# CS:4350 Logic in Computer Science

# Natural Deduction for Propositional Logic

Cesare Tinelli

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#### **Credits**

Part of these slides are based on Chap. 2 of *Logic in Computer Science* by M. Huth and M. Ryan, Cambridge University Press, 2nd edition, 2004.

## Outline

#### **Natural Deduction**

Derivation Rules Soundness and Completeness

There are many derivation systems for propositional logic

Natural deduction is a family of derivation systems with derivation rules designed to mimic the way people reason deductively

There are many derivation systems for propositional logic

Natural deduction is a family of derivation systems with derivation rules designed to mimic the way people reason deductively

#### Note

- "Natural" here is meant in contraposition to "mechanical / automated"
- Other derivation systems for PL are more machine-oriented and so arguably not as natural for people
- Natural deduction is actually automatable but less conveniently than other, more machine-oriented derivation systems

There are many derivation systems for propositional logic

Natural deduction is a family of derivation systems with derivation rules designed to mimic the way people reason deductively

#### Note

For simplicity but without loss of generality, we will

- not use  $\top$  (as  $\top \equiv \neg \bot$ )
- not use  $\leftrightarrow$  (as  $A \leftrightarrow B \equiv (A \rightarrow B) \land (B \rightarrow A)$ )
- use  $\land$  only with two arguments (as  $A \land B \land C \equiv (A \land B) \land C$ )
- use  $\lor$  only with two arguments (as  $A \lor B \lor C \equiv (A \lor B) \lor C$ )

There are many derivation systems for propositional logic

Natural deduction is a family of derivation systems with derivation rules designed to mimic the way people reason deductively

We will write

$$\underbrace{A_1,\ldots,A_n\vdash A}_{sequent}$$

to indicate that A is derivable from  $A_1, \ldots, A_n$  using the rules of natural deduction

$$\frac{A \quad B}{A \wedge B} \wedge i \qquad \qquad \frac{A \wedge B}{A} \wedge e_1$$

$$\frac{A \wedge B}{A} \wedge e_1$$

$$\frac{A \wedge B}{B} \wedge e_2$$

$$\frac{A \quad B}{A \wedge B} \wedge i \qquad \frac{A \wedge B}{A} \wedge e_1 \qquad \frac{A \wedge B}{B} \wedge e_2$$

$$\frac{A \wedge B}{A} \wedge e_1$$

$$\frac{A \wedge B}{B} \wedge e_2$$

Given: A set S of formulas Usage

 $\wedge$ i: for any two formulas A and B in S, add  $A \wedge B$  to S

 $\wedge e_1$ : for any formula of the form  $A \wedge B$  in S, add A to S

 $\triangle e_2$ : for any formula of the form  $A \triangle B$  in S. add A to S

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$p \wedge q$$
,  $r \vdash q \wedge r$ 

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$\underbrace{p \land q, r}_{premises} \vdash \underbrace{q \land r}_{conclusion}$$

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$\underbrace{p \land q, r}_{premises} \vdash \underbrace{q \land r}_{conclusion}$$

I like cats and (like) dogs, Jill likes birds ⊢ I like dogs and Jill likes birds

$$\frac{A \quad B}{A \land B} \land i \quad \frac{A \land B}{A} \land e_1 \quad \frac{A \land B}{B} \land e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$\underbrace{p \land q, r}_{premises} \vdash \underbrace{q \land r}_{conclusion}$$

$$p \land q$$
 premise

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$\underbrace{p \land q, r}_{premises} \vdash \underbrace{q \land r}_{conclusion}$$

- $p \land q$  premise
- <sub>2</sub> r premise

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$\underbrace{p \land q, r}_{premises} \vdash \underbrace{q \land r}_{conclusion}$$

- $p \wedge q$  premise
- <sub>2</sub> r premise
- $_3$  q  $\wedge e_2$  applied to 1

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$\begin{array}{c}
p \land q, r \\
premises
\end{array}
\qquad \begin{array}{c}
q \land r \\
conclusion
\end{array}$$

- $p \wedge q$  premise
- <sub>2</sub> r premise
- $_3$  q  $\wedge e_2$  applied to 1
- $q \wedge r \wedge i$  applied to 3, 2

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive  $q \wedge r$  from  $p \wedge q$  and r, i.e., that

$$\underbrace{p \land q, r}_{premises} \vdash \underbrace{q \land r}_{conclusion}$$

#### (Linear) Proof

- $p \wedge q$  premise
- <sub>2</sub> r premise
- $_3$  q  $\wedge e_2$  applied to 1
- $q \wedge r \wedge i$  applied to 3, 2

#### **Proof tree**

$$\frac{p \wedge q}{q} \wedge e_2 \over q \wedge r} \wedge$$

$$\frac{A}{\neg \neg A}$$
  $\neg \neg i$   $\frac{\neg \neg A}{A}$   $\neg \neg e$ 

**Example** Prove 
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

$$\frac{A}{\neg \neg A} \neg \neg i \qquad \frac{\neg \neg A}{A} \neg \neg e$$

**Example** Prove  $p, \neg \neg (q \land r) \vdash \neg \neg p \land r$ 

$$\frac{A}{\neg \neg A}$$
  $\neg \neg i$   $\frac{\neg \neg A}{A}$   $\neg \neg e$ 

**Example** Prove 
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

premise 
$$p \rightarrow q(q \land r)$$
 premise

$$\frac{A}{\neg \neg A}$$
  $\neg \neg i$   $\frac{\neg \neg A}{A}$   $\neg \neg e$ 

**Example** Prove 
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

premise 
$$p \rightarrow q \wedge r$$
 premise

$$_3$$
  $q \wedge r$   $\neg \neg e 2$ 

$$\frac{A}{\neg \neg A}$$
  $\neg \neg i$   $\frac{\neg \neg A}{A}$   $\neg \neg e$ 

**Example** Prove 
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

premise 
$$p \rightarrow q \wedge r$$
 premise  $p \rightarrow q \wedge r$   $p \rightarrow q \wedge r$ 

$$\frac{A}{\neg \neg A}$$
  $\neg \neg i$   $\frac{\neg \neg A}{A}$   $\neg \neg e$ 

**Example** Prove 
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

premise
$$premise$$

$$q \wedge r$$

$$r \wedge e_2$$

$$r \wedge e_2$$

$$r \wedge e_3$$

$$\frac{A}{\neg \neg A}$$
  $\neg \neg i$   $\frac{\neg \neg A}{A}$   $\neg \neg e$ 

