues are negative, then the transformed system H' is asymptotically stable, and so is H
- If all eigenvalues are non-positive, then H' is stable, and so is H

Theorem: A system H with dynamics $dS = A\ S$ is asymptotically stable iff each eigenvalue of A has a negative real part

Continuous-time Component Car



- Let $S = (x \ \ v)^T$
- The matrix A is $(\,(0 \ \ 0)^T \ \ (1 \ \ {-k/m})^T\,)$
- Eigenvalues: 0 and $-k/m$
- Stable but not asymptotically stable
- If we consider only the dimension v, then asymptotically stable

Exercise: Set $\mathbf{F}(t) = 0$ for all t, and analyze what happens if we perturb the system from the equilibrium $(0 \ \ 0)^T$

Lyapunov Stability vs BIBO Stability

Consider linear component H with dynamics given by

$$dS = A\,S + B\,I \qquad O = C\,S + D\,I$$

BIBO stability: Starting from initial state 0, if the input is a bounded signal, output must be a bounded signal

Theorem: For linear components, asymptotic stability implies BIBO stability

Proof of the theorem relies of analysis of dynamical systems using transfer functions

Note: Asymptotic stability depends only on the properties of matrix A

Credits

Notes based on Chapter 6 of

Principles of Cyber-Physical Systems
by Rajeev Alur
MIT Press, 2015