CS:4350 Logic in Computer Science

Semantic Tableaux

Cesare Tinelli

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The University of Iowa

Credits

These slides are largely based on slides originally developed by **Andrei Voronkov** at the University of Manchester. Adapted by permission.

Outline

Semantic tableaux

Signed formula

- *Signed formula*: an expression A^b, where A is a formula and b a boolean value
- A signed formula A^b is *satisfied* by an interpretation $\mathcal{I}$, written $\mathcal{I} \models A^b$, if $\mathcal{I}(A) = b$; it is *falsified* otherwise
- If $\mathcal{I} \models A^b$, we also say that *$\mathcal{I}$ is a model of A^b*
- A signed formula is *satisfiable* if it has a model

Note:

1. For every formula A and interpretation $\mathcal{I}$ exactly one of the signed formulas A^1 and A^0 is satisfied by $\mathcal{I}$
2. A formula A is satisfiable iff A^1 is satisfiable
3. A formula A is falsifiable iff A^0 is satisfiable

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How to find a model of a signed formula?

Example: $A \wedge B$

		B	B
	$\wedge$	0	1
A	0	0	0
A	1	0	1

$(A \wedge B)^1$ — We can make $A \wedge B$ true iff we make A true (A^1) and B true (B^1)

$(A \wedge B)^0$ — We can make $A \wedge B$ false iff we make A false (A^0) or B false (B^0)

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Tableau

The search for a model of a formula can be expressed by an AND-OR tree

Tableau: a tree having signed formulas at nodes (plural: tableaux)

A tableau for a signed formula A^b has A^b as a root

Alternatively, we can regard a tableau as a collection of branches; each branch is a set of signed formulas

Notation for branches: $A_1^{b_1} \mid \cdots \mid A_n^{b_n}$

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Constructing a semantic tableau

$$(\neg(q \vee p \to p \vee q))^1$$

$$(q \vee p \to p \vee q)^0$$

$$(q \vee p)^1$$
$$(p \vee q)^0$$

$$p^0$$
$$q^0$$

q^1 closed | p^1 closed

Rules to grow a tree branch:

$$(A_1 \vee A_2)^0 \quad \rightsquigarrow \quad A_1^0, A_2^0$$
$$(A_1 \vee A_2)^1 \quad \rightsquigarrow \quad A_1^1 \mid A_2^1$$
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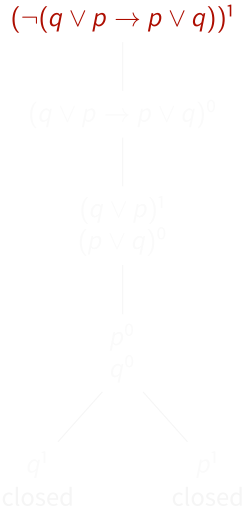

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$$q^1 \qquad\qquad p^1$$

closed   closed

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$$\begin{array}{rcl} (A_1 \vee A_2)^0 & \rightsquigarrow & A_1^0, A_2^0 \\ (A_1 \vee A_2)^1 & \rightsquigarrow & A_1^1 \mid A_2^1 \\ (A_1 \to A_2)^0 & \rightsquigarrow & A_1^1, A_2^0 \\ (\neg A_1)^1 & \rightsquigarrow & A_1^0 \end{array}$$

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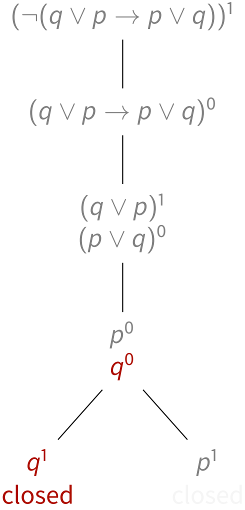

Rules to grow a tree branch:

$$(A_1 \lor A_2)^0 \rightsquigarrow A_1^0, A_2^0$$
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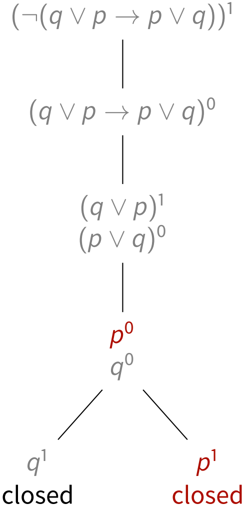