**Example** Prove 
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

6  $\neg \neg p \wedge r \wedge i 5, 4$ 

$$\frac{A \qquad A \to B}{B} \to e$$

$$\frac{A \qquad A \to B}{B} \to e$$

$$\frac{A \qquad A \to B}{B} \to e$$

- p premise
- $p \rightarrow q$  premise  $q \rightarrow r$  premise

$$\frac{A \qquad A \to B}{B} \to e$$

$$\begin{array}{ccc} & p & & \text{premise} \\ & p \rightarrow q & \text{premise} \\ & q \rightarrow r & \text{premise} \end{array}$$

$$_4$$
  $q$   $\rightarrow$ e 1,2

$$\frac{A \qquad A \to B}{B} \to \mathrm{e}$$

$$A \longrightarrow B \longrightarrow \Theta$$

$$\frac{A \qquad A \to B}{B} \to e \qquad \qquad \frac{A \to B \qquad \neg B}{\neg A} \text{ MT}$$

$$\frac{A \longrightarrow B}{B} \rightarrow \epsilon$$

$$\frac{A \qquad A \to B}{B} \to e \qquad \frac{A \to B \qquad \neg B}{\neg A} \text{ MT}$$

- $\rightarrow$ e is also known as *Modus Ponens*
- MT is known as Modus Tollens





**Example** Prove  $p \rightarrow q \vdash \neg q \rightarrow \neg p$ 



$$\frac{\begin{vmatrix} A \\ \vdots \\ B \end{vmatrix}}{A \to B} \to i$$

## **Example** Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

If x equals 10 then x is positive  $\vdash$  If x is not positive then x does not equal 10

$$\frac{A \to B \quad \neg B}{\neg A} \quad \mathsf{MT}$$

$$\begin{array}{c|c}
A \\
\vdots \\
B
\end{array}$$

$$A \to B \to i$$

**Example** Prove 
$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

 $p \rightarrow q$  premise

$$\frac{A \to B \quad \neg B}{\neg A} \quad \mathsf{MT}$$

$$\begin{array}{c}
A \\
\vdots \\
B
\end{array}$$

$$A \to B \to i$$

## **Example** Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

- $p \rightarrow q$  premise assumption

$$\frac{A \to B \quad \neg B}{\neg A} \quad \mathsf{MT}$$

$$\begin{array}{c}
A \\
\vdots \\
B
\end{array}$$

$$A \to B \to i$$

## **Example** Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

 $p \rightarrow q$  premise  $p \rightarrow q$  premise  $p \rightarrow q$  assumption  $p \rightarrow q$  MT 1,2

### $\rightarrow$ introduction rule





### **Example** Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2

### $\rightarrow$ introduction rule





## **Example** Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	ightarrowi 2-3

Prove  $\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$ 



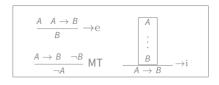
Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

 $q \rightarrow r$ 

assumption

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

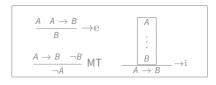
- $_1$   $q \rightarrow r$
- $_2$   $\neg q \rightarrow \neg p$



assumption assumption

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

- $a \rightarrow b$
- $_2$   $\neg q \rightarrow \neg p$ 
  - 3 P



assumption

assumption

assumption

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

 $\frac{A \quad A \to B}{B} \to e$   $\frac{A \to B \quad \neg B}{\neg A} \quad MT$   $\frac{A}{B} \to B \quad \rightarrow i$ 

- $_1$   $q \rightarrow r$
- $_2$   $\neg q \rightarrow \neg p$
- <sub>3</sub> p
- $_4$   $\neg \neg p$

assumption

assumption

assumption

¬¬i 3

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

 $\frac{A \quad A \to B}{B} \to e$   $\frac{A \to B \quad \neg B}{\neg A} \quad MT$   $\frac{A}{B} \to B \quad \rightarrow i$ 

- $_1$   $q \rightarrow r$
- $_2$   $\neg q \rightarrow \neg p$ 
  - p
- $_4$   $\neg \neg p$
- 5 779

assumption

assumption

assumption

 $\neg \neg i$  3

MT 2,4

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

$$\frac{A \quad A \to B}{B} \to e$$

$$\frac{A \to B \quad \neg B}{\neg A} \quad MT$$

$$\frac{A}{\vdots}$$

$$\vdots$$

$$B$$

$$A \to B$$

- $q \rightarrow r$
- $_2$   $\neg q \rightarrow \neg p$ 
  - p
- $_4$   $\neg \neg p$
- $5 \neg \neg q$
- 6 **C**

assumption

assumption

assumption

 $\neg \neg i$  3

MT 2,4

¬¬е 5

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

$$\frac{A \quad A \to B}{B} \to e \qquad \qquad \begin{vmatrix} A \\ \vdots \\ B \\ \neg A \end{vmatrix} \to i$$

$$\frac{A \to B \quad \neg B}{\neg A} \quad \mathsf{MT} \qquad \frac{B}{A \to B} \to i$$

1  $q \rightarrow r$ 2  $\neg q \rightarrow \neg p$ 3 p4  $\neg \neg p$ 5  $\neg \neg q$ 6 q7 r assumption assumption assumption ¬¬i 3 MT 2,4 ¬¬e 5  $\rightarrow$ e 1,6

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

$$\frac{A \quad A \to B}{B} \to e \qquad \qquad \begin{vmatrix} A \\ \vdots \\ B \end{vmatrix} \\
\xrightarrow{\neg A} MT \qquad \frac{B}{A \to B} \to i$$

1	$q \rightarrow r$	assumption
2	$\neg q  ightarrow  eg p$	assumption
3	p	assumption
4	$\neg\neg p$	¬¬і з
5	$\neg \neg q$	MT 2,4
6	q	¬¬e 5
7	r	→e 1,6

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

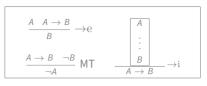
$$\frac{A \quad A \to B}{B} \to e \qquad \qquad \begin{vmatrix} A \\ \vdots \\ B \end{vmatrix} \\
\xrightarrow{-A} MT \qquad \frac{B}{A \to B} \to i$$

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$

$$\begin{array}{c}
A & A \to B \\
B & \\
A \to B & \neg B \\
\hline
 & \neg A
\end{array}$$
MT
$$\begin{array}{c}
A \\
\vdots \\
B \\
A \to B
\end{array}$$
 $\rightarrow i$ 

1	q  o r	assumption
2	$\neg q \rightarrow \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	¬¬і з
5	$\neg \neg q$	MT 2,4
6	q	¬¬е 5
7	r	→e 1,6
8	$p \rightarrow r$	→і 3-7

