CS:4350 Logic in Computer Science

Natural Deduction for Propositional Logic

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Credits

Part of these slides are based on Chap. 2 of *Logic in Computer Science* by M. Huth and M. Ryan, Cambridge University Press, 2nd edition, 2004.

Outline

Natural Deduction
- Derivation Rules
- Soundness and Completeness

Natural deduction

There are many derivation systems for propositional logic

Natural deduction is a family of derivation systems with derivation rules *designed to mimic the way people reason deductively*

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Natural deduction is a family of derivation systems with derivation rules *designed to mimic the way people reason deductively*

Note

- “Natural” here is meant in contraposition to “mechanical / automated”
- Other derivation systems for PL are more machine-oriented and so arguably not as natural for people
- Natural deduction is actually automatable but less conveniently than other, more machine-oriented derivation systems

Natural deduction

There are many derivation systems for propositional logic

Natural deduction is a family of derivation systems with derivation rules *designed to mimic the way people reason deductively*

Note

For simplicity but without loss of generality, we will

- not use $\top$ (as $\top \equiv \neg\bot$)
- not use $\leftrightarrow$ (as $A \leftrightarrow B \equiv (A \to B) \land (B \to A)$)
- use $\land$ only with two arguments (as $A \land B \land C \equiv (A \land B) \land C$)
- use $\lor$ only with two arguments (as $A \lor B \lor C \equiv (A \lor B) \lor C$)

Natural deduction

There are many derivation systems for propositional logic

Natural deduction is a family of derivation systems with derivation rules *designed to mimic the way people reason deductively*

We will write

$$\underbrace{A_1, \ldots, A_n \ \vdash \ A}_{\textit{sequent}}$$

to indicate that A is derivable from $A_1, \ldots, A_n$ using the rules of natural deduction

$\wedge$ introduction and elimination rules

$$\frac{A \qquad B}{A \wedge B}\,\wedge\mathrm{i} \qquad\qquad \frac{A \wedge B}{A}\,\wedge\mathrm{e}_1 \qquad\qquad \frac{A \wedge B}{B}\,\wedge\mathrm{e}_2$$

Usage Given: A set **S** of formulas

- $\wedge\mathrm{i}$: for any two formulas A and B in **S**, add $A \wedge B$ to **S**
- $\wedge\mathrm{e}_1$: for any formula of the form $A \wedge B$ in **S**, add A to **S**
- $\wedge\mathrm{e}_2$: for any formula of the form $A \wedge B$ in **S**, add A to **S**

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Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$p \wedge q,\ r \vdash q \wedge r$$

Example derivation

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$$\underbrace{p \wedge q, r}_{\textit{premises}} \vdash \underbrace{q \wedge r}_{\textit{conclusion}}$$

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$$\underbrace{p \wedge q, r}_{\textit{premises}} \vdash \underbrace{q \wedge r}_{\textit{conclusion}}$$

I like cats and (like) dogs, Jill likes birds $\vdash$ I like dogs and Jill likes birds

Example derivation

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Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

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(Linear) Proof

1 $p \wedge q$ premise

Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

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$$\underbrace{p \wedge q, r}_{\textit{premises}} \vdash \underbrace{q \wedge r}_{\textit{conclusion}}$$

(Linear) Proof

1	$p \wedge q$	premise
2	r	premise

Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

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(Linear) Proof

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge\mathrm{e}_2$ applied to 1

Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\textit{premises}} \vdash \underbrace{q \wedge r}_{\textit{conclusion}}$$

(Linear) Proof

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge\mathrm{e}_2$ applied to 1
4	$q \wedge r$	$\wedge\mathrm{i}$ applied to 3, 2

Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\textit{premises}} \vdash \underbrace{q \wedge r}_{\textit{conclusion}}$$

(Linear) Proof

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge\mathrm{e}_2$ applied to 1
4	$q \wedge r$	$\wedge\mathrm{i}$ applied to 3, 2

Proof tree

$$\dfrac{\dfrac{p \wedge q}{q}\wedge\mathrm{e}_2 \qquad r}{q \wedge r}\wedge\mathrm{i}$$

$\neg$ introduction and elimination rules

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

$\neg$ introduction and elimination rules

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

$\neg$ introduction and elimination rules

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise

$\neg$ introduction and elimination rules

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$q \wedge r$	$\neg\neg\mathrm{e}$ 2

