CS:4980
Foundations of Embedded Systems

Dynamical Systems
Part II

Properties of Dynamical Systems

Correctness requirements for dynamical systems:

- Safety
- Liveness
- Stability

Cruise controller example:

- **Safety**: speed should always be within certain bounds
- **Liveness**: actual speed should eventually converge to desired speed
- **Stability**: as the road slope changes, speed should change gradually

Stability of Dynamical Systems

Intuitively, a dynamical system is *stable* if

small perturbations in the input ➔ small changes in the output

Classical mathematical formalization of stability:

- Lyapunov stability of equilibria
- Bounded-Input-Bounded-Output stability of response

Stability is studied for *closed* continuous-time components, i.e., components with with no inputs

- If H has inputs, we can analyze it by fixing them to a constant

Equilibria of Dynamical Systems

Consider a closed continuous-time component H

- Assume state x is n-dimensional, and its dynamics is Lipschitz-continuous and given by $dx/dt = f(x)$

Definition: A state x_e is an *equilibrium* of H if $f(x_e) = 0$

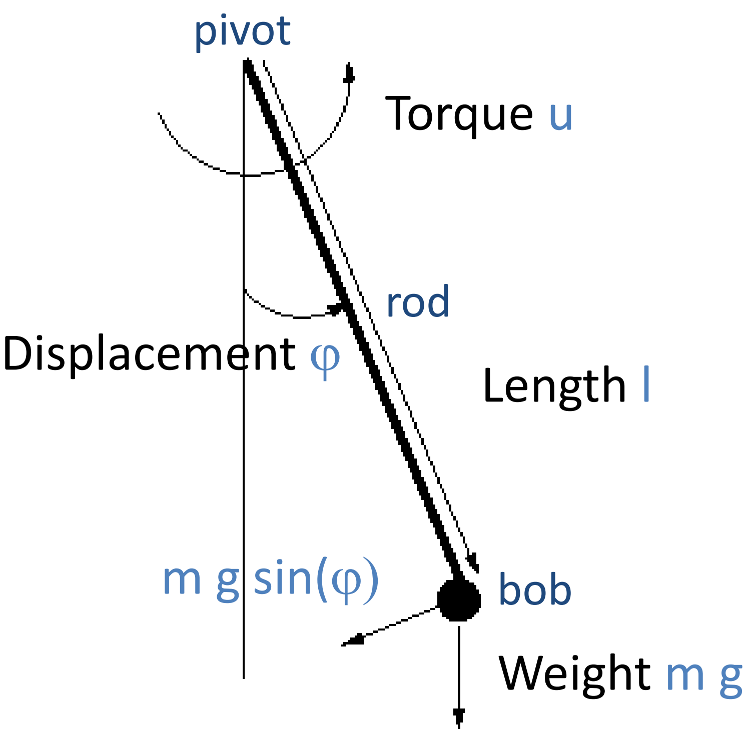
Note: if a component H starts in an equilibrium state x_e, it stays in this state at all times

Pendulum Equilibria

Dynamics when $u(t) = 0$

$d\varphi = \nu$

$d\nu = -g \sin(\varphi) / l$



Equilibrium states are solutions of:

$\nu = 0$

$\sin(\varphi) = 0$

$-\pi \leq \varphi < \pi$

Equilibrium state 1: $\nu = 0, \varphi = 0$ Pendulum is vertically downwards

Equilibrium state 2: $\nu = 0, \varphi = -\pi$ Pendulum is vertically upwards

Lyapunov Stability

- Consider a closed continuous-time component H with Lipschitz-continuous dynamics $dx/dt = f(x)$
- Given an initial state s_0, let $\mathbf{x}[s]$ denote the *response signal*, the unique solution for the initial value problem

$$x(0) = s_0 \; ; \;\; dx/dt = f(x)$$

Stability of an equilibrium: if the system is in an equilibrium state and we perturb its state slightly, as time passes,