Prove 
$$\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$$



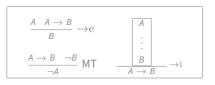
1	$q \rightarrow r$	assumption
2	$\neg q  ightarrow  eg p$	assumption
3	p	assumption
4	$\neg\neg p$	¬¬і з
5	$\neg \neg q$	MT 2,4
6	q	¬¬е 5
7	r	→e 1,6
8	p  o r	→i 3-7
9	$(\neg q  o \neg p)  o (p  o r)$	ightarrowi 2-8

Prove  $\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$ 

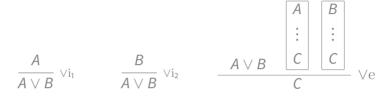
$$\frac{A \quad A \to B}{B} \to e \qquad \qquad \begin{vmatrix} A \\ \vdots \\ B \end{vmatrix} \\
\xrightarrow{\neg A} MT \qquad \frac{B}{A \to B} \to i$$

1	$q \rightarrow r$	assumption
2	$\neg q \rightarrow \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	¬¬і з
5	$\neg \neg q$	MT 2,4
6	q	¬¬е 5
7	r	→e 1,6
8	$p \rightarrow r$	→i 3-7
9	$(\neg q \to \neg p) \to (p \to r)$	→i 2-8

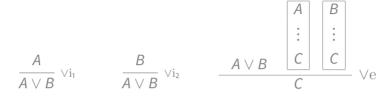
Prove  $\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$ 



1	$q \rightarrow r$	assumption
2	$\neg q \rightarrow \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	¬¬і з
5	$\neg \neg q$	MT 2,4
6	q	¬¬e 5
7	r	→e 1,6
8	p  o r	→i 3-7
9	$(\neg q  o \neg p)  o (p  o r)$	→i 2-8
10	$(q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q))$	→i 1-9



$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_2} \qquad \frac{A \vee B}{C} \vee_{i_2} \vee_{e}$$



**Example 1** Prove  $p \lor q \vdash q \lor p$ 

 $p \lor q$  premise

$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_2} \qquad \frac{A \vee B}{C} \vee_{i_2} \qquad \qquad \vee_{i_3}$$

- $p \lor q$  premise
- <sub>2</sub> *p* assumption

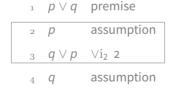
$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_2} \qquad \frac{A \vee B}{C} \vee_{i_2} \qquad \qquad C$$

- $p \lor q$  premise
- <sub>2</sub> *p* assumption
- $_3$   $q \lor p \lor i_2 2$

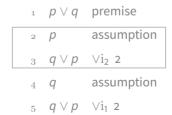


$$p \lor q$$
 premise 
$$\begin{array}{cccc} & p \lor q & \text{premise} \\ & 2 & p & \text{assumption} \\ & 3 & q \lor p & \lor i_2 & 2 \end{array}$$

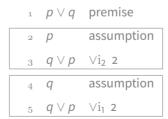




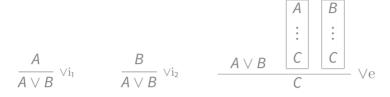
$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_2} \qquad \frac{A \vee B}{C} \vee_{i_2} \qquad \frac{A \vee B}{C} \vee_{i_2} \vee_{i_3}$$

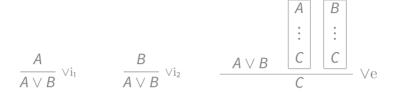












**Example 2** Prove  $p \lor q$ ,  $p \to r$ ,  $q \to r \vdash r$ 

$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_2} \qquad \frac{A \vee B}{C} \vee_{i_2} \qquad \vee_{e}$$

**Example 2** Prove 
$$p \lor q$$
,  $p \to r$ ,  $q \to r \vdash r$ 

- $p \lor q$  premise
- $_2$   $p \rightarrow r$  premise
- $_3$   $q \rightarrow r$  premise

$$\frac{A}{A \vee B} \vee i_1 \qquad \frac{B}{A \vee B} \vee i_2 \qquad \frac{A \vee B}{C} \vee i_2$$

#### **Example 2** Prove $p \lor q$ , $p \to r$ , $q \to r \vdash r$

$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_2} \qquad \frac{A \vee B}{C} \vee_{i_2} \qquad \vee_{i_3}$$

#### **Example 2** Prove $p \lor q$ , $p \to r$ , $q \to r \vdash r$

$$\frac{L}{A}$$
  $\perp$ e  $\frac{A}{\Box}$ 

$$\frac{\perp}{A}$$
 \perp e  $\frac{A}{\perp}$  \tag{-e}

**Example** Prove 
$$\neg p \lor q \vdash p \rightarrow q$$

I will not need a ride; otherwise, I will tell you ⊢ If I need a ride I will tell you

$$\frac{\bot}{A}$$
  $\bot$ e  $\frac{A}{\bot}$   $\neg$ e

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

premise

$$\frac{\bot}{4}$$
  $\bot$ e  $\frac{A}{\bot}$   $\neg e$ 

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

 $\neg p \lor q$   $2 \neg p$  assumption

premise

$$\frac{\bot}{A}$$
  $\bot$ e  $\frac{A}{\bot}$   $\neg$ e

premise

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

$$\neg p \lor q$$

 $_2$   $\neg p$  assumption  $_3$  p assumption

$$\frac{\bot}{A}$$
  $\bot$ e  $\frac{A}{\bot}$   $\neg$ e

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

 $_{1}$   $\neg p \lor q$ 

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

 $\neg p \lor q$ 

$$\frac{\perp}{\Lambda}$$
  $\perp$ e

$$\frac{A}{\bot}$$
  $\neg \epsilon$ 

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

$$\neg p \lor q$$

2	$\neg p$	assumption
3	p	assumption
4	$\perp$	¬e 3,2
5	9	⊥е 4

$$\frac{\bot}{A}$$
  $\bot$ e  $\frac{A}{\bot}$   $\neg$ e

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

 $_{1}$   $\neg p \lor q$ 

2	$\neg p$	assumption
3	р	assumption
4	$\perp$	¬e 3,2
5	q	⊥е 4
6	$p \rightarrow q$	→і 3-5

$$\frac{\perp}{A}$$
  $\perp$ e  $\frac{A}{\perp}$   $\neg$ e

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

 $_{1} \neg p \lor q$ 

2 ¬p assumption

premise

assumption

_	P	
3	р	assumptio
4	$\perp$	¬е 3,2
5	9	⊥е 4
6	$p \rightarrow a$	→i 3-5

$$\frac{\perp}{A}$$
 \( \text{Le} \)  $\frac{A}{\perp}$  \( \sqrt{A} \)