$\neg$ introduction and elimination rules

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$

1	p	premise
2	$\neg\neg(q \land r)$	premise
3	$q \land r$	$\neg\neg\mathrm{e}$ 2
4	r	$\land\mathrm{e}_2$ 3

$\neg$ introduction and elimination rules

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p,\ \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$q \wedge r$	$\neg\neg\mathrm{e}$ 2
4	r	$\wedge\mathrm{e}_2$ 3
5	$\neg\neg p$	$\neg\neg\mathrm{i}$ 1

$\neg$ introduction and elimination rules

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$q \wedge r$	$\neg\neg\mathrm{e}$ 2
4	r	$\wedge\mathrm{e}_2$ 3
5	$\neg\neg p$	$\neg\neg\mathrm{i}$ 1
6	$\neg\neg p \wedge r$	$\wedge\mathrm{i}$ 5, 4

$\rightarrow$ elimination rules

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow \mathrm{e}$$

$\rightarrow$ elimination rules

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e}$$

Example Prove $p,\ p \rightarrow q,\ q \rightarrow r \vdash r$

$\to$ elimination rules

$$\frac{A \qquad A \to B}{B} \to\text{e}$$

Example Prove $p, p \to q, q \to r \vdash r$

1	p	premise
2	$p \to q$	premise
3	$q \to r$	premise

$\to$ elimination rules

$$\frac{A \qquad A \to B}{B} \to\mathrm{e}$$

Example Prove $p, p \to q, q \to r \vdash r$

1	p	premise
2	$p \to q$	premise
3	$q \to r$	premise
4	q	$\to$e 1,2

$\rightarrow$ elimination rules

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e}$$

Example Prove $p, p \rightarrow q, q \rightarrow r \vdash r$

1	p	premise
2	$p \rightarrow q$	premise
3	$q \rightarrow r$	premise
4	q	$\rightarrow\mathrm{e}$ 1,2
5	r	$\rightarrow\mathrm{e}$ 4,3

$\rightarrow$ elimination rules

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e} \qquad\qquad \frac{A \rightarrow B \qquad \neg B}{\neg A} \text{ MT}$$

$\rightarrow$ elimination rules

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e} \qquad\qquad \frac{A \rightarrow B \qquad \neg B}{\neg A} \mathrm{MT}$$

- $\rightarrow\mathrm{e}$ is also known as *Modus Ponens*
- MT is known as *Modus Tollens*

$$\frac{A \to B \quad \neg B}{\neg A}\ \text{MT}$$

$\to$ introduction rule

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B}\ {\to}\mathrm{i}$$

Example Prove $p \to q \vdash \neg q \to \neg p$

$\rightarrow$ introduction rule

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ \rightarrow\mathrm{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

If x equals 10 then x is positive $\vdash$ If x is not positive then x does not equal 10

$\rightarrow$ introduction rule

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \rightarrow B}\ \rightarrow\text{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise

$\rightarrow$ introduction rule

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B} \rightarrow\text{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption

$\rightarrow$ introduction rule

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B} \rightarrow\text{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2

$\rightarrow$ introduction rule

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ \rightarrow\text{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2

$\rightarrow$ introduction rule

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ \rightarrow\text{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	$\rightarrow$i 2–3

Longer Example

Prove $\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$

$$\frac{A \quad A \rightarrow B}{B} \rightarrow\mathrm{e}$$

$$\frac{A \rightarrow B \quad \neg B}{\neg A} \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B} \rightarrow\mathrm{i}$$

Longer Example

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to\mathrm{i}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption

Longer Example

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{A \to B \quad \neg B}{\neg A} \text{ MT}$$

$$\frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to i$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption

Longer Example

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{A \to B \quad \neg B}{\neg A} \text{MT} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to i$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption

Longer Example

$$\frac{A \quad A \to B}{B}\ {\to}\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B}\ {\to}\mathrm{i}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3

Longer Example

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to\mathrm{i}$$

$$\frac{A \to B \quad \neg B}{\neg A} \ \mathrm{MT}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4

Longer Example

$$\frac{A \quad A \to B}{B}\ \to\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B}\ \to\mathrm{i}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg\mathrm{e}$ 5

Longer Example

$$\frac{A \quad A \to B}{B}\ {\to}\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B}\ {\to}\mathrm{i}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg\mathrm{e}$ 5
7	r	${\to}\mathrm{e}$ 1,6

Longer Example

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{\boxed{A \; \vdots \; B}}{A \to B} \to i$$

$$\frac{A \to B \quad \neg B}{\neg A} \text{MT}$$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$e 1,6

