CS:4350 Logic in Computer Science

Quantified Boolean Formulas

Cesare Tinelli

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The University of Iowa

Credits

These slides are largely based on slides originally developed by **Andrei Voronkov** at the University of Manchester. Adapted by permission.

Outline

Two-Player Games

Does she have a winning strategy?

Two-Player Games: The Satisfiability Game

Given: a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$

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Player P wins if after n rounds the chosen values satisfy formula G

Suppose Both Players Make no Mistakes: Who Wins?

Consider several special cases:

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5.	$p_1 \wedge \neg p_1$	G is unsatisfiable, Q always wins!
6.	$p_1 \leftrightarrow q_1$	each move by P can be beaten by Q

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The existence of a winning strategy can be expressed by the *quantified Boolean formula*

$$\exists p_1 \forall q_1 \exists p_2 \forall q_2 \cdots \exists p_n \forall q_n \, G$$

Quantified Boolean Formulas

Propositional Formula:

- Every Boolean variable is a (propositional) formula
- $\top$ and $\bot$ are formulas
- If F is a formula, then $\neg F$ is a formula
- If $F_1, \ldots, F_n$ are formulas, where $n \geq 2$,
  then $F_1 \wedge \cdots \wedge F_n$ and $F_1 \vee \cdots \vee F_n$ are formulas
- If F and G are formulas, then $F \rightarrow G$ and $F \leftrightarrow G$ are formulas

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Quantified Boolean Formulas (QBFs):

- Every propositional formula is a QBF
- If p is a Boolean variable and F is a QBF, then $\forall p\, F$ and $\exists p\, F$ are QBFs

Quantifiers

- $\forall$ is called the *universal quantifier* (symbol)
- $\exists$ is called the *existential quantifier* (symbol)
- $\forall p\, F$ is read as "for all p, F"
- $\exists p\, F$ is read as "there exists p such that F" or "for some p, F"

For every variable p, we treat $\forall p$ and $\exists p$ as unary operators applied to a formula F

$\forall p$ and $\exists p$ have the highest precedence (like $\neg$), e.g.:

$$\forall p\, p \to q \quad \equiv \quad (\forall p\, p) \to q \quad \not\equiv \quad \forall p\, (p \to q)$$

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Changing interpretations pointwise

Let $\mathcal{I}$ be an interpretation

Notation:

$$\mathcal{I}[p \mapsto b](q) \stackrel{\text{def}}{=} \begin{cases} \mathcal{I}(q), & \text{if } p \neq q \\ b, & \text{if } p = q \end{cases}$$

Example: $\mathcal{I} = \{ p \mapsto 1, q \mapsto 0, r \mapsto 1 \}$

$$\begin{array}{rcl} \mathcal{I}[q \mapsto 1] & = & \{p \mapsto 1, q \mapsto 1, r \mapsto 1\} \\ \mathcal{I}[q \mapsto 0] & = & \{p \mapsto 1, q \mapsto 0, r \mapsto 1\} \quad = \quad \mathcal{I} \\ \mathcal{I}[p \mapsto 0] & = & \{p \mapsto 0, q \mapsto 0, r \mapsto 1\} \end{array}$$

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QBF Semantics

1. $\mathcal{I}(\top) = 1$ and $\mathcal{I}(\bot) = 0$
2. $\mathcal{I}(F_1 \wedge \cdots \wedge F_n) = 1$ iff $\mathcal{I}(F_i) = 1$ for all i
3. $\mathcal{I}(F_1 \vee \cdots \vee F_n) = 1$ iff $\mathcal{I}(F_i) = 1$ for some i
4. $\mathcal{I}(\neg F) = 1$ iff $\mathcal{I}(F) = 0$
5. $\mathcal{I}(F \rightarrow G) = 1$ iff $\mathcal{I}(F) = 0$ or $\mathcal{I}(G) = 1$
6. $\mathcal{I}(F \leftrightarrow G) = 1$ iff $\mathcal{I}(F) = \mathcal{I}(G)$
7. $\mathcal{I}(\forall p\, F) = 1$ iff $\mathcal{I}[p \mapsto 0](F) = 1$ and $\mathcal{I}[p \mapsto 1](F) = 1$
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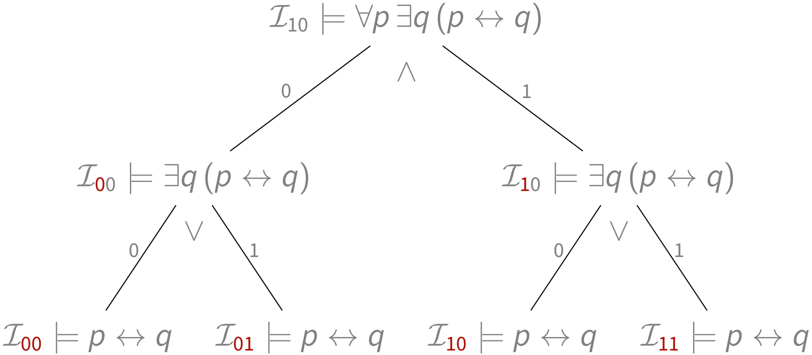

Evaluating a formula

Notation: Denote any interpretation $\{ p \mapsto b_1, q \mapsto b_2 \}$ by $\mathcal{I}_{b_1 b_2}$
Use wildcards $*$ to denote *any* Boolean value

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The variables p and q are *bound* by the quantifiers $\forall p$ and $\exists q$, so
the value of the formula does not depend on the values p and q

Subformula

Propositional formulas:

- F is the immediate subformula of $\neg F$
- $F_1, \ldots, F_n$ are the immediate subformulas of $F_1 \wedge \cdots \wedge F_n$
- $F_1, \ldots, F_n$ are the immediate subformulas of $F_1 \vee \cdots \vee F_n$
- F_1 and F_2 are the immediate subformulas of $F_1 \rightarrow F_2$
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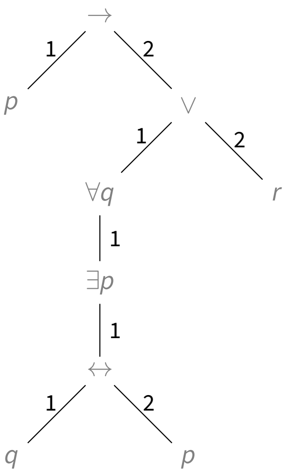
Quantified Boolean formulas:

- F is the immediate subformula of $\forall p\, F$ and of $\exists p\, F$

Positions and polarity by example



$p \to \forall q \, \exists p (q \leftrightarrow p) \vee r$

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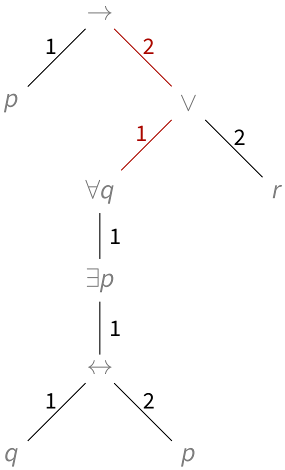

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$F|_{2.1} = \forall q\, \exists p(q \leftrightarrow p)$

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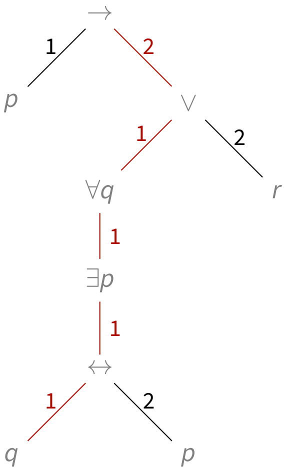

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$F|_{2.1} = \forall q\, \exists p(q \leftrightarrow p)$

$F|_{2.1.1.1.1} = q$

Positions and Polarity

Let $F|_\pi = A$

Propositional formulas:

- If A has the form $\neg A_1$,
  then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} A_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} -pol(F, \pi)$
- If A has the form $A_1 \land \cdots \land A_n$ or $A_1 \lor \cdots \lor A_n$ and $i \in \{ 1, \ldots, n \}$,
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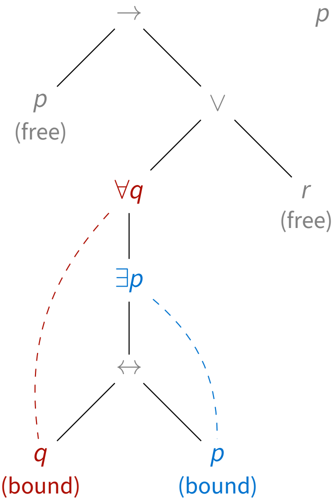
Quantified Boolean formulas:

- If A has the form $\forall p\, B$ or $\exists p\, B$,
  then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} B$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} pol(F, \pi)$

Free and bound variables by example



$p \to \forall q \, \exists p \, (q \leftrightarrow p) \lor r$

Free and bound occurrences in programs

- Free variables in formulas are analogous to global variables in programs
- Bound variables in formulas are analogous to local variables in programs

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int offset_sym_diff(int i, int j)
{
  int k = i > j ?  i - j :  j - i;
  return a + k
}


sum = i + offset_sym_diff(3,4);
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Free and bound occurrences of variables

Let F be a QBF and p be atom of at position π

The occurrence of p at position π in F is *bound* if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall p\, G$ or $\exists p\, G$

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Closed formula: formula with no free variables

Only free variables matter for truth

The truth value of a QBF formula F depends only on the values of its free variables:

Lemma 1

Suppose $\mathcal{I}_1(p) = \mathcal{I}_2(p)$ *for all free variables* p *of* F. *Then*

$$\mathcal{I}_1 \models F \textit{ iff } \mathcal{I}_2 \models F$$

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Theorem 2
Let F be a closed formula and let $\mathcal{I}_1, \mathcal{I}_2$ be two interpretations. Then

$$\mathcal{I}_1 \models F \text{ iff } \mathcal{I}_2 \models F$$

Truth, Validity and Satisfiability

Validity and satisfiability are defined as for propositional formulas

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For every interpretation $\mathcal{I}$ and closed formula F the following statements are equivalent: (i) $\mathcal{I} \models F$; (ii) F is satisfiable; and (iii) F is valid.

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Satisfiability can be expressed through satisfiability/validity of closed formulas:

Lemma 4

Let F be a formula with free variables $p_1, \ldots, p_n$.

- *F is satisfiable iff $\exists p_1 \cdots \exists p_n\, F$ is satisfiable/valid*
- *F is valid iff the formula $\forall p_1 \cdots \forall p_n\, F$ is satisfiable/valid*

Substitutions for propositional formulas

Substitution: F_p^G: denotes the formula obtained from F by replacing all occurrences of variable p by G

Example:

$$((p \vee s) \wedge (q \rightarrow p))_p^{(l \wedge s)} = ((l \wedge s) \vee s) \wedge (q \rightarrow (l \wedge s))$$

Property: Applying any substitution to a valid formula results in a valid formula

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Substitutions for **quantified** formulas

Some problems …

Substitutions for quantified formulas

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Consider $\exists q\, (\neg p \leftrightarrow q)$

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Free variables are parameters: we can only substitute for parameters.
But a variable can have both free and bound occurrences in a formula, e.g.,

$$\forall p\,((p \to q) \lor \neg p) \land (q \lor (q \to p))$$

Renaming bound variables

Notation: $\exists\!\!\forall$: any of $\exists$, $\forall$

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Renaming bound variables in $F[\ \exists\!\!\!\forall pG\ ]$:

1. Take a *fresh* variable q (i.e., a variable not occurring in F)
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$\exists r\, (\forall p\, ((p \to r) \wedge p)\,) \vee p$ rename p to q, obtaining

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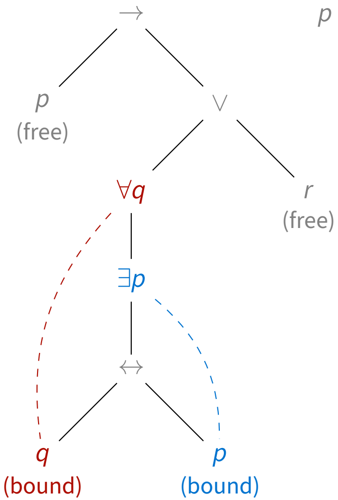
Lemma 5

$F[\,\exists\!\!\!\forall p G\,] \equiv F[\,\exists\!\!\!\forall q G'\,]$

Free and bound variables by example



$p \rightarrow \forall q \exists p (q \leftrightarrow p) \vee r$

Rectified formulas

Rectified formula F:

1. no variable appears both free and bound in F
2. for every variable p, there is at most one occurrence of quantifier $\exists\!\!\forall p$ in F

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Rectification: Example

$$p \to \exists p\,(p \land \forall p\,(p \lor r \to \neg p)) \Rightarrow$$

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Renaming each bound variable to a fresh one preserves equivalence

Another problem

$\exists q\, (\neg p \leftrightarrow q)$ This formula is valid (whatever value p has, choose the opposite for q)

substitute p by q

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Substitutions below a quantifier should not lead to *variable capturing*

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Suppose we want to substitute G for p in $F[p]$

Requirement: no free variables in G become bound in F_p^G

(In previous example, $(\exists q\,(\neg p \leftrightarrow q))_p^q$ does not satisfy this requirement)

Uniform solution: renaming of bound variables before application of substitution

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Since $\exists q\,(\neg p \leftrightarrow q) \equiv \exists r(\neg p \leftrightarrow r)$

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Unifor tution

Examp

Since $\exists$

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From now on, we always assume that:

1. formulas are rectified
2. all substitutions satisfy the requirement above

Equivalent replacement

Lemma 6
Let $\mathcal{I}$ be an interpretation and $\mathcal{I} \models F_1 \leftrightarrow F_2$. Then $\mathcal{I} \models G[F_1] \leftrightarrow G[F_2]$.

Theorem 7 (Equivalent Replacement)
Let $F_1 \equiv F_2$. Then $G[F_1] \equiv G[F_2]$.

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More equivalences

Theorem 8

The following holds for all QBFs F:

1. $\forall p_1 \forall p_2\, F \;\equiv\; \forall p_2 \forall p_1\, F$
2. $\exists p_1 \exists p_2\, F \;\equiv\; \exists p_2 \exists p_1\, F$
3. $\exists\!\!\!\forall p\, F \;\equiv\; F$ *if p does not occur free in F*
4. $\forall p\, F \;\equiv\; F_p^{\perp} \wedge F_p^{\top}$
5. $\exists p\, F \;\equiv\; F_p^{\perp} \vee F_p^{\top}$

Note: In general, $\exists p_1 \forall p_2 F \;\not\equiv\; \forall p_2 \exists p_1 F$!