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Branch expansion rules

$$
\begin{array}{rcl}
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(A_1 \wedge \cdots \wedge A_n)^1 & \rightsquigarrow & A_1^1, \ldots, A_n^1 \\
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(A_1 \vee \cdots \vee A_n)^0 & \rightsquigarrow & A_1^0, \ldots, A_n^0 \\
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\\
(\neg A)^0 & \rightsquigarrow & A^1 \\
(\neg A)^1 & \rightsquigarrow & A^0 \\
\\
(A_1 \leftrightarrow A_2)^0 & \rightsquigarrow & A_1^0, A_2^1 \mid A_1^1, A_2^0 \\
(A_1 \leftrightarrow A_2)^1 & \rightsquigarrow & A_1^0, A_2^0 \mid A_1^1, A_2^1
\end{array}
$$

Open and closed branches

A branch is *closed* in any of the following cases:

- it contains both p^0 and p^1 for some atom p
- it contains $\top^0$
- it contains $\bot^1$

It is *open* otherwise.

A tableau is *closed* if all of its branches are closed

Note: The formulas on a closed branch are jointly unsatisfiable

A branch is *complete* (or *saturated*) if it cannot be expanded further without adding a formula already in it

Note: From the signed atoms of an complete open branch it is possible to construct a model of the root formula

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Example 2

$(\neg((p \to q) \land (p \land q \to r) \to (\neg p \to r)))^1$

$$
\begin{aligned}
(A_1 \land A_2)^0 &\quad\leadsto\quad A_1^0 \mid A_2^0 \\
(A_1 \land A_2)^1 &\quad\leadsto\quad A_1^1, A_2^1 \\[1ex]
(A_1 \to A_2)^0 &\quad\leadsto\quad A_1^1, A_2^0 \\
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$$(A_1 \to A_2)^1 \quad \rightsquigarrow \quad A_1^0 \mid A_2^1$$

$$(\neg A_1)^1 \quad \rightsquigarrow \quad A_1^0$$

Example 2

$$(\neg((p \to q) \land (p \land q \to r) \to (\neg p \to r)))^1$$

$$((p \to q) \land (p \land q \to r) \to (\neg p \to r))^0$$

$$((p \to q) \land (p \land q \to r))^1$$
$$(\neg p \to r)^0$$

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$$|$$

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$$|$$

$$((p \to q) \land (p \land q \to r))^1$$
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$$|$$

$$(p \to q)^1$$
$$(p \land q \to r)^1$$

$$|$$

$$(\neg p)^1$$
$$r^0$$

$$(A_1 \land A_2)^0 \quad \rightsquigarrow \quad A_1^0 \mid A_2^0$$
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$$(\neg p)^1$$
$$r^0$$

$$p^0 \qquad\qquad q^1$$

$$\begin{aligned}
(A_1 \land A_2)^0 &\quad\rightsquigarrow\quad A_1^0 \mid A_2^0 \\
(A_1 \land A_2)^1 &\quad\rightsquigarrow\quad A_1^1, A_2^1 \\[1ex]
(A_1 \to A_2)^0 &\quad\rightsquigarrow\quad A_1^1, A_2^0 \\
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$$(p \to q)^1$$
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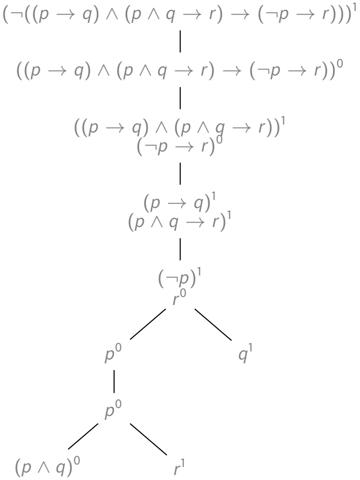
Example 2



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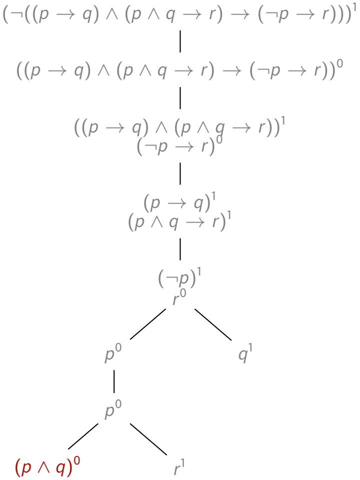

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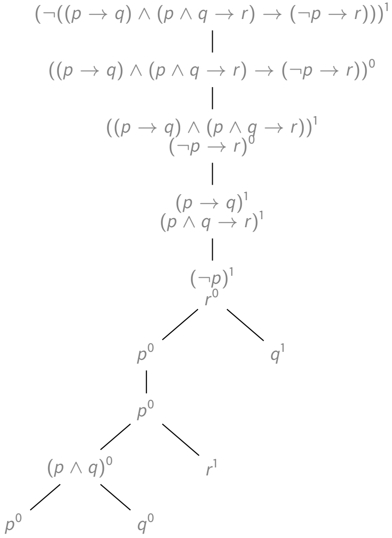