Longer Example

$$\frac{A \quad A \to B}{B}\ {\to}\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{A \ \vdots\ B}}{A \to B}\ {\to}\mathrm{i}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$e 1,6
8	$p \to r$	$\to$i 3–7

Longer Example

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix}A\\ \vdots\\ B\end{matrix}}}{A \to B} \to\mathrm{i}$$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$e 1,6
8	$p \to r$	$\to$i 3–7

Longer Example

$$\frac{A \quad A \to B}{B}\ {\to}\mathrm{e} \qquad \frac{\boxed{\begin{array}{c}A\\ \vdots\\ B\end{array}}}{A \to B}\ {\to}\mathrm{i}$$

$$\frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$e 1,6
8	$p \to r$	$\to$i 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	$\to$i 2–8

Longer Example

$$\frac{A \quad A \rightarrow B}{B} \rightarrow\mathrm{e} \qquad \frac{A \rightarrow B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{A \ \vdots\ B}}{A \rightarrow B} \rightarrow\mathrm{i}$$

Prove $\vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r))$

1	$q \rightarrow r$	assumption
2	$\neg q \rightarrow \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\rightarrow$e 1,6
8	$p \rightarrow r$	$\rightarrow$i 3–7
9	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow$i 2–8

Longer Example

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A} \text{MT} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to\mathrm{i}$$

Prove $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$e 1,6
8	$p \to r$	$\to$i 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	$\to$i 2–8
10	$(q \to r) \to ((\neg q \to \neg p) \to (p \to q))$	$\to$i 1–9

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B} \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B} \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C} \vee\mathrm{e}$$

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee i_1 \qquad \frac{B}{A \vee B}\ \vee i_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee e$$

Example 1 Prove $p \vee q \vdash q \vee p$

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B} \vee i_1 \qquad \frac{B}{A \vee B} \vee i_2 \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C} \vee e$$

Example 1 Prove $p \vee q \vdash q \vee p$

1 $p \vee q$ premise

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee i_1 \qquad \frac{B}{A \vee B}\ \vee i_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee e$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B} \vee i_1 \qquad \frac{B}{A \vee B} \vee i_2 \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C} \vee e$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee i_2$ 2

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee i_1 \qquad \frac{B}{A \vee B}\ \vee i_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee e$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee i_2$ 2
4	q	assumption

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2
4	q	assumption
5	$q \vee p$	$\vee\mathrm{i}_1$ 2

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee i_1 \qquad\qquad \frac{B}{A \vee B}\ \vee i_2 \qquad\qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee e$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee i_2$ 2
4	q	assumption
5	$q \vee p$	$\vee i_1$ 2

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2
4	q	assumption
5	$q \vee p$	$\vee\mathrm{i}_1$ 2
6	$q \vee p$	$\vee\mathrm{e}$ 1, 2–3, 4–5

Lines 2–3 and lines 4–5 are each enclosed in a box.

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_2 \qquad \frac{A \vee B \qquad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \qquad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee\mathrm{e}$$

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \rightarrow r,\ q \rightarrow r \vdash r$

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B} \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B} \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C} \vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \to r,\ q \to r \vdash r$

1	$p \vee q$	premise
2	$p \to r$	premise
3	$q \to r$	premise

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \rightarrow r,\ q \rightarrow r \vdash r$

1	$p \vee q$			premise
2	$p \rightarrow r$			premise
3	$q \rightarrow r$			premise
4	p	assumption	q	assumption
5	r	$\rightarrow$e 4, 2	r	$\rightarrow$e 4, 3

$\vee$ introduction and elimination rules

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_2 \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \to r,\ q \to r \vdash r$

1	$p \vee q$			premise
2	$p \to r$			premise
3	$q \to r$			premise
4	p	assumption	q	assumption
5	r	$\to$e 4, 2	r	$\to$e 4, 3
6	r			$\vee$e 1, 4–5

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

I will not need a ride; otherwise, I will tell you $\vdash$ If I need a ride I will tell you

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot e \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg e$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$	premise

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot e \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg e$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$	premise
2	$\neg p$	assumption

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot e \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg e$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$	premise
2	$\neg p$	assumption
3	p	assumption

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$		premise
2	$\neg p$	assumption	
3	p	assumption	
4	$\bot$	$\neg$e 3,2	