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

1	$\neg p \lor q$			premise
2	$\neg p$	assumption	q	assumption
3	р	assumption	p	assumption
4	$\perp$	¬е 3,2		
5	q	⊥е 4		
6	p  o q	→i 3-5		

$$\frac{\perp}{A}$$
  $\perp$ e  $\frac{A}{\perp}$   $\neg$ e

#### **Example** Prove $\neg p \lor q \vdash p \rightarrow q$

1	$\neg \rho \lor q$			premise
2	$\neg p$	assumption	q	assumption
3	р	assumption	p	assumption
4	$\perp$	¬e 3,2	q	сору 2
5	9	⊥е 4		
6	$p \rightarrow q$	→i 3-5		

$$\frac{\perp}{A}$$
 \( \text{Le} \)  $\frac{A}{\perp}$  \( \sqrt{\pi} \)

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

 $\neg p \lor q$ 

9	assamption
p	assumption
q	сору 2

$$\frac{\perp}{A}$$
 \( \perp \) \( \frac{A}{\pm} \)  $\emptyseta \)$ 

**Example** Prove  $\neg p \lor q \vdash p \rightarrow q$ 

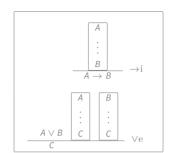
 $\neg p \lor q$ 

2	$\neg p$	assumption
3	р	assumption
4	$\perp$	¬е 3,2
5	q	⊥е <b>4</b>
6	p  o a	→i 3-5

q	assumption
p	assumption
q	сору 2
$p \rightarrow q$	ightarrowi 3-4

$$\frac{\perp}{\Delta}$$
  $\perp e$ 

$$\frac{A}{---}$$
 ¬e



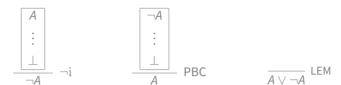
#### **Example** Prove $\neg p \lor q \vdash p \rightarrow q$

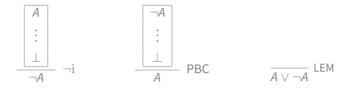
$$\neg p \lor q$$

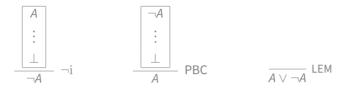
2	$\neg p$	assumption
3	р	assumption
4	$\perp$	¬е 3,2
5	q	⊥e 4
6	$p \rightarrow q$	→i 3-5

q	assumption
p	assumption
q	сору 2
$p \rightarrow q$	→i 3-4

 $_{7}$  p 
ightarrow q ee 1,2-6

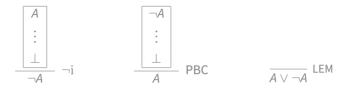






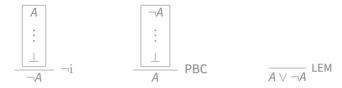
**Example 1** Prove 
$$p \rightarrow q$$
,  $p \rightarrow \neg q \vdash \neg p$ 

- $p \rightarrow q$  premise
- $p \rightarrow \neg q$  premise

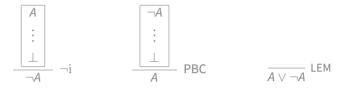


**Example 1** Prove 
$$p \rightarrow q$$
,  $p \rightarrow \neg q \vdash \neg p$ 

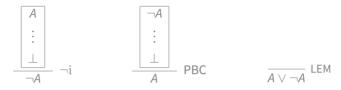
- $p \rightarrow q$  premise
- $_{2}$   $p \rightarrow \neg q$  premise
- <sub>3</sub> *p* assumption



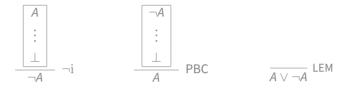
- p o q premise
  - $_{2}$   $p 
    ightarrow \neg q$  premise
- <sub>3</sub> *p* assumption
- $_4$  q  $\rightarrow$ e 1,3



$$p \rightarrow q$$
 premise  
 $p \rightarrow \neg q$  premise  
 $p \rightarrow \neg q$  premise  
 $p \rightarrow q$  assumption  
 $p \rightarrow q$   $p \rightarrow q$   $p \rightarrow q$  assumption

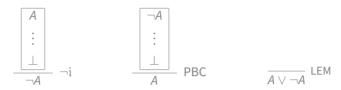


$$p \rightarrow q$$
 premise  
 $p \rightarrow q$  premise  
 $p \rightarrow \neg q$  premise  
 $p \rightarrow \neg q$  premise  
 $p \rightarrow q$  assumption  
 $p \rightarrow q$  assumption

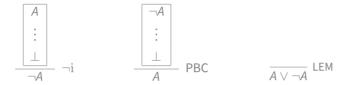


$$p \rightarrow q$$
 premise  $p \rightarrow \neg q$  premise

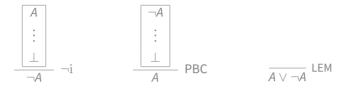
3	р	assumption
4	q	ightarrowe 1,3
5	$\neg q$	ightarrowe 2,3
6	$\perp$	¬e 4,5



$$\begin{array}{cccc}
 & p \rightarrow q & \text{premise} \\
 & p \rightarrow \neg q & \text{premise} \\
 & p & \text{assumption} \\
 & q & \rightarrow e \ 1, 3 \\
 & 5 & \neg q & \rightarrow e \ 2, 3 \\
 & 6 & \bot & \neg e \ 4, 5 \\
 & 7 & \neg p & \neg i \ 2-4
\end{array}$$

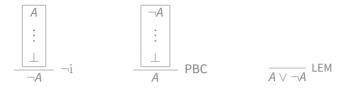


**Example 2** Prove  $\neg p \rightarrow \bot \vdash p$ 



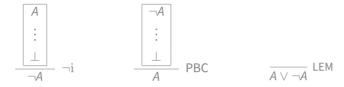
**Example 2** Prove  $\neg p \rightarrow \bot \vdash p$ 

 $_{\scriptscriptstyle 1}$   $\neg p \rightarrow \bot$  premise



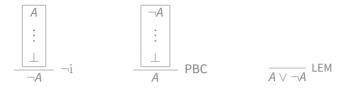
#### **Example 2** Prove $\neg p \rightarrow \bot \vdash p$

- $\neg p \rightarrow \bot$  premise
- ₂ ¬p assumption



#### **Example 2** Prove $\neg p \rightarrow \bot \vdash p$

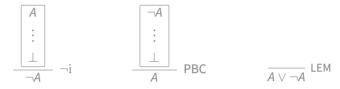
- $\neg p \rightarrow \bot$  premise
- 2 ¬p assumption
- $_3$   $\perp$   $\rightarrow$ e 1,2



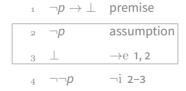
#### **Example 2** Prove $\neg p \rightarrow \bot \vdash p$