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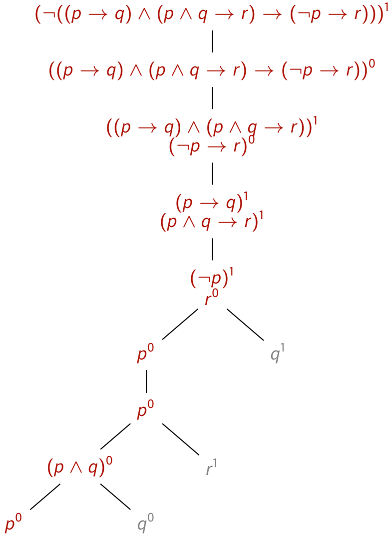

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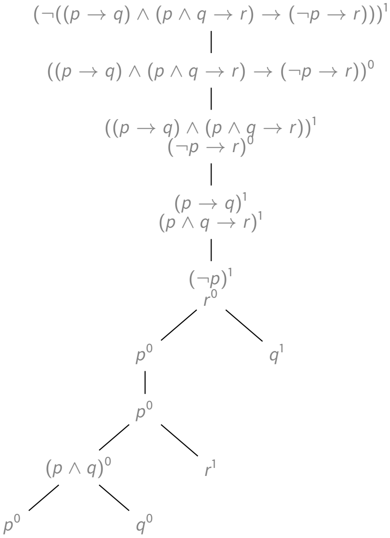
\end{array}
$$

The leftmost branch is complete (nothing new can be added) but still open

Finding Models Using Tableaux



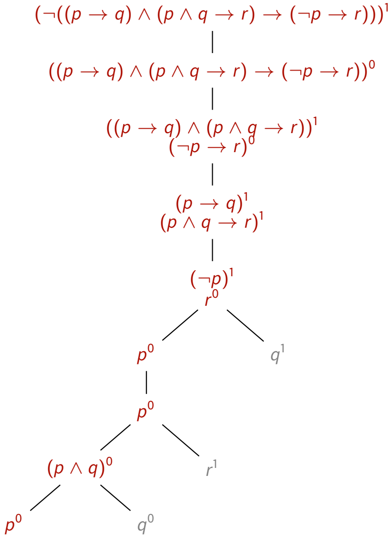
Build a complete branch

Select the signed atoms on it

They give us a (possibly partial) model of the root formula:

$$\{\, r \mapsto 0, p \mapsto 0, q \mapsto \cdots \,\}$$

Finding Models Using Tableaux



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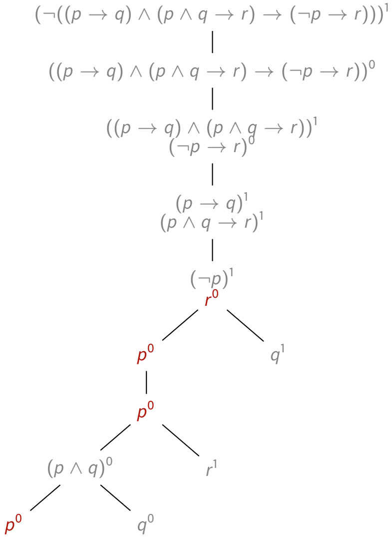

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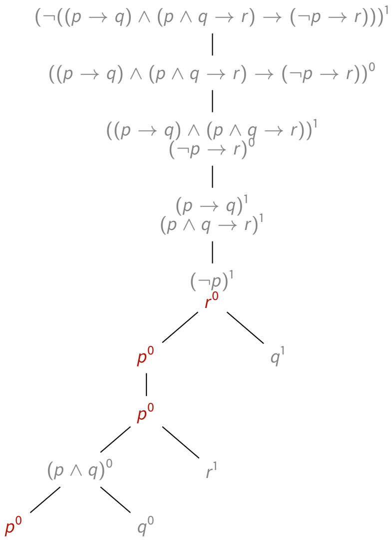

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Soundness and completeness of tableaux

Theorem 1 (Soundness and completeness)

A formula A is valid iff there is a closed tableau for A^0 (iff every tableau for A^0 is closed)

Corollary 2

1. *A formula A is satisfiable iff there is a tableau for A^1 with a complete open branch (iff every tableau for A^1 contains a complete open branch)*
2. *Formulas A and B are equivalent iff there is a closed tableau for $(A \leftrightarrow B)^0$ (iff every tableau for $(A \leftrightarrow B)^0$ is closed)*