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\text{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\text{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\bot$	$\neg$e 3,2
5	q	$\bot$e 4

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\text{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\text{e}$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\bot$	$\neg$e 3,2
5	q	$\bot$e 4

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot e \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg e$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\bot$	$\neg$e 3,2
5	q	$\bot$e 4
6	$p \to q$	$\to$i 3–5

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\text{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\text{e}$$

Example Prove $\neg p \vee q \vdash p \rightarrow q$

1	$\neg p \vee q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption		
4	$\bot$	$\neg$e 3,2		
5	q	$\bot$e 4		
6	$p \rightarrow q$	$\rightarrow$i 3–5		

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\bot$	$\neg$e 3,2		
5	q	$\bot$e 4		
6	$p \to q$	$\to$i 3–5		

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\bot$	$\neg$e 3,2	q	copy 2
5	q	$\bot$e 4		
6	$p \to q$	$\to$i 3–5		

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\bot$	$\neg$e 3,2	q	copy 2
5	q	$\bot$e 4		
6	$p \to q$	$\to$i 3–5		

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\bot$	$\neg$e 3,2	q	copy 2
5	q	$\bot$e 4	$p \to q$	$\to$i 3–4
6	$p \to q$	$\to$i 3–5		

$\bot$ elimination and $\neg$ elimination rules

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

$$\frac{\begin{array}{|c|}A\\ \vdots\\ B\end{array}}{A \to B}\ \to\mathrm{i}$$

$$\frac{A \vee B \quad \begin{array}{|c|}A\\ \vdots\\ C\end{array} \quad \begin{array}{|c|}B\\ \vdots\\ C\end{array}}{C}\ \vee\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1 $\neg p \vee q$ premise

2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\bot$	$\neg$e 3,2	q	copy 2
5	q	$\bot$e 4	$p \to q$	$\to$i 3–4
6	$p \to q$	$\to$i 3–5		

7 $p \to q$ $\vee$e 1, 2–6

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \rightarrow q,\ p \rightarrow \neg q \vdash \neg p$

¬ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise

$\neg$ introduction and proof by contradiction rules

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c}A\\ \vdots\\ \bot\end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c}\neg A\\ \vdots\\ \bot\end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3
6	$\bot$	$\neg$e 4, 5

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\text{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3
6	$\bot$	$\neg$e 4, 5

¬ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3
6	$\bot$	$\neg$e 4, 5
7	$\neg p$	$\neg$i 2–4

(Lines 3–6 are in a box.)

$\neg$ introduction and proof by contradiction rules

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\text{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \dfrac{}{A \vee \neg A}\ \text{LEM}$$

Example 2 Prove $\neg p \rightarrow \bot \vdash p$

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\text{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 2 Prove $\neg p \rightarrow \bot \vdash p$

1	$\neg p \rightarrow \bot$	premise

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2
4	$\neg\neg p$	$\neg$i 2–3

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2
4	$\neg\neg p$	$\neg$i 2–3
5	p	$\neg\neg$e 4

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2
4	$\neg\neg p$	$\neg$i 2–3
5	p	$\neg\neg$e 4

PBC can be simulated

$\neg$ introduction and proof by contradiction rules

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

¬ introduction and proof by contradiction rules

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \dfrac{}{A \vee \neg A}\ \text{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1 $\neg(p \vee \neg p)$ assumption

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee\mathrm{i}_1$ 2

¬ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee i_1$ 2
4	$\bot$	$\neg e$ 3, 1

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1
5	$\neg p$	$\neg\mathrm{i}$ 2–4

(Lines 2–4 are enclosed in a box.)

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1
5	$\neg p$	$\neg\mathrm{i}$ 2–4
6	$p \vee \neg p$	$\vee\mathrm{i}_2$ 5

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c}A\\ \vdots\\ \bot\end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c}\neg A\\ \vdots\\ \bot\end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1
5	$\neg p$	$\neg\mathrm{i}$ 2–4
6	$p \vee \neg p$	$\vee\mathrm{i}_2$ 5
7	$\bot$	$\neg\mathrm{e}$ 6, 1

$\neg$ introduction and proof by contradiction rules

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee \mathrm{i}_1$ 2
4	$\bot$	$\neg$e 3, 1
5	$\neg p$	$\neg$i 2–4
6	$p \vee \neg p$	$\vee \mathrm{i}_2$ 5
7	$\bot$	$\neg$e 6, 1

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8	$p \vee \neg p$	PBC 7

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LEM can be simulated too

¬ introduction and proof by contradiction rules

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PBC and LEM are *derived* rules

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PBC and LEM are *derived* rules

MT and $\neg\neg\mathrm{i}$ are derived rules too

Soundness of natural deduction

We will prove a crucial property of natural deduction:

Any formula A derived from a set **S** of premises is a logical consequence of **S**

Theorem 1 (Soundness)
For all formulas $A_1, \ldots, A_n$ *and* A *such that* $A_1, \ldots, A_n \vdash A$, *we have that* $A_1, \ldots, A_n \models A$.