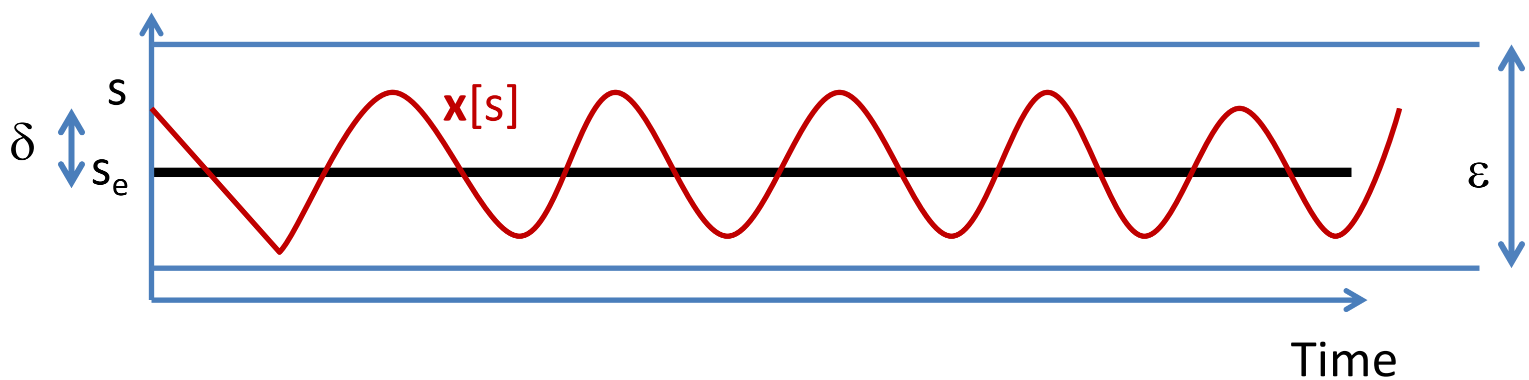
- will the system stay close to the equilibrium state ?
- will the system eventually return to that equilibrium state?

Lyapunov Stability Conditions

Note: if an initial state s_e is an equilibrium state then $\mathbf{x}[s_e](t) = s_e$ for all times t (i.e., it is a constant function)

Suppose another initial state s is close to s_e, do the states along the signal $\mathbf{x}[s]$ stay close to s_e as well? If so, s_e is said to be *stable*

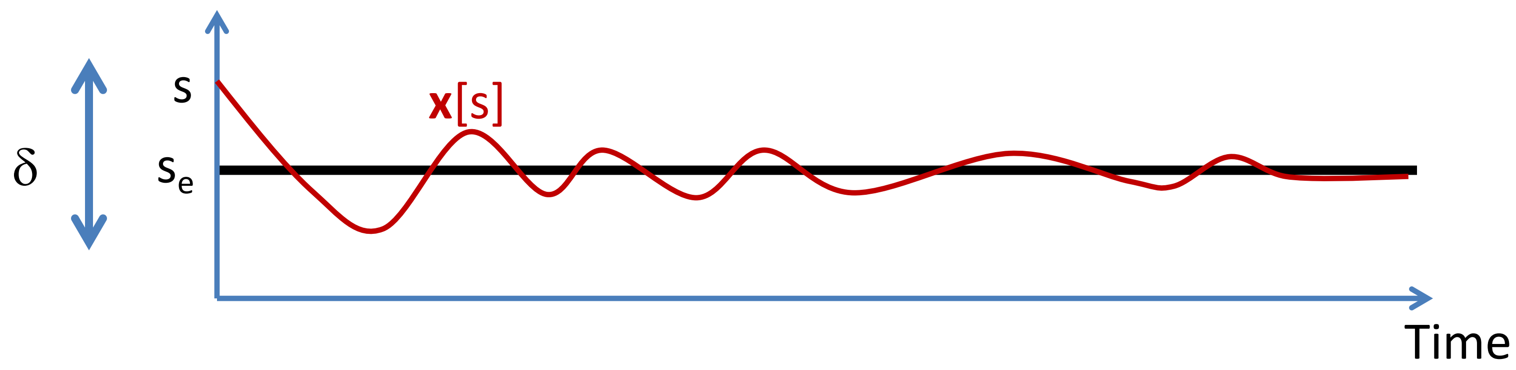
Formally, s_e is *stable* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all states s with $||s_e - s|| < \delta$ and times t, $||\mathbf{x}[s](t) - s_e|| < \varepsilon$



Lyapunov Stability Conditions

If, in addition, the response signal $\mathbf{x}[s]$ converges to the equilibrium state s_e, then s_e is *asymptotically stable*