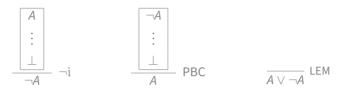
1	7 -	premise
2	$\neg p$	assumption
3	$\perp$	ightarrowe 1,2

 $\neg n \rightarrow \bot$  premise

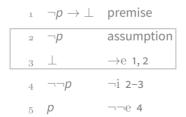


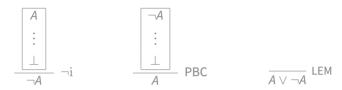
#### **Example 2** Prove $\neg p \rightarrow \bot \vdash p$





#### **Example 2** Prove $\neg p \rightarrow \bot \vdash p$

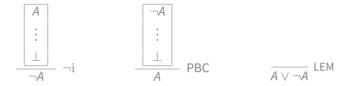




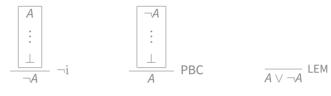
#### **Example 2** Prove $\neg p \rightarrow \bot \vdash p$

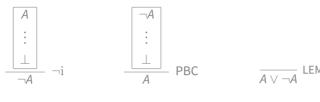
$$\begin{array}{ccc}
 & \neg p \rightarrow \bot & \text{premise} \\
\hline
 & 2 & \neg p & \text{assumption} \\
 & 3 & \bot & \rightarrow e & 1, 2 \\
\hline
 & 4 & \neg \neg p & \neg i & 2-3 \\
\hline
 & 5 & p & \neg \neg e & 4
\end{array}$$

PBC can be simulated



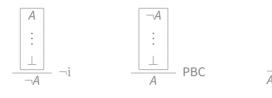
**Example 3** Prove  $\vdash p \lor \neg p$ 





**Example 3** Prove  $\vdash p \lor \neg p$   $_1 \neg (p \lor \neg p)$  assumption

<sub>2</sub> p assumption



**Example 3** Prove 
$$\vdash p \lor \neg p$$
 1  $\neg (p \lor \neg p)$  assumption

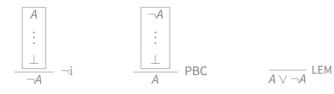
$$\neg (p \lor \neg p)$$
 assumption



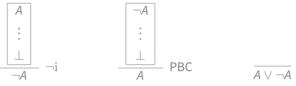
Example 3 Prove 
$$\vdash p \lor \neg p$$

**Example 3** Prove 
$$\vdash p \lor \neg p$$
 1  $\neg (p \lor \neg p)$  assumption

$$_3$$
  $p \lor \neg p$   $\lor i_1$  2  
 $_4$   $\bot$   $\neg e$  3,1

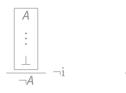


**Example 3** Prove  $\vdash p \lor \neg p$   $\neg (p \lor \neg p)$  assumption



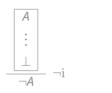
**Example 3** Prove  $\vdash p \lor \neg p$   $_1 \neg (p \lor \neg p)$  assumption

 $\neg (p \lor \neg p)$  assumption p assumption  $p \lor \neg p$   $\lor i_1 2$  $p \lor \neg p$   $\Rightarrow i_2 1$ 



**Example 3** Prove 
$$\vdash p \lor \neg p$$
  $_1 \neg (p \lor \neg p)$  assumption

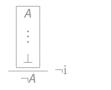
$$\neg (p \lor \neg p)$$
 assumption



**Example 3** Prove 
$$\vdash p \lor \neg p$$
  $_1 \neg (p \lor \neg p)$  assumption

$$\neg (p \lor \neg p)$$
 assumption

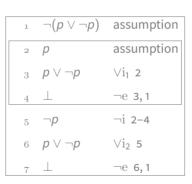
$$6 \quad p \lor \neg p \qquad \lor i_2 \ 5$$

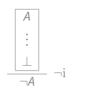




LEM

**Example 3** Prove 
$$\vdash p \lor \neg p$$

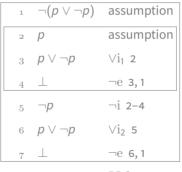




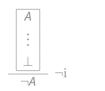
: \_\_\_\_\_ PBC

LEN

**Example 3** Prove  $\vdash p \lor \neg p$ 



 $p \lor \neg p$  PBC 7

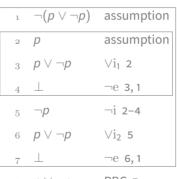


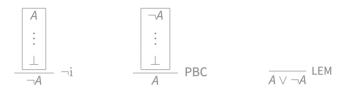


LEN

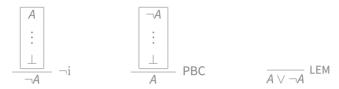
**Example 3** Prove  $\vdash p \lor \neg p$ 

LEM can be simulated too





PBC and LEM are derived rules



PBC and LEM are derived rules

MT and  $\neg \neg i$  are derived rules too

#### Soundness of natural deduction

We will prove a crucial property of natural deduction:

Any formula A derived from a set S of premises is a logical consequence of S

#### Theorem 1 (Soundness)

For all formulas  $A_1, \ldots, A_n$  and A such that  $A_1, \ldots, A_n \vdash A$ , we have that  $A_1, \ldots, A_n \models A$ .

For the proof of the theorem, we will rely on this lemma

#### Lemma 2

For all formulas  $A_1, \ldots, A_n$ , A and B,

- 1.  $A_1, \ldots, A_n, A \models B \text{ iff } A_1, \ldots, A_n \models A \rightarrow B$
- 2.  $A_1, \ldots, A_n \models B \text{ iff } A_1, \ldots, A_n, \neg B \models \bot$

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- **2.**  $A_1, \ldots, A_n \models B$  iff  $A_1, \ldots, A_n, \neg B \models \bot$

The proof of Theorem 1 is by induction on proof length

The *length* of a natural deduction proof is the number of lines in it

**Proof of Theorem 1.** (if  $A_1, \ldots, A_n \vdash A$  then  $A_1, \ldots, A_n \models A$ 

Let  $\Pi$  be the a proof of  $A_1, \ldots, A_n \vdash A$ , seen as a sequence of formulas

Assume, without loss of generality, that A is the last formula in the sequence

By induction on the length l of  $\Pi$ .

(Base case: l = n

Then  $A = A_i$  for some  $i \in \{1, \ldots, n\}$ . Trivially,  $A_1, \ldots, A_n \models A_i$ 

**Proof of Theorem 1.** (if  $A_1, \ldots, A_n \vdash A$  then  $A_1, \ldots, A_n \models A$ )

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Then  $A = A_i$  for some  $i \in \{1, ..., n\}$ . Trivially,  $A_1, ..., A_n \models A_i$ .