Example:

- $\forall p \exists q\, (p \leftrightarrow q) \;\equiv\; \top$

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Prenex form

Quantifier-free formula: no quantifiers (that is, propositional)

Prenex formula: formula of the form

$$\overbrace{\exists\forall_1 p_1 \cdots \exists\forall_n p_n}^{\text{quantifier prefix}} \underbrace{G}_{\text{matrix}}$$

with G quantifier-free

Outermost prefix of $\exists\forall_1 p_1 \cdots \exists\forall_n p_n G$: the longest subsequence $\exists\forall_1 p_1 \cdots \exists\forall_k p_k$ of $\exists\forall_1 p_1 \cdots \exists\forall_n p_n$ such that $\exists\forall_1 = \cdots = \exists\forall_k$

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Example

- outermost prefix of $\forall p\, \forall q\, \exists r (r \land p \to q)$: $\forall p\, \forall q$
- outermost prefix of $\exists p\, \forall q\, \exists r (r \land p \to q)$: $\exists p$

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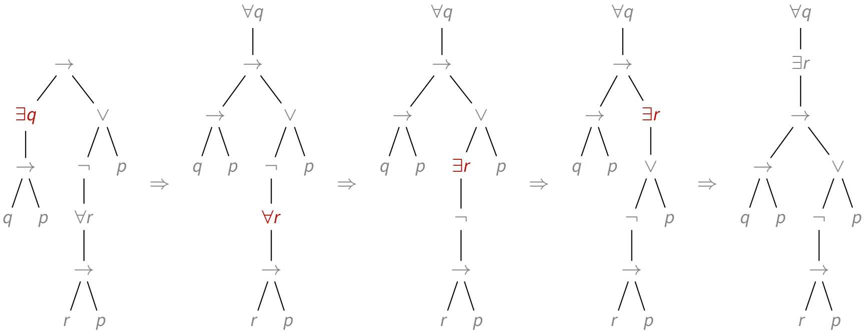
A formula F is a *prenex form* of a formula G if F is prenex and $F \equiv G$

Conversion to prenex form, Example I



Conversion to prenex form, Example I

Same conversion:

$$
\begin{aligned}
&(\exists q\,(q \rightarrow p)) \rightarrow \neg\forall r\,(r \rightarrow p) \vee p && \Rightarrow\\
&\forall q\,((q \rightarrow p) \rightarrow \neg\forall r\,(r \rightarrow p) \vee p) && \Rightarrow\\
&\forall q\,((q \rightarrow p) \rightarrow \exists r\,\neg(r \rightarrow p) \vee p) && \Rightarrow\\
&\forall q\,((q \rightarrow p) \rightarrow \exists r\,(\neg(r \rightarrow p) \vee p)) && \Rightarrow\\
&\forall q\,\exists r\,((q \rightarrow p) \rightarrow \neg(r \rightarrow p) \vee p)
\end{aligned}
$$

Prenexing rules

$$
\begin{array}{c}
(\exists\!\!\!\forall p\, F_1) \wedge \cdots \wedge F_n \;\Rightarrow\; \exists\!\!\!\forall p\, (F_1 \wedge \cdots \wedge F_n) \\
(\exists\!\!\!\forall p\, F_1) \vee \cdots \vee F_n \;\Rightarrow\; \exists\!\!\!\forall p\, (F_1 \vee \cdots \vee F_n)
\end{array}
$$

$$
\begin{array}{cc}
(\forall p\, F_1) \rightarrow F_2 \;\Rightarrow\; \exists p\, (F_1 \rightarrow F_2) & F_1 \rightarrow (\exists p\, F_2) \;\Rightarrow\; \exists p\, (F_1 \rightarrow F_2) \\
(\exists p\, F_1) \rightarrow F_2 \;\Rightarrow\; \forall p\, (F_1 \rightarrow F_2) & F_1 \rightarrow (\forall p\, F_2) \;\Rightarrow\; \forall p\, (F_1 \rightarrow F_2) \\
\neg \forall p\, F \;\Rightarrow\; \exists p\, \neg F & \neg \exists p\, F \;\Rightarrow\; \forall p\, \neg F
\end{array}
$$

Conversion to prenex form, Example II

$$\exists q\,(q \to p) \to \neg\forall r\,(r \to p) \lor p \qquad \Rightarrow$$
$$\exists q\,(q \to p) \to \exists r\,\neg(r \to p) \lor p \qquad \Rightarrow$$
$$\exists q\,(q \to p) \to \exists r\,(\neg(r \to p) \lor p) \qquad \Rightarrow$$
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$$\exists r\,\forall q\,((q \to p) \to \neg(r \to p) \lor p)$$

Checking the satisfiability of QBFs

The algorithms for propositional satisfiability or validity can be adapted to QBF

We will see:

- Splitting
- DPLL

Recall:

1. $F(p_1, \ldots, p_n)$ is satisfiable iff $\exists p_1 \cdots \exists p_n\, F(p_1, \ldots, p_n)$ is satisfiable
2. $F(p_1, \ldots, p_n)$ is valid iff $\forall p_1 \cdots \forall p_n\, F(p_1, \ldots, p_n)$ is satisfiable
3. A closed QBF is either always true (valid) or always false (unsatisfiable)

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Splitting: foundations

Lemma 9

- *A closed formula $\forall p\, F$ evaluates to 1 iff both $F_p^{\perp}$ and $F_p^{\top}$ evaluate to 1.*
- *A closed formula $\exists p\, F$ evaluates to true iff either $F_p^{\perp}$ or $F_p^{\top}$ evaluates to 1.*

Splitting

Simplification rules for $\top$:

$$\neg\top \;\Rightarrow\; \bot$$
$$\top \wedge F_1 \wedge \cdots \wedge F_n \;\Rightarrow\; F_1 \wedge \cdots \wedge F_n$$
$$\top \vee F_1 \vee \cdots \vee F_n \;\Rightarrow\; \top$$
$$F \rightarrow \top \;\Rightarrow\; \top \qquad \top \rightarrow F \;\Rightarrow\; F$$
$$F \leftrightarrow \top \;\Rightarrow\; F \qquad \top \leftrightarrow F \;\Rightarrow\; F$$

Simplification rules for $\bot$:

$$\neg\bot \;\Rightarrow\; \top$$
$$\bot \wedge F_1 \wedge \cdots \wedge F_n \;\Rightarrow\; \bot$$
$$\bot \vee F_1 \vee \cdots \vee F_n \;\Rightarrow\; F_1 \vee \cdots \vee F_n$$
$$F \rightarrow \bot \;\Rightarrow\; \neg F \qquad \bot \rightarrow F \;\Rightarrow\; \top$$
$$F \leftrightarrow \bot \;\Rightarrow\; \neg F \qquad \bot \leftrightarrow F \;\Rightarrow\; \neg F$$

Splitting

Simplification rules for $\top$:

$$
\begin{gathered}
\neg\top \;\Rightarrow\; \bot \\
\top \wedge F_1 \wedge \cdots \wedge F_n \;\Rightarrow\; F_1 \wedge \cdots \wedge F_n \\
\top \vee F_1 \vee \cdots \vee F_n \;\Rightarrow\; \top \\
F \rightarrow \top \;\Rightarrow\; \top \qquad \top \rightarrow F \;\Rightarrow\; F \\
F \leftrightarrow \top \;\Rightarrow\; F \qquad \top \leftrightarrow F \;\Rightarrow\; F \\
\forall p\, \top \;\Rightarrow\; \top \\
\exists p\, \top \;\Rightarrow\; \top
\end{gathered}
$$

Simplification rules for $\bot$:

$$
\begin{gathered}
\neg\bot \;\Rightarrow\; \top \\
\bot \wedge F_1 \wedge \cdots \wedge F_n \;\Rightarrow\; \bot \\
\bot \vee F_1 \vee \cdots \vee F_n \;\Rightarrow\; F_1 \vee \cdots \vee F_n \\
F \rightarrow \bot \;\Rightarrow\; \neg F \qquad \bot \rightarrow F \;\Rightarrow\; \top \\
F \leftrightarrow \bot \;\Rightarrow\; \neg F \qquad \bot \leftrightarrow F \;\Rightarrow\; \neg F \\
\forall p\, \bot \;\Rightarrow\; \bot \\
\exists p\, \bot \;\Rightarrow\; \bot
\end{gathered}
$$

Splitting, Example

$$\forall p \, \exists q \, (p \leftrightarrow q)$$

Splitting, Example

$$\begin{tikzcd} & \forall p\,\exists q\,(p \leftrightarrow q) \arrow[dl, "p = 0"'] \\ \exists q\,(\bot \leftrightarrow q) & \wedge \end{tikzcd}$$