For the proof of the theorem, we will rely on this lemma:

Lemma 2
For all formulas $A_1, \ldots, A_n, A$ *and* B,

1. $A_1, \ldots, A_n, A \models B$ *iff* $A_1, \ldots, A_n \models A \rightarrow B$
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Soundness proof

The proof of Theorem 1 is by induction on proof length

The *length* of a natural deduction proof is the number of lines in it

Proof of Theorem 1. (if $A_1, \ldots, A_n \vdash A$ then $A_1, \ldots, A_n \models A$)

Let Π be the a proof of $A_1, \ldots, A_n \vdash A$, seen as a sequence of formulas.

Assume, without loss of generality, that A is the last formula in the sequence.

By induction on the length l of Π.

(Base case: $l = n$)

Then $A = A_i$ for some $i \in \{1, \ldots, n\}$. Trivially, $A_1, \ldots, A_n \models A_i$.

(continued)

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(continued)

Soundness proof (continued)

(Inductive step: $l > n$)

Assume by induction that the theorem holds for all proofs of length $l' < l$.

The proof depends on the final rule used to derive A.

($\wedge e_1$) If A was derived by $\wedge e_1$, then Π looks like:

$$\begin{array}{ll} A_1 & \text{premise} \\ \vdots & \\ A \wedge B & \ldots \\ \vdots & \\ A & \wedge e_1 \end{array}$$

for some formula B.

Note that the subsequence of Π from A_1 to $A \wedge B$ is a proof of $A \wedge B$ of length $< l$.

Then, by inductive hypothesis, $A_1, \ldots, A_n \models A \wedge B$. Hence, $A_1, \ldots, A_n \models A$.

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($\wedge$i)

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A_1	premise		A_1	premise
$\vdots$			$\vdots$	
B_1	...		B_2	...
$\vdots$		or	$\vdots$	
B_2	...		B_1	...
$\vdots$			$\vdots$	
$B_1 \wedge B_2$	$\wedge$i		$B_1 \wedge B_2$	$\wedge$i

Soundness proof (continued)

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$$\begin{array}{ll} A_1 & \text{premise} \\ \vdots & \\ B_1 & \ldots \\ \vdots & \\ B_2 & \ldots \\ \vdots & \\ B_1 \wedge B_2 & \wedge\mathrm{i} \end{array} \quad \text{or} \quad \begin{array}{ll} A_1 & \text{premise} \\ \vdots & \\ B_2 & \ldots \\ \vdots & \\ B_1 & \ldots \\ \vdots & \\ B_1 \wedge B_2 & \wedge\mathrm{i} \end{array}$$

This implies that Π contains a (shorter) proof of B_1 and of B_2.

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$$
\begin{array}{ll}
A_1 & \text{premise} \\
\vdots & \\
B_1 & \ldots \\
\vdots & \\
B_2 & \ldots \\
\vdots & \\
B_1 \wedge B_2 & \wedge\mathrm{i}
\end{array}
\quad \text{or} \quad
\begin{array}{ll}
A_1 & \text{premise} \\
\vdots & \\
B_2 & \ldots \\
\vdots & \\
B_1 & \ldots \\
\vdots & \\
B_1 \wedge B_2 & \wedge\mathrm{i}
\end{array}
$$

This implies that Π contains a (shorter) proof of B_1 and of B_2.

Then, by inductive hypothesis, $A_1, \ldots, A_n \models B_i$ for $i = 1, 2$.

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This implies that Π contains a (shorter) proof of B_1 and of B_2.

Then, by inductive hypothesis, $A_1, \ldots, A_n \models B_i$ for $i = 1, 2$.

Hence, $A_1, \ldots, A_n \models B_1 \wedge B_2$.