Note: A fully expanded tableau for A^1 gives us all models of A

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Tableaux as derivation systems

Main idea:

1. Represent a tableau as the set of its branches
2. Represent a branch as the set of the signed formulas on it
3. Turn the tableaux expansion rules into derivation rules
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5. To check a signed formula A^b start with the tableau $\{\{A^b\}\}$

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Tableau expansion rules — $\neg$

p	atom	$\mathbf{B}$	a branch (set of signed formulas)
A_i	formula	$\mathbf{T}$	a tableaux (set of branches)

$$\frac{\{\{(\neg A)^0\} \cup \mathbf{B}\} \cup \mathbf{T}}{\{\{A^1\} \cup \mathbf{B}\} \cup \mathbf{T}}\ \neg_0 \qquad \frac{\{\{(\neg A)^0\} \cup \mathbf{B}\} \cup \mathbf{T}}{\{\{A^1\} \cup \mathbf{B}\} \cup \mathbf{T}}\ \neg_1$$

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shorthand notation

Tableau expansion rules — $\wedge$ and $\vee$

p	atom	$\mathbf{B}$	a branch (set of signed formulas)
A_i	formula	$\mathbf{T}$	a tableaux (set of branches)

$$\frac{(A_1 \wedge \cdots \wedge A_n)^0, \mathbf{B} \mid \mathbf{T}}{A_1^0, \mathbf{B} \mid \cdots \mid A_n^0, \mathbf{B} \mid \mathbf{T}} \wedge_0 \qquad \frac{(A_1 \wedge \cdots \wedge A_n)^1, \mathbf{B} \mid \mathbf{T}}{A_1^1, \ldots, A_n^1, \mathbf{B} \mid \mathbf{T}} \wedge_1$$

$$\frac{(A_1 \vee \cdots \vee A_n)^0, \mathbf{B} \mid \mathbf{T}}{A_1^0, \ldots, A_n^0, \mathbf{B} \mid \mathbf{T}} \vee_0 \qquad \frac{(A_1 \vee \cdots \vee A_n)^1, \mathbf{B} \mid \mathbf{T}}{A_1^1, \mathbf{B} \mid \cdots \mid A_n^1, \mathbf{B} \mid \mathbf{T}} \vee_1$$

Tableau expansion rules — $\rightarrow$ and $\leftrightarrow$

p	atom	$\mathbf{B}$	a branch (set of signed formulas)
A_i	formula	$\mathbf{T}$	a tableaux (set of branches)

$$\frac{(A_1 \rightarrow A_2)^0, \mathbf{B} \mid \mathbf{T}}{A_1^1, A_2^0, \mathbf{B} \mid \mathbf{T}} \rightarrow_0 \qquad \frac{(A_1 \rightarrow A_2)^1, \mathbf{B} \mid \mathbf{T}}{A_1^0, \mathbf{B} \mid A_2^1, \mathbf{B} \mid \mathbf{T}} \rightarrow_1$$

$$\frac{(A_1 \leftrightarrow A_2)^0, \mathbf{B} \mid \mathbf{T}}{A_1^0, A_2^1, \mathbf{B} \mid A_1^1, A_2^0, \mathbf{B} \mid \mathbf{T}} \leftrightarrow_0 \qquad \frac{(A_1 \leftrightarrow A_2)^1, \mathbf{B} \mid \mathbf{T}}{A_1^0, A_2^0, \mathbf{B} \mid A_1^1, A_2^1, \mathbf{B} \mid \mathbf{T}} \leftrightarrow_1$$

Tableau closure rules

p	atom	$\mathbf{B}$	a branch (set of signed formulas)
A_i	formula	$\mathbb{T}$	a tableaux (set of branches)

$$\frac{p^0, p^1, \mathbf{B} \mid \mathbb{T}}{\mathbb{T}}\ \textsc{atom} \qquad \frac{\bot^1, \mathbf{B} \mid \mathbb{T}}{\mathbb{T}}\ \bot \qquad \frac{\top^0, \mathbf{B} \mid \mathbb{T}}{\mathbb{T}}\ \top$$

Note: A tableau is closed iff it is the empty set

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$$\frac{p^0, p^1, \mathbf{B} \mid \mathsf{T}}{\mathsf{T}}\ \text{ATOM} \qquad \frac{\bot^1, \mathbf{B} \mid \mathsf{T}}{\mathsf{T}}\ \bot \qquad \frac{\top^0, \mathbf{B} \mid \mathsf{T}}{\mathsf{T}}\ \top$$

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