Splitting, Example

$$\begin{tikzcd}
& \forall p\,\exists q\,(p \leftrightarrow q) \arrow[dl, "p = 0"'] \\
\exists q\,(\neg q) & \wedge
\end{tikzcd}$$

Splitting, Example

$$
\begin{tikzcd}
\forall p \, \exists q \, (p \leftrightarrow q) \arrow[d, "p = 0"', "\wedge"] \\
\exists q \, (\neg q) \arrow[d, "q = 0"', "\vee"] \\
(\neg \bot)
\end{tikzcd}
$$

Splitting, Example

$$
\begin{tikzcd}[column sep=small]
 & & \forall p\, \exists q\, (p \leftrightarrow q) \arrow[dl, "p = 0"'] \\
 & \exists q\, (\neg q) \arrow[dl, "q = 0"'] & \wedge \\
\boxed{1}\ \top & \vee &
\end{tikzcd}
$$

Splitting, Example

$$
\begin{tikzcd}
 & & \forall p\, \exists q\, (p \leftrightarrow q) \arrow[dl, "p = 0"'] \\
 & & \wedge \\
 & \boxed{1}\ \exists q\, (\neg q) \arrow[dl, "q = 0"'] & \\
 & \vee & \\
\boxed{1}\ \top & &
\end{tikzcd}
$$

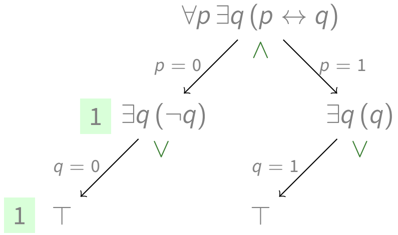
Splitting, Example

$$\begin{tikzcd}
& & \forall p\, \exists q\, (p \leftrightarrow q) \arrow[dl, "p = 0"'] \arrow[dr, "p = 1"] \arrow[d, phantom, "\wedge"] & \\
& \boxed{1}\ \exists q\, (\neg q) \arrow[dl, "q = 0"'] \arrow[d, phantom, "\vee"] & & \exists q\, (\top \leftrightarrow q) \\
\boxed{1}\ \top & & &
\end{tikzcd}$$

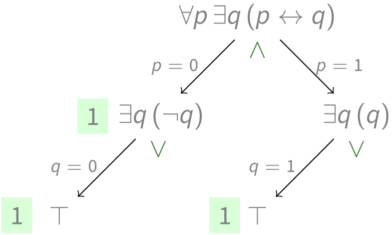
Splitting, Example

$$
\begin{tikzcd}
& & \forall p\,\exists q\,(p \leftrightarrow q) \arrow[dl, "p = 0"'] \arrow[dr, "p = 1"] \arrow[d, phantom, "\wedge"] & \\
& \boxed{1}\ \exists q\,(\neg q) \arrow[dl, "q = 0"'] \arrow[d, phantom, "\vee"] & & \exists q\,(q) \\
\boxed{1}\ \top & & &
\end{tikzcd}
$$

Splitting, Example



Splitting, Example



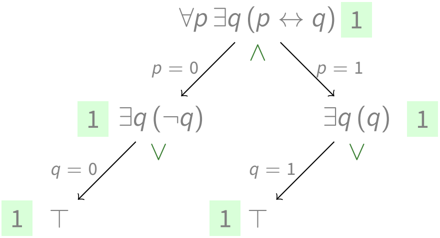
Splitting, Example

$$
\begin{tikzcd}
& & \forall p\, \exists q\, (p \leftrightarrow q) \arrow[dl, "p = 0"'] \arrow[dr, "p = 1"] & & \\
& \boxed{1}\ \exists q\, (\neg q) \arrow[dl, "q = 0"'] & \land & \exists q\, (q)\ \boxed{1} \arrow[dl, "q = 1"'] & \\
\boxed{1}\ \top & \lor & \boxed{1}\ \top & \lor &
\end{tikzcd}
$$

Splitting, Example

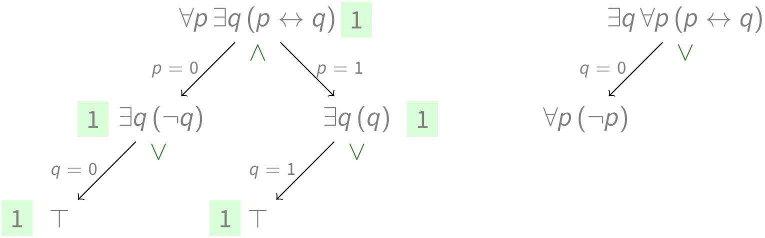
$$
\begin{tikzcd}
 & & & \forall p\, \exists q\, (p \leftrightarrow q)\ \boxed{1} \arrow[dl, "p = 0"'] \arrow[dr, "p = 1"] \arrow[d, phantom, "\wedge"] & \\
 & & \boxed{1}\ \exists q\, (\neg q) \arrow[dl, "q = 0"'] \arrow[d, phantom, "\vee"] & {} & \exists q\, (q)\ \boxed{1} \arrow[dl, "q = 1"'] \arrow[d, phantom, "\vee"] \\
 & \boxed{1}\ \top & {} & \boxed{1}\ \top & {}
\end{tikzcd}
$$

Splitting, Example

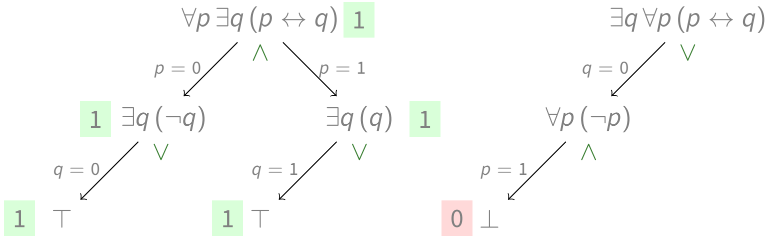


$\exists q \, \forall p \, (p \leftrightarrow q)$

Splitting, Example



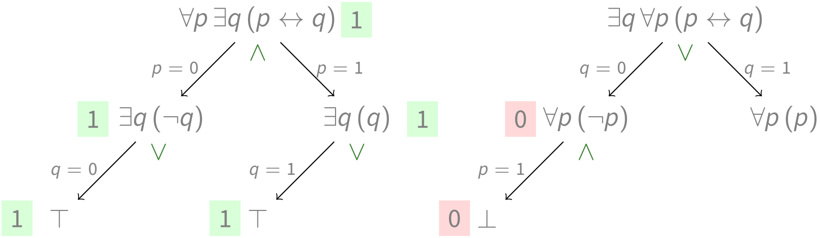
Splitting, Example



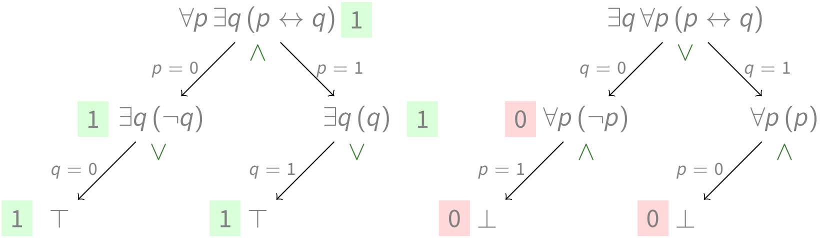
Splitting, Example

$$
\begin{tikzcd}[column sep=small]
& & \forall p\,\exists q\,(p \leftrightarrow q)\ \boxed{1} \arrow[dl, "p=0"'] \arrow[dr, "p=1"] \arrow[d, phantom, "\wedge"] & \\
& \boxed{1}\ \exists q\,(\neg q) \arrow[dl, "q=0"'] \arrow[d, phantom, "\vee"] & \ & \exists q\,(q)\ \boxed{1} \arrow[dl, "q=1"'] \arrow[d, phantom, "\vee"] \\
\boxed{1}\ \top & \ & \boxed{1}\ \top & \
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & \exists q\,\forall p\,(p \leftrightarrow q) \arrow[dl, "q=0"'] \arrow[d, phantom, "\vee"] \\
& \boxed{0}\ \forall p\,(\neg p) \arrow[dl, "p=1"'] \arrow[d, phantom, "\wedge"] & \ \\
\boxed{0}\ \bot & \ &
\end{tikzcd}
$$