(Inductive step: l > n)

Assume by induction that the theorem holds for all proofs of length l' < l.

The proof depends on the final rule used to derive A.

 $(Ae_1)$  If A was derived by  $Ae_1$ , then  $\Pi$  looks like:

$$A_1$$
 premise  $\vdots$   $A \wedge B \dots$   $\vdots$   $A \wedge A \wedge B \dots$ 

for some formula B

Note that the subsequence of  $\Pi$  from  $A_1$  to  $A \wedge B$  is a proof of  $A \wedge B$  of length < l.

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for some formula B

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(Inductive step: l > n)
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for some formula B.

Note that the subsequence of  $\Pi$  from  $A_1$  to  $A \wedge B$  is a proof of  $A \wedge B$  of length < l. Then, by inductive hypothesis,  $A_1, \ldots, A_n \models A \wedge B$ . Hence,  $A_1, \ldots, A_n \models A$ .

```
(Inductive step: l > n)
```

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(∧i)

( $\wedge$ i) Then A has the form  $B_1 \wedge B_2$ 

( $\wedge$ i) Then A has the form  $B_1 \wedge B_2$  and  $\Pi$  looks like:

$A_1$	premise		$A_1$	premise
:			:	
$B_1$			$B_2$	
:		or	:	
$B_2$			$B_1$	
:			:	
$B_1 \wedge B_2$	$\wedge i$		$B_1 \wedge B_2$	$\wedge i$

( $\wedge$ i) Then A has the form  $B_1 \wedge B_2$  and  $\Pi$  looks like:

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:			:	
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Then, by inductive hypothesis,  $A_1, \ldots, A_n \models B_i$  for i = 1, 2.

Hence,  $A_1, \ldots, A_n \models B_1 \wedge B_2$ .

(→i)

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□ looks like:

```
<sup>1</sup> A<sub>1</sub> premise
```

2

$$_3$$
  $B_1$  assumption

4:

$$_5$$
  $B_2$  ...

6 
$$B_1 \rightarrow B_2 \rightarrow i$$

 $(\rightarrow i)$  Then A has the form  $B_1 \rightarrow B_2$  and

□ looks like:

1	$A_1$	premise
2	:	
3	B <sub>1</sub>	assumption
4	:	
5	B <sub>2</sub>	
6	$B_1 \rightarrow B_2$	$\rightarrow$ i

but then

 $(\rightarrow i)$  Then A has the form  $B_1 \rightarrow B_2$  and

□ looks like:	1	$A_1$	premise	but then	1	$A_1$	premise
	2	:			2	:	
	3	$B_1$	assumption		3	$B_1$	premise
	4	:			4	:	
	5	$B_2$			5	$B_2$	
	6	$B_1 \rightarrow B_2$	$\rightarrow$ i				

is a proof of  $B_2$  from  $A_1, \ldots, A_n, B_1$  that is shorter than  $\Pi$ .

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	3	B <sub>1</sub>	assumption		3	$B_1$	premise
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	4	:			4	:	
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Then, by inductive hypothesis,  $A_1, \ldots, A_n, B_1 \models B_2$ .

It follows from Lemma 2(1) that  $A_1, \ldots, A_n \models B_1 \rightarrow B_2$ .

(¬i)

 $(\neg i)$  Then A has the form  $\neg B$  and

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 $\Pi$  looks like: 

1  $A_1$  premise
2 :

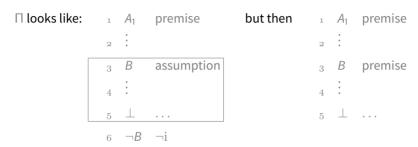
3 B assumption
4 :
5  $\bot$  ...

 $(\neg i)$  Then A has the form  $\neg B$  and

 $\Pi$  looks like: 1  $A_1$  premise 2 : 3 B assumption 4 : 5  $\bot$  ...

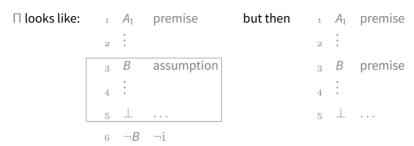
but then

 $(\neg i)$  Then A has the form  $\neg B$  and



is a proof of  $\perp$  from  $A_1, \ldots, A_n, B$  that is shorter than  $\square$ .

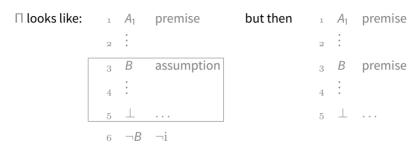
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Then, by inductive hypothesis,  $A_1, \ldots, A_n, B \models \bot$ .

It follows from Lemma 2 that  $A_1, \ldots, A_n \models \neg B$ .

```
(\wedge i_2) Analogous to \wedge i_2 case.
(\vee i_1) Exercise.
(\vee i_1) Exercise.
(∨e) Exercise.
(\rightarrow e) Exercise.
(\neg e) Exercise.
(\perp e) Exercise.
(\neg \neg e) Exercise.
```

We will now prove another important property of natural deduction:

Any logical consequence A of a set S of formulas has a proof with premises S

Theorem 3 (Completeness)

For all formulas  $A_1, \ldots, A_n$  and A such that  $A_1, \ldots, A_n \models A$ , we have that  $A_1, \ldots, A_n \models A$ .

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To prove this theorem, we will rely on several intermediate results

### Lemma 4

For all formulas  $A_1, \ldots, A_n$  and A the following holds:

- 1.  $A_1, A_2, \ldots, A_n \models A$  implies  $\models A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$
- **2.**  $\vdash A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$  implies  $A_1, A_2, \dots, A_n \vdash A$

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### Proof.

By induction on n in both cases (see Huth & Ryan).

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All valid formulas B are provable in natural deduction: if  $\models$  B then  $\vdash$  B.

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**Proof of Theorem 3**  $(A_1, \ldots, A_n \models A \text{ implies } A_1, \ldots, A_n \vdash A)$ .

Assume  $A_1, \ldots, A_n \models A$ , prove  $A_1, A_2, \ldots, A_n \vdash A$ .

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By Lemma 4(1),  $\models$   $A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$ .

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By Lemma 4(2),  $A_1, A_2, ..., A_n \vdash A$ .

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So we are left with proving Theorem 5

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Let A be a formula over variables  $p_1, \ldots, p_n$  with  $n \ge 0$  and let  $\mathcal{I}$  be an interpretation. Let  $\hat{p}_i = p$  if  $\mathcal{I} \models p$  and  $\hat{p}_i = \neg p$  otherwise. Then,  $\hat{p}_1, \ldots, \hat{p}_n \vdash A$  if  $\mathcal{I} \models A$  and  $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$  if  $\mathcal{I} \not\models A$ .