Formally, s_e is *asymptotically stable* if it is stable and there exists a $\delta > 0$ such that for all states s with $||s_e - s|| < \delta$, $\lim_{t \to \infty} \mathbf{x}[s](t) = s_e$



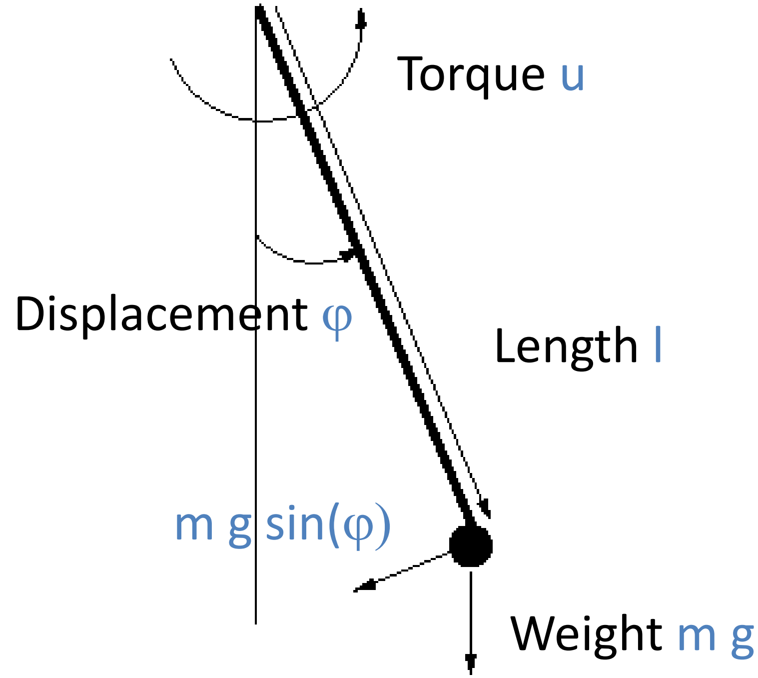
Pendulum Equilibria

Equilibrium state 1: $\nu = 0$; $\varphi = 0$

Pendulum is vertically downwards

Stable, but not asymptotically stable

(in frictionless model)

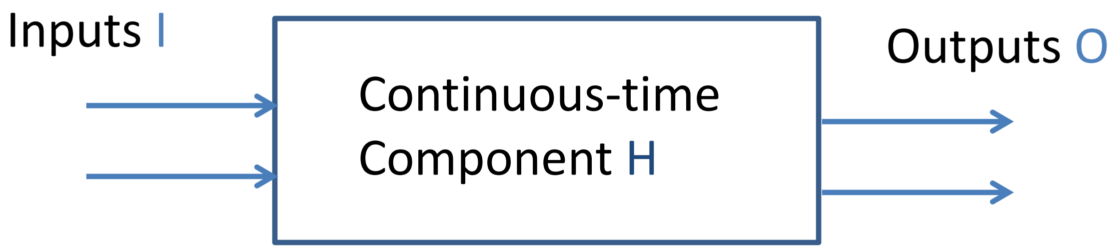


Equilibrium state 2: $\nu = 0$; $\varphi = -\pi$

Pendulum is vertically upwards

Unstable !

Input-Output Stability



- A continuous-time component H maps input signals **I**(t) to output signals **O**(t)
- *Input-output stability:* If we change the input signal slightly, the output signal should change only slightly
- Suffices to focus on bounded signals

Input-Output Stability

Definition: A signal $\mathbf{x}(t)$ is *bounded* if there exists constant Δ such that

$||\mathbf{x}(t)|| \leq \Delta$ at all times t

Examples:

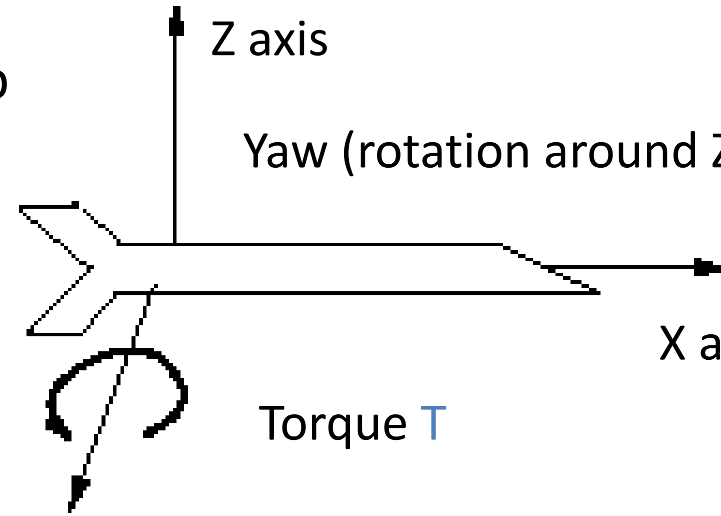
- Constant signal $\mathbf{x}(t) = a$: bounded
- Linearly increasing signal $\mathbf{x}(t) = a + bt$ with $b \neq 0$: not bounded
- Exponential signal $\mathbf{x}(t) = a + e^{bt}$ with $b \leq 0$: bounded
- Sinusoidal signals $\mathbf{x}(t) = a \sin(bt)$: bounded

A continuous-time component H with Lipschitz-continuous dynamics is *Bounded-Input-Bounded-Output (BIBO) stable* if

for every bounded input signal $\mathbf{I}(t)$, the output response signal $\mathbf{O}(t)$ from initial state $\mathbf{x}(0) = 0$ is bounded

Helicopter Model (Simplified)

Design problem: What torque should the tail rotor apply to keep helicopter from spinning?

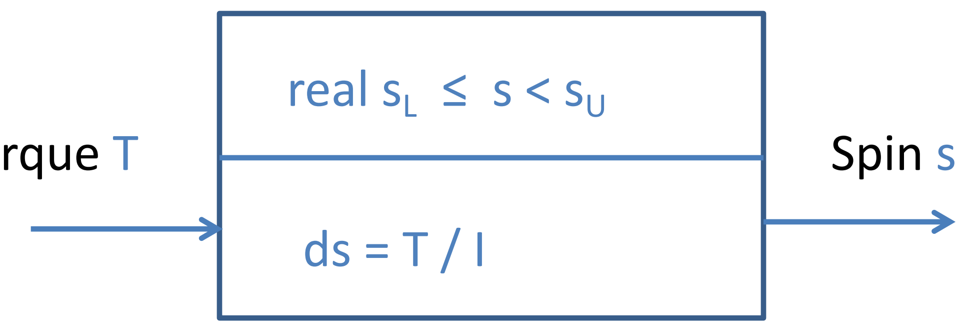


Spin ($d\theta/dt$): s

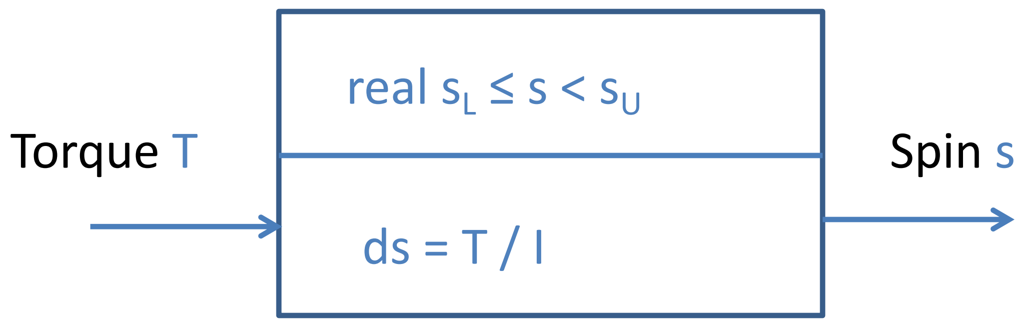
Moment of inertia: I

Equation of motion:

$$ds/dt = T / I$$



Stability of Helicopter Model



- Is the system BIBO stable?
- Consider bounded constant input signal $\mathbf{T}(t) = T_0$
- Output response from initial state 0 not bounded: $\mathbf{s}(t) = T_0 t / I$
- Not BIBO stable!
- What are the equilibria?
  - Set input torque to 0. If initial spin is c, it will stay c. Thus every initial state is an equilibrium state
  - Each such state c is stable but not asymptotically stable!

Credits

Notes based on Chapter 6 of

Principles of Cyber-Physical Systems
by Rajeev Alur
MIT Press, 2015