Splitting, Example



Splitting, Example



Splitting, Example

$$
\begin{tikzcd}[column sep=small]
 & & \forall p\, \exists q\, (p \leftrightarrow q)\ \boxed{1} \arrow[dl, "p = 0"'] \arrow[dr, "p = 1"] \arrow[d, phantom, "\wedge"] & & \\
 & \boxed{1}\ \exists q\, (\neg q) \arrow[dl, "q = 0"'] \arrow[d, phantom, "\vee"] & {} & \exists q\, (q)\ \boxed{1} \arrow[dl, "q = 1"'] \arrow[d, phantom, "\vee"] & \\
\boxed{1}\ \top & {} & \boxed{1}\ \top & {} &
\end{tikzcd}
$$

$$
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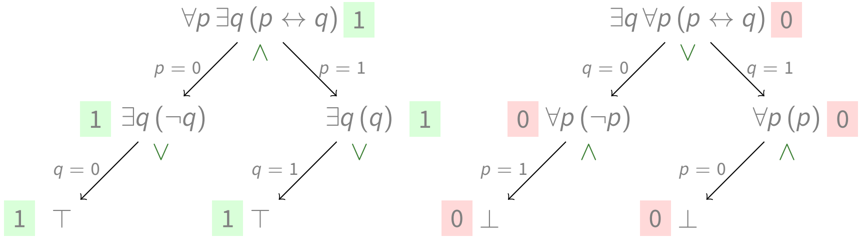
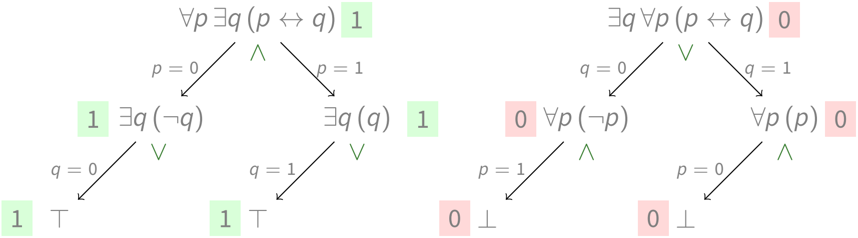
\boxed{0}\ \bot & {} & \boxed{0}\ \bot & {} &
\end{tikzcd}
$$

Splitting, Example



Splitting, Example



To minimize search, the selection of variable values is best seen as a two-player game:

- by selecting a value for $\exists q$ one is trying to make the formula true
- by selecting a value for $\forall p$ one is trying to make the formula false

Splitting algorithm

Notation: if $\boldsymbol{p} = (p_1, \ldots, p_k)$ then $\exists\!\forall \boldsymbol{p} F$ denotes $\exists\!\forall p_1 \cdots \exists\!\forall p_k F$

Splitting algorithm

procedure $splitting(F)$
input: closed rectified prenex formula F
output: 0 or 1
parameters: function $select\_variable\_value$ // selects a variable from the outermost prefix
begin // of F as well as a Boolean value for it
$F := simplify(F)$ // apply extended simplification rules to completion
if $F = \bot$ **then return** 0
if $F = \top$ **then return** 1
// else F has the form $\exists\forall \boldsymbol{p}F'$ where $\boldsymbol{p}$ is F's outermost prefix
$(p, b) := select\_variable\_value(F)$
Let G be obtained from F by deleting p from $\boldsymbol{p}$
if $b = 0$ **then** $A := \bot$; $B := \top$ **else** $A := \top$; $B := \bot$
$b := splitting(G_p^A)$
case $(b, \exists\forall)$ **of**
$(0, \forall) \Rightarrow$ **return** 0
$(0, \exists) \Rightarrow$ **return** $splitting(G_p^B)$
$(1, \forall) \Rightarrow$ **return** $splitting(G_p^B)$
$(1, \exists) \Rightarrow$ **return** 1
end

Conjunctive Normal Form

For more efficient algorithms we need QBFs to be in a convenient normal form

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A quantified Boolean formula F is in *Conjunctive Normal Form (CNF)*, if

- it is either $\bot$, or $\top$, or
- it has the form
$$\exists\!\!\!\forall_1 p_1 \cdots \exists\!\!\!\forall_n p_n \, (C_1 \wedge \cdots \wedge C_m)$$
  where $C_1, \ldots, C_m$ are clauses

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  where $C_1, \ldots, C_m$ are clauses

Example:

$$\forall p \, \exists q \, \exists s \, ((\neg p \vee s \vee q) \wedge (s \vee \neg q) \wedge \neg s)$$

CNF rules

Prenexing rules

+

propositional CNF rules:

$$
\begin{array}{rcl}
F \leftrightarrow G & \Rightarrow & (\neg F \vee G) \wedge (\neg G \vee F) \\
F \rightarrow G & \Rightarrow & \neg F \vee G \\
\neg(F \wedge G) & \Rightarrow & \neg F \vee \neg G \\
\neg(F \vee G) & \Rightarrow & \neg F \wedge \neg G \\
\neg\neg F & \Rightarrow & F \\
(F_1 \wedge \cdots \wedge F_m) \vee G_1 \vee \cdots \vee G_n & \Rightarrow & (F_1 \vee G_1 \vee \cdots \vee G_n) \quad \wedge \\
 & & \cdots \quad \wedge \\
 & & (F_m \vee G_1 \vee \cdots \vee G_n)
\end{array}
$$

DPLL for quantified Boolean formulas

Input:

Q: quantifier sequence $\exists\!\!\forall_1 \boldsymbol{p}_1 \cdots \exists\!\!\forall_n \boldsymbol{p}_n$

$\boldsymbol{S}$: set of clauses with variables from $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$

Main components:

Unit propagation

Splitting on literals

Unit Propagation

Q: quantifier sequence
$\boldsymbol{S}$: current clause set

Propositional formulas:

For each unit clause L in $\boldsymbol{S}$

1. remove all clauses containing literal L from $\boldsymbol{S}$
2. remove every literal $\overline{L}$ from remaining clauses

Quantified Boolean formulas:

For each unit clause L in $\boldsymbol{S}$ of the form p or $\neg p$

- If Q does not contain p or contains $\exists p$,
  1. remove all clauses containing literal L from $\boldsymbol{S}$
  2. remove every literal $\overline{L}$ from remaining clauses
- otherwise (Q contains $\forall p$), add $\square$ to $\boldsymbol{S}$

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- If Q does not contain p or contains $\exists p$,
    1. remove all clauses containing literal L from $\boldsymbol{S}$
    2. remove every literal $\overline{L}$ from remaining clauses
- otherwise (Q contains $\forall p$), add $\Box$ to $\boldsymbol{S}$

DPLL algorithm

Why do we add $\Box$ to $\boldsymbol{S}$ when Q is $\forall p \, \exists\!\!\!\forall_1 p_1 \cdots \exists\!\!\!\forall_m \, p_m$ and
$\boldsymbol{S}$ is $\{ \, p, C_1, \ldots, C_n \, \}$?