Proof of Lemma 6. By structural induction on A

### (Base case)

If *A* is just a variable, say  $p_1$ , then it is immediate that  $p_1 \vdash p_1$  and  $\neg p_1 \vdash \neg p_1$ . If *A* is  $\bot$  then n = 0 and  $\mathcal{I} \not\models A$ . We can prove  $\neg \bot$  from no premises by  $\neg i$ .

(Inductive Step) If A is not a variable or  $\bot$ , assume the result holds for all proper subformulas of A.

We reason by cases on the form of A.

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**Proof of Lemma 6.** ( $\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$ ) (continued)

 $(A = \neg B)$  (that is, suppose A has the form  $\neg B$ )

- If  $\mathcal{I} \models A$  then  $\mathcal{I} \not\models B$ . By inductive hypothesis,  $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg B$ .
- If I ⊭ A then I ⊨ B. By inductive hypothesis, p̂<sub>1</sub>,..., p̂<sub>n</sub> ⊢ B.
   Take a proof of B from p̂<sub>1</sub>,..., p̂<sub>n</sub> and apply ¬¬i to B.
   The resulting proof is a proof of ¬A.

**Proof of Lemma 6.**  $(\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$  (continued)

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• If  $\mathcal{I} \models A$  then  $\mathcal{I} \models B_1$  and  $\mathcal{I} \models B_2$ . By inductive hypothesis,  $\hat{p}_1, \dots, \hat{p}_n \vdash B_1$  and  $\hat{p}_1, \dots, \hat{p}_n \vdash B_2$ .

A proof of A from  $\hat{p}_1, \ldots, \hat{p}_n$  is obtained by chaining a proof of  $B_1$  and a proof of  $B_2$  and applying  $\wedge$ i to  $B_1$  and  $B_2$ .

Proof of Lemma 6.  $(\hat{p}_1, ..., \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, ..., \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$  (continued)

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If \( \mu \) \( \mu \) A then \( \mu \) \( \mu \) B<sub>k</sub> for some \( k \in \) {1, 2}. Say \( k = 1 \) (the other case is similar).
 By inductive hypothesis, \( \hat{\rho}\_1, \ldots, \hat{\rho}\_n \) ⊢ \( B\_1. \)
 A proof of \( \sigma\_B \) can be extended to a proof of \( \sigma\_A \) as follows:

$$\begin{array}{cccc}
 & \vdots & & & \\
 & 2 & \neg B_1 & & \\
 & 3 & B_1 \wedge B_2 & \text{assumption} \\
 & 4 & B_1 & & \wedge e_1 3 \\
 & 5 & \bot & & \bot i 4, 2 \\
 & 6 & \neg (B_1 \wedge B_2) & \bot i 3, 5
\end{array}$$

```
Proof of Lemma 6. (\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A) (continued)
```

$$(A = B_1 \vee B_2)$$

Proof of Lemma 6.  $(\hat{p}_1, ..., \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, ..., \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$  (continued)

$$(A = B_1 \vee B_2)$$

If \( \mathcal{I} \) |= \( A\) then \( \mathcal{I} \) |= \( B\_k\) for some \( k \) ∈ \( \{1,2 \) .
 A proof of \( A\) from \( \hat{p}\_1, \ldots, \hat{p}\_n\) is obtained from a proof of \( B\_k\) by applying \( \neq i\_k\) to \( B\_k\) to get \( B\_1 \lor B\_2\).

Proof of Lemma 6.  $(\hat{p}_1, ..., \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, ..., \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$  (continued)

$$(A = B_1 \vee B_2)$$

If \( \mu \models A\) then \( \mu \models B\_1\) and \( \mu \models B\_2\).
 A proof of ¬A from \( \hat{\rho}\_1, \ldots, \hat{\rho}\_n\) is obtained by chaining a proof of ¬B<sub>1</sub> and a proof of ¬B<sub>2</sub> and continuing as follows:

**Proof of Lemma 6.** ( $\hat{p}_1, \ldots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \ldots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$ ) (continued)

$$(A = B_1 \to B_2)$$

- If  $\mathcal{I} \models A$  then  $\mathcal{I} \not\models B_1$  or  $\mathcal{I} \models B_2$ . (exercise)
- If  $\mathcal{I} \not\models A$  then  $\mathcal{I} \models B_1$  and  $\mathcal{I} \not\models B_2$ . (exercise)

**Proof of Lemma 6.** ( $\hat{p}_1, \ldots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \ldots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$ ) (continued)

$$(A = B_1 \to B_2)$$

- If  $\mathcal{I} \models A$  then  $\mathcal{I} \not\models B_1$  or  $\mathcal{I} \models B_2$ . (exercise)
- If  $\mathcal{I} \not\models A$  then  $\mathcal{I} \models B_1$  and  $\mathcal{I} \not\models B_2$ . (exercise)

**Proof of Lemma 6.** ( $\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$ ) (continued)

$$(A = B_1 \to B_2)$$

- If  $\mathcal{I} \models A$  then  $\mathcal{I} \not\models B_1$  or  $\mathcal{I} \models B_2$ . (exercise)
- If  $\mathcal{I} \not\models A$  then  $\mathcal{I} \models B_1$  and  $\mathcal{I} \not\models B_2$ . (exercise)

### Lemma 7

Let  $L_2, ..., L_n$ , A be formulas and let p be one of A's variables. If  $p, L_2, ..., L_n \vdash A$  and  $\neg p, L_2, ..., L_n \vdash A$  then  $L_2, ..., L_n \vdash A$ .

## **Proof of Lemma 7.** $(p, L_2, \ldots, L_n \vdash A \text{ and } \neg p, L_2, \ldots, L_n \vdash A \text{ implies } L_2, \ldots, L_n \vdash A)$

Suppose we have the proofs:

1 
$$p$$
 premise and 1  $\neg p$  premise  
2  $L_2$  premise  
3  $\vdots$  3  $\vdots$  4  $A$  ... 4  $A$  ...