DPLL algorithm

Why do we add $\Box$ to $\boldsymbol{S}$ when Q is $\forall p \, \exists\forall_1 p_1 \cdots \exists\forall_m \, p_m$ and $\boldsymbol{S}$ is $\{ \, p, C_1, \ldots, C_n \, \}$?

Because

1. The intended input formula is
   $G \; = \; \forall p \, \exists\forall_1 q_1 \cdots \exists\forall_m q_m \, (p \wedge C_1 \wedge \cdots \wedge C_m)$
2. $G \; = \; \exists\forall_1 q_1 \cdots \exists\forall_m q_m \, ((p \wedge C_1 \wedge \cdots \wedge C_m)^{\perp}_{p} \wedge (p \wedge C_1 \wedge \cdots \wedge C_m)^{\top}_{p})$

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2. $$\begin{aligned} G \;&\equiv\; \exists\forall_1 q_1 \cdots \exists\forall_m q_m \,((p \wedge C_1 \wedge \cdots \wedge C_m)_p^{\perp} \wedge (p \wedge C_1 \wedge \cdots \wedge C_m)_p^{\top}) \\ &=\; \exists\forall_1 q_1 \cdots \exists\forall_m q_m \,(\perp \wedge (C_1 \wedge \cdots \wedge C_m)_p^{\perp} \wedge (p \wedge C_1 \wedge \cdots \wedge C_m)_p^{\top}) \end{aligned}$$

DPLL algorithm

Why do we add $\square$ to $\boldsymbol{S}$ when Q is $\forall p\, \exists\!\!\!\forall_1 p_1 \cdots \exists\!\!\!\forall_m\, p_m$ and $\boldsymbol{S}$ is $\{\, p, C_1, \ldots, C_n \,\}$?

Because

1. The intended input formula is
$$G \;=\; \forall p\, \exists\!\!\!\forall_1 q_1 \cdots \exists\!\!\!\forall_m q_m\, (p \wedge C_1 \wedge \cdots \wedge C_m)$$
2. $$\begin{aligned} G &\equiv \exists\!\!\!\forall_1 q_1 \cdots \exists\!\!\!\forall_m q_m \left((p \wedge C_1 \wedge \cdots \wedge C_m)_p^{\bot} \wedge (p \wedge C_1 \wedge \cdots \wedge C_m)_p^{\top} \right) \\ &= \exists\!\!\!\forall_1 q_1 \cdots \exists\!\!\!\forall_m q_m \left(\bot \wedge (C_1 \wedge \cdots \wedge C_m)_p^{\bot} \wedge (p \wedge C_1 \wedge \cdots \wedge C_m)_p^{\top} \right) \\ &\equiv \exists\!\!\!\forall_1 q_1 \cdots \exists\!\!\!\forall_m q_m\, \bot \\ &\equiv \bot \end{aligned}$$

DPLL algorithm

Why do we add $\Box$ to $\boldsymbol{S}$ when Q is $\forall p \, \exists\!\!\forall_1 p_1 \cdots \exists\!\!\forall_m p_m$ and $\boldsymbol{S}$ is $\{ p, C_1, \ldots, C_n \}$?

Alternatively, using the game metaphor, because

the $\forall$-player wants to falsify the formula

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Winning move for the $\forall$-player:

select the value for p that falsifies the unit clause p, and hence the whole CNF

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(argument is similar for $\{\, \neg p, C_1, \ldots, C_n \,\}$)

DPLL, Example

$\exists p \, \forall q \, \exists r$

$p \vee q \vee \neg r$

$p \vee \neg q \vee r$

$\neg p \vee q \vee r$

$\neg p \vee q \vee \neg r$

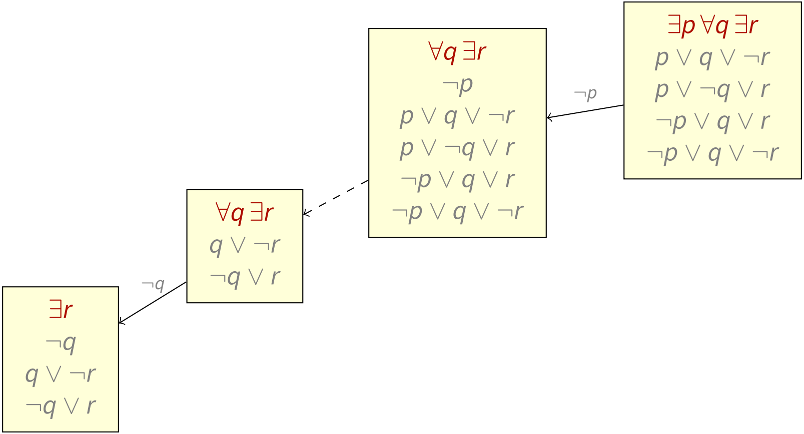
DPLL, Example

$$\begin{tikzcd}
\boxed{\begin{array}{c} \forall q\,\exists r \\ \neg p \\ p \lor q \lor \neg r \\ p \lor \neg q \lor r \\ \neg p \lor q \lor r \\ \neg p \lor q \lor \neg r \end{array}} & \boxed{\begin{array}{c} \exists p\,\forall q\,\exists r \\ p \lor q \lor \neg r \\ p \lor \neg q \lor r \\ \neg p \lor q \lor r \\ \neg p \lor q \lor \neg r \end{array}} \arrow[l, "\neg p"']
\end{tikzcd}$$

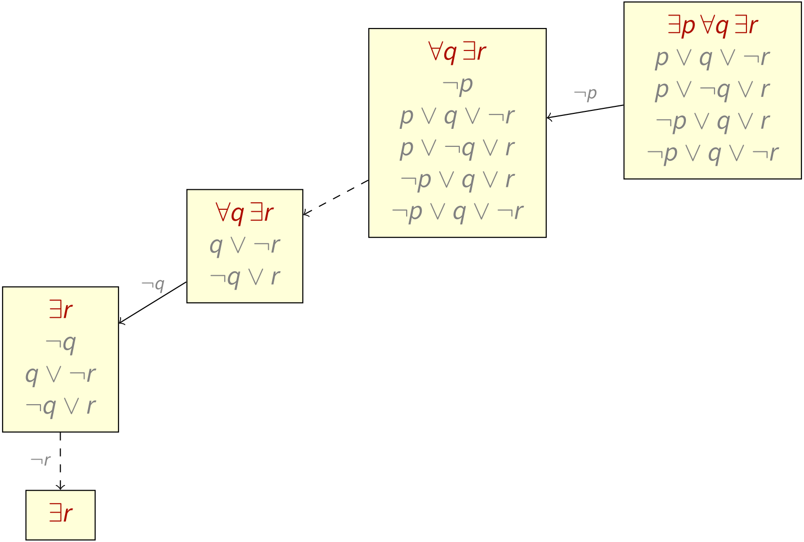
DPLL, Example

$$
\begin{tikzcd}
\boxed{\begin{array}{c} \forall q\, \exists r \\ q \lor \neg r \\ \neg q \lor r \end{array}} & \boxed{\begin{array}{c} \forall q\, \exists r \\ \neg p \\ p \lor q \lor \neg r \\ p \lor \neg q \lor r \\ \neg p \lor q \lor r \\ \neg p \lor q \lor \neg r \end{array}} \arrow[l, dashed] & \boxed{\begin{array}{c} \exists p\, \forall q\, \exists r \\ p \lor q \lor \neg r \\ p \lor \neg q \lor r \\ \neg p \lor q \lor r \\ \neg p \lor q \lor \neg r \end{array}} \arrow[l, "\neg p"']
\end{tikzcd}
$$