The following is a proof of A from  $L_2, \ldots, L_n$ :

# **Proof of Lemma 7.** ( $p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$ )

Suppose we have the proofs:

The following is a proof of A from  $L_2, \ldots, L_n$ :

## **Proof of Lemma 7.** ( $p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$ )

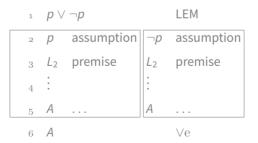
Suppose we have the proofs:

The following is a proof of A from  $L_2, \ldots, L_n$ :

### **Proof of Lemma 7.** $(p, L_2, \ldots, L_n \vdash A \text{ and } \neg p, L_2, \ldots, L_n \vdash A \text{ implies } L_2, \ldots, L_n \vdash A)$

#### Suppose we have the proofs:

#### The following is a proof of A from $L_2, \ldots, L_n$ :



Let  $p_1, \ldots, p_n$  be all of A's variables and consider the set

$$S = \{ p_1, \neg p_1 \} \times \cdots \times \{ p_n, \neg p_n \},$$

of all tuples  $(\hat{p}_1, \dots, \hat{p}_n)$  where each  $\hat{p}_i$  is either  $p_i$  or  $\neg p_i$ . We prove by induction on  $i = 1, \dots, n+1$  that

$$\hat{p}_{l},\ldots,\hat{p}_{n}\vdash A \ \ ext{for every}\ (\hat{p}_{1},\ldots,\hat{p}_{n})\in \mathbf{S}\ .$$

The theorem then follows from Property (1) for i=n+1.

(i=1) Property (1) holds by Lemma 6 since every  $(\hat{p}_1,\ldots,\hat{p}_n)\in \mathbf{S}$  corresponds to an interpretation of A and all interpretations satisfy A (by def. of validity).

$$(i>1)$$
 Suppose  $\hat{p}_i,\ldots,\hat{p}_n\vdash A$  for all  $(\hat{p}_1,\ldots,\hat{p}_n)\in \mathbb{S}$ . We prove that  $\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  for all  $(\hat{p}_1,\ldots,\hat{p}_n)\in \mathbb{S}$ . Let  $(\hat{p}_1,\ldots,p_i,\hat{p}_{i+1},\ldots,\hat{p}_n),(\hat{p}_1,\ldots,\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n)\in \mathbb{S}$ . By induction hypothesis,  $p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  and  $\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  Then  $\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  by Lemma 7.

Let  $p_1, \ldots, p_n$  be all of A's variables and consider the set

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of all tuples  $(\hat{p}_1, \dots, \hat{p}_n)$  where each  $\hat{p}_i$  is either  $p_i$  or  $\neg p_i$ .

We prove by induction on i = 1, ..., n + 1 that

$$\hat{p}_{l}, \dots, \hat{p}_{n} \vdash A \text{ for every } (\hat{p}_{1}, \dots, \hat{p}_{n}) \in \mathbf{S}$$
 . (1)

The theorem then follows from Property (1) for i = n + 1.

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Let  $p_1, \ldots, p_n$  be all of A's variables and consider the set

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The theorem then follows from Property (1) for i = n + 1.

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ho}_1,\dots,\hat{
ho}_i,\hat{
ho}_{i+1},\dots,\hat{
ho}_n), (\hat{
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ho}_{i+1},\dots,\hat{
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(i=1) Property (1) holds by Lemma 6 since every  $(\hat{p}_1,\ldots,\hat{p}_n)\in S$  corresponds to an interpretation of A and all interpretations satisfy A (by def. of validity).

 $\begin{array}{l} (i>1) \ \text{Suppose} \ \hat{\rho}_i, \dots, \hat{\rho}_n \ \vdash \ A \ \text{for all} \ (\hat{\rho}_1, \dots, \hat{\rho}_n) \in \mathbf{S}. \\ \text{We prove that} \ \hat{\rho}_{i+1}, \dots, \hat{\rho}_n \ \vdash \ A \ \text{for all} \ (\hat{\rho}_1, \dots, \hat{\rho}_n) \in \mathbf{S}. \\ \text{Let} \ (\hat{\rho}_1, \dots, p_i, \hat{\rho}_{i+1}, \dots, \hat{\rho}_n), \ (\hat{\rho}_1, \dots, \neg p_i, \hat{\rho}_{i+1}, \dots, \hat{\rho}_n) \in \mathbf{S}. \\ \text{By induction hypothesis,} \ p_i, \hat{\rho}_{i+1}, \dots, \hat{\rho}_n \ \vdash \ A \ \text{and} \ \neg p_i, \hat{\rho}_{i+1}, \dots, \hat{\rho}_n \ \vdash \ A \\ \text{Then} \ \hat{\rho}_{i+1}, \dots, \hat{\rho}_n \ \vdash \ A \ \text{by Lemma 7}. \end{array}$ 

Let  $p_1, \ldots, p_n$  be all of A's variables and consider the set

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The theorem then follows from Property (1) for i = n + 1.

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(
$$i > 1$$
) Suppose  $\hat{p}_i, \dots, \hat{p}_n \vdash A$  for all  $(\hat{p}_1, \dots, \hat{p}_n) \in S$ .

We prove that  $\hat{p}_{l+1},\ldots,\hat{p}_n\vdash A$  for all  $(\hat{p}_1,\ldots,\hat{p}_n)\in \mathbf{S}$ . Let  $(\hat{p}_1,\ldots,p_l,\hat{p}_{l+1},\ldots,\hat{p}_n),(\hat{p}_1,\ldots,\neg p_l,\hat{p}_{l+1},\ldots,\hat{p}_n)\in \mathbf{S}$ . By induction hypothesis,  $p_l,\hat{p}_{l+1},\ldots,\hat{p}_n\vdash A$  and  $\neg p_l,\hat{p}_{l+1},\ldots,\hat{p}_n\vdash A$ Then  $\hat{p}_{l+1},\ldots,\hat{p}_n\vdash A$  by Lemma 7.

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Let  $(p_1,\ldots,p_i,\hat{p}_{i+1},\ldots,\hat{p}_n),(p_1,\ldots,\neg p_i,\hat{p}_{i+1},\ldots,p_n)\in \mathbb{S}.$ By induction hypothesis,  $p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  and  $\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$ Then  $\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  by Lemma 7.

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(
$$i > 1$$
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Let  $(\hat{p}_1,\ldots,p_i,\hat{p}_{i+1},\ldots,\hat{p}_n),(\hat{p}_1,\ldots,\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n) \in \mathbf{S}.$ 

By induction hypothesis,  $p_i, p_{i+1}, \ldots, p_n \vdash A$  and  $\neg p_i, p_{i+1}, \ldots, p_n \vdash A$ 

Then  $p_{i+1}, \ldots, p_n \vdash A$  by Lemma 7.

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(
$$i>1$$
) Suppose  $\hat{p}_i,\ldots,\hat{p}_n\vdash A$  for all  $(\hat{p}_1,\ldots,\hat{p}_n)\in \mathbf{S}$ . We prove that  $\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  for all  $(\hat{p}_1,\ldots,\hat{p}_n)\in \mathbf{S}$ . Let  $(\hat{p}_1,\ldots,p_i,\hat{p}_{i+1},\ldots,\hat{p}_n),(\hat{p}_1,\ldots,\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n)\in \mathbf{S}$ . By induction hypothesis,  $p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  and  $\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  Then  $\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$  by Lemma 7.