DPLL, Example



DPLL, Example



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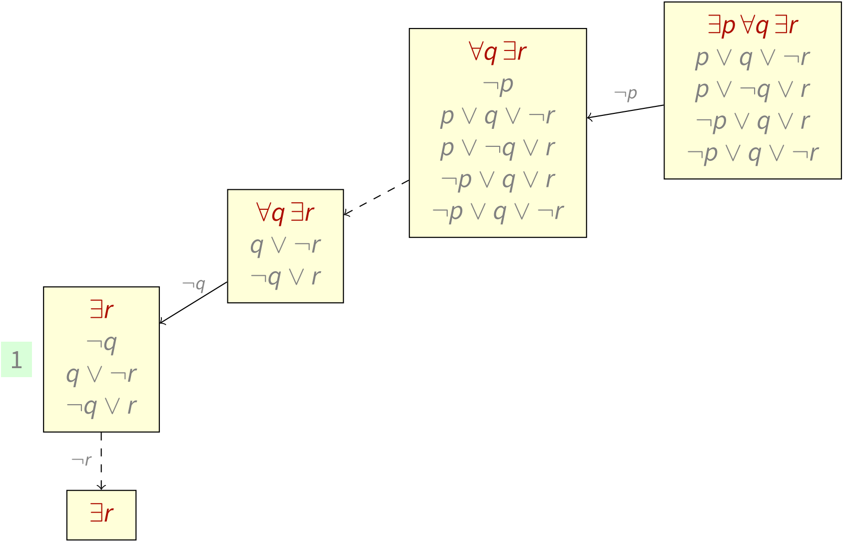

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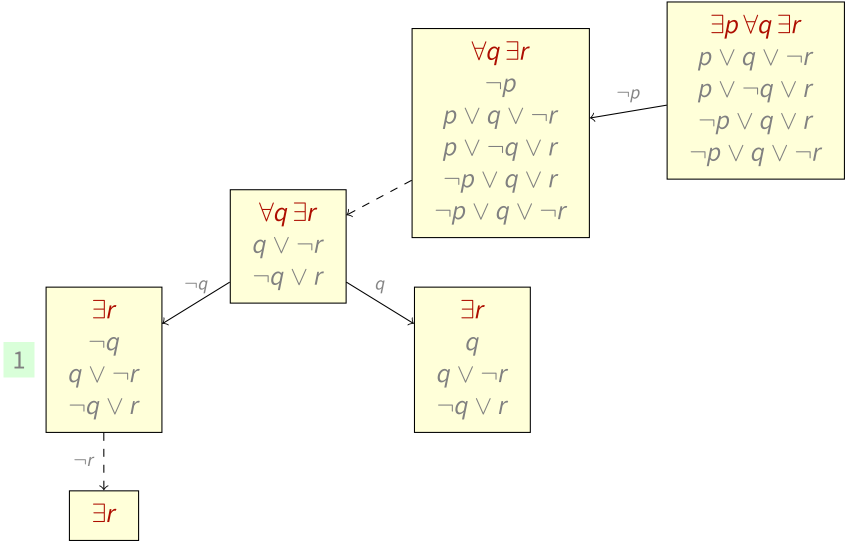

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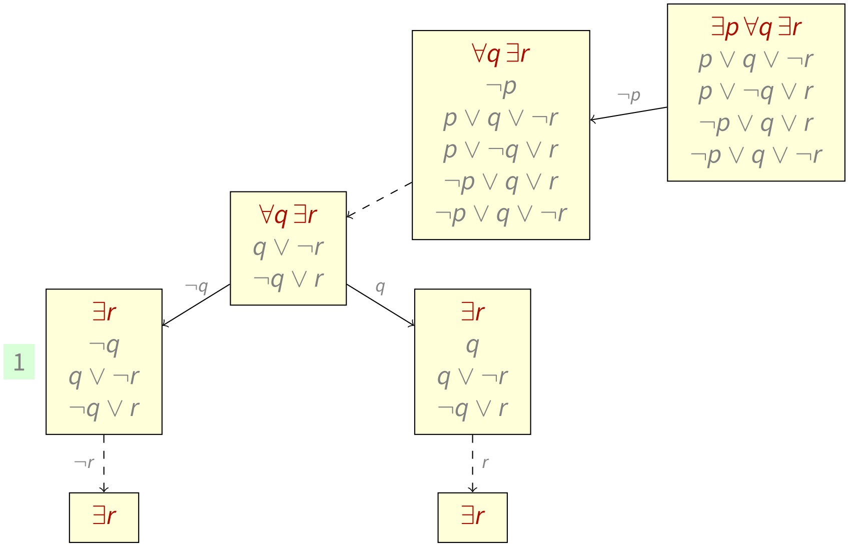
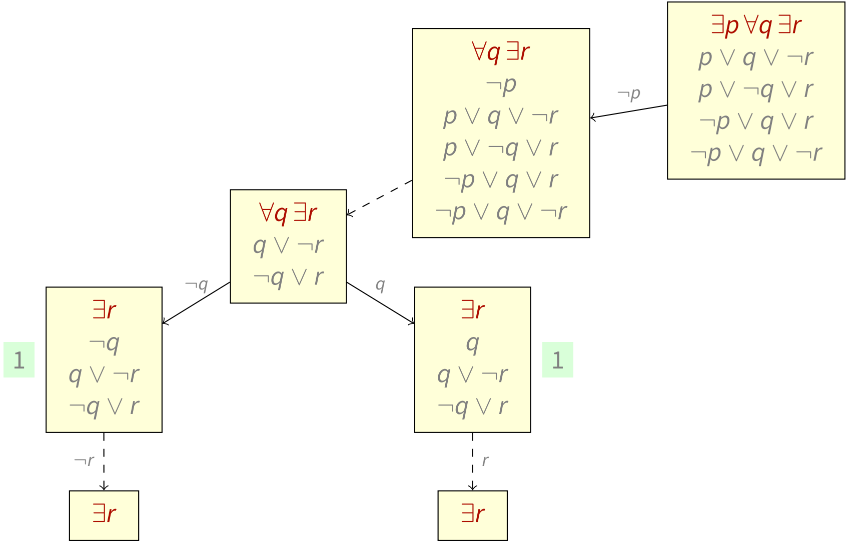


DPLL, Example



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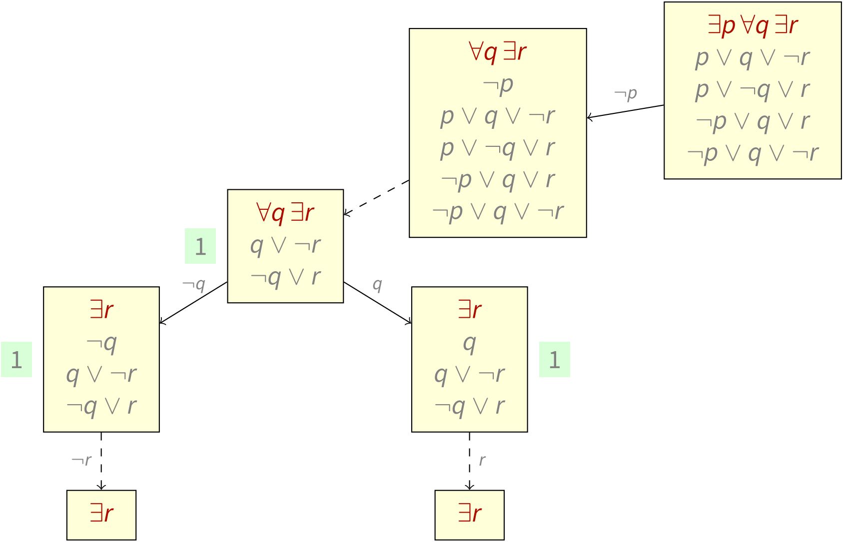

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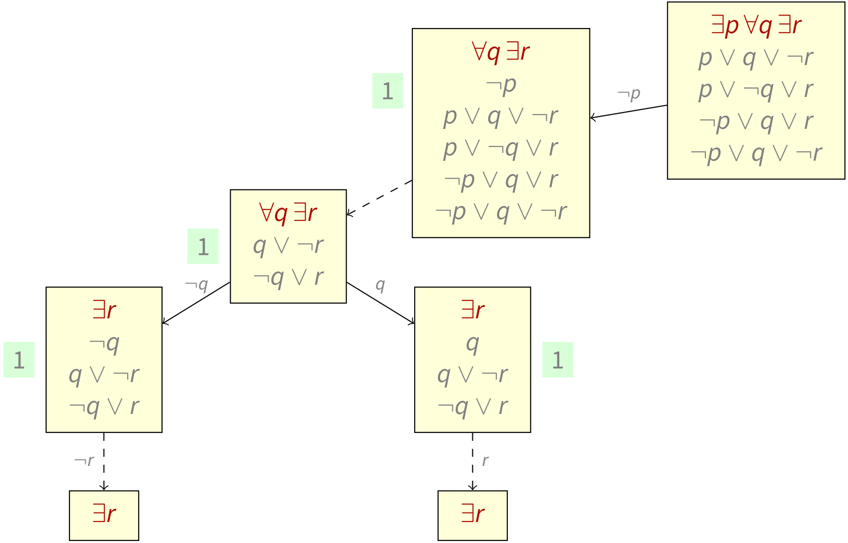
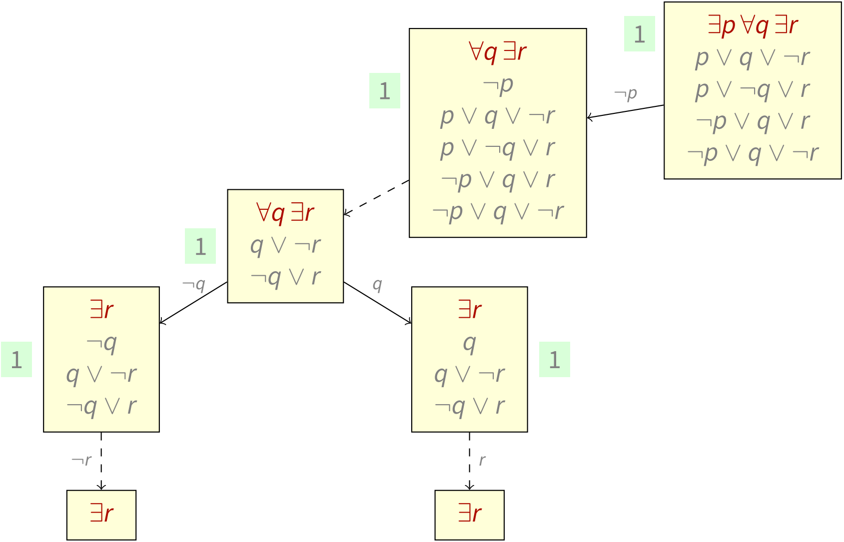


DPLL, Example



DPLL algorithm

procedure $DPLL(Q, S)$
input: quantifier sequence $Q = \exists\!\forall_1 \boldsymbol{p}_1 \cdots \exists\!\forall_n \boldsymbol{p}_n$,
clause set $\boldsymbol{S}$ with vars from Q
output: 0 or 1
parameters: function $select\_variable\_value$
begin
$\boldsymbol{S} \ := \ unit\_propagate(Q, S)$
if $\boldsymbol{S}$ is empty **then return** 1
if $\boldsymbol{S}$ contains $\Box$ **then return** 0
$(p, b) \ := \ select\_variable\_value(\boldsymbol{p}_1, S)$
Let Q' be obtained from Q by deleting $\exists\!\forall_1 p$ from $\exists\!\forall_1 \boldsymbol{p}_1$
if $b = 0$ **then** $L \ := \ \neg p$
else $L \ := \ p$
case $(DPLL(Q', S \cup \{ L \}), \ \exists\!\forall)$ **of**
$(0, \forall) \Rightarrow$ **return** 0
$(0, \exists) \Rightarrow$ **return** $DPLL(Q', S \cup \{ \bar{L} \})$
$(1, \forall) \Rightarrow$ **return** $DPLL(Q', S \cup \{ \bar{L} \})$
$(1, \exists) \Rightarrow$ **return** 1
end

Improving DPLL with further simplifications

$$\exists p\,\exists q\,\forall r\,\exists s\,((p \lor \neg r) \land (\neg q \lor r) \land (\neg p \lor q \lor s) \land (\neg p \lor q \lor r \lor \neg s))$$

- We can treat $\neg r$ in $p \lor \neg r$ as 0 without loss of generality
- We can apply unit propagation
- We can treat r as 0 everywhere without loss of generality
- We can apply unit propagation with $\neg q$
- We can apply unit propagation with s

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Pure literal rule

Q: quantifier sequence
$\boldsymbol{S}$: current clause set
L: literal of the form p or $\neg p$

Suppose L is *pure* in $\boldsymbol{S}$ (i.e., $\overline{L}$ does not occur in $\boldsymbol{S}$). Then:

- If p is existentially quantified in Q, we can remove all clauses containing L

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- if p is universally quantified in Q, we can remove L from all clauses

Why?

- The $\exists$-player will make L true (satisfying all clauses containing L)
- The $\forall$-player will make L false (so it can be removed from all clauses containing L)

Universal literal deletion

Q: quantifier sequence
$\boldsymbol{S}$: clause set
p, q: variables

- p is *existential in* Q if Q contains $\exists p$
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Example: In $Q = \forall q \, \exists p \, \forall r$
q is quantified before both p and r; and p is quantified before r

Universal literal deletion

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Theorem 10
Suppose that

1. C *is a clause in* $\boldsymbol{S}$;
2. *a variable* q *in a literal* L *of* C *is universal in* Q;
3. *all existential variables of* Q *in* C *are quantified before* q.

Then deleting L *from* C *does not change the truth value of* $Q\,S$.

Universal literal deletion

Intuition behind Theorem 10

Consider a clause C from $\boldsymbol{S}$ of the form

$$L_1 \vee \cdots \vee L_n \vee (\neg)q_1 \vee \cdots \vee (\neg)q_m$$

where all existential variables of Q in C are quantified before $q_1, \ldots, q_m$

- If at least one of $L_1, \ldots, L_n$ is true,
  then C is true regardless of the truth value of of $(\neg)q_1, \ldots, (\neg)q_m$
- If all of $L_1, \ldots, L_n$ are false,
  the $\forall$-player will make all $(\neg)q_1, \ldots, (\neg)q_m$ false and win the game

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Consider the position before the $q_1, \ldots, q_m$-moves of the $\forall$-player

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In either case, the deletion of $(\neg)q_1, \ldots, (\neg)q_m$ will not change the final outcome

Example revisited

$$\exists p\, \exists q\, \forall r\, \exists s\, ((p \vee \neg r) \wedge (\neg q \vee r) \wedge (\neg p \vee q \vee s) \wedge (\neg p \vee q \vee r \vee \neg s))$$

Example revisited

$$\exists p\, \exists q\, \forall r\, \exists s\, ((p \vee \neg r) \wedge (\neg q \vee r) \wedge (\neg p \vee q \vee s) \wedge (\neg p \vee q \vee r \vee \neg s))$$

- Apply *universal literal deletion* to $p \vee \neg r$

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$$\exists p\, \exists q\, \forall r\, \exists s\, ((p \lor \neg r) \land (\neg q \lor r) \land (\neg p \lor q \lor s) \land (\neg p \lor q \lor r \lor \neg s)) \Rightarrow$$
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- Apply *universal literal deletion* to $p \vee \neg r$
- Apply unit propagation

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- Apply *universal literal deletion* to $p \vee \neg r$
- Apply unit propagation
- Apply the *pure literal rule* to r

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