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1	A_1	premise
2	$\vdots$	
3	B_1	assumption
4	$\vdots$	
5	B_2	...
6	$B_1 \to B_2$	$\to$i

Soundness proof (continued)

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3	B_1	assumption
4	$\vdots$	
5	B_2	$\ldots$
6	$B_1 \to B_2$	$\to$i

(Lines 3–5 are enclosed in a box.)

but then

Soundness proof (continued)

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2	$\vdots$	
3	B_1	assumption
4	$\vdots$	
5	B_2	...
6	$B_1 \to B_2$	→i

but then

1	A_1	premise
2	$\vdots$	
3	B_1	premise
4	$\vdots$	
5	B_2	...

is a proof of B_2 from $A_1, \ldots, A_n, B_1$ that is shorter than Π.

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Then, by inductive hypothesis, $A_1, \ldots, A_n, B_1 \models B_2$.

Soundness proof (continued)

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It follows from Lemma 2(1) that $A_1, \ldots, A_n \models B_1 \rightarrow B_2$.

Soundness proof (continued)

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but then

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2	$\vdots$	
3	B	premise
4	$\vdots$	
5	$\bot$	...

is a proof of $\bot$ from $A_1, \dots, A_n, B$ that is shorter than Π.

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Then, by inductive hypothesis, $A_1, \dots, A_n, B \models \bot$.

It follows from Lemma 2 that $A_1, \dots, A_n \models \neg B$.

Soundness proof (continued)

($\wedge i_2$) Analogous to $\wedge i_2$ case.

($\vee i_1$) Exercise.

($\vee i_1$) Exercise.

($\vee e$) Exercise.

($\rightarrow e$) Exercise.

($\neg e$) Exercise.

($\bot e$) Exercise.

($\neg\neg e$) Exercise.

□

Completeness of natural deduction

We will now prove another important property of natural deduction:

Any logical consequence A of a set **S** of formulas has a proof with premises **S**

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Lemma 4

For all formulas $A_1, \ldots, A_n$ *and* A *the following holds:*

1. $A_1, A_2, \ldots, A_n \models A$ *implies* $\models A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$
2. $\vdash A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$ *implies* $A_1, A_2, \ldots, A_n \vdash A$

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Proof.

By induction on n in both cases (see Huth & Ryan). □

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All valid formulas B *are provable in natural deduction: if* $\models B$ *then* $\vdash B$.

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Theorem 5 (Completeness for validity)

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Proof of Theorem 3 ($A_1, \ldots, A_n \models A$ implies $A_1, \ldots, A_n \vdash A$).

Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

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Theorem 5 (Completeness for validity)

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Proof of Theorem 3 ($A_1, \ldots, A_n \models A$ implies $A_1, \ldots, A_n \vdash A$).

Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

By Lemma 4(1), $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

Completeness of natural deduction

Lemma 4

For all formulas $A_1, \ldots, A_n$ *and* A *the following holds:*

1. $A_1, A_2, \ldots, A_n \models A$ *implies* $\models A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$
2. $\vdash A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$ *implies* $A_1, A_2, \ldots, A_n \vdash A$

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Theorem 5 (Completeness for validity)

All valid formulas B *are provable in natural deduction: if* $\models B$ *then* $\vdash B$.

Proof of Theorem 3 ($A_1, \ldots, A_n \models A$ implies $A_1, \ldots, A_n \vdash A$).

Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

By Lemma 4(1), $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

By Theorem 5, $\vdash A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

By Lemma 4(2), $A_1, A_2, \ldots, A_n \vdash A$.

Completeness of natural deduction

Lemma 4

For all formulas $A_1, \dots, A_n$ *and* A *the following holds:*

1. $A_1, A_2, \dots, A_n \models A$ *implies* $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$
2. $\vdash A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$ *implies* $A_1, A_2, \dots, A_n \vdash A$

Theorem 5 (Completeness for validity)

All valid formulas B *are provable in natural deduction: if* $\models B$ *then* $\vdash B$.

So we are left with proving Theorem 5

Towards a proof of Theorem 5

Lemma 6

Let A be a formula over variables $p_1, \ldots, p_n$ with $n \geq 0$ and let $\mathcal{I}$ be an interpretation. Let $\hat{p}_i = p$ if $\mathcal{I} \models p$ and $\hat{p}_i = \neg p$ otherwise. Then,

$$\hat{p}_1, \ldots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \quad \text{and} \quad \hat{p}_1, \ldots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A\,.$$

Proof of Lemma 6. By structural induction on A.

(Base case)
If A is just a variable, say p_1, then it is immediate that $p_1 \vdash p_1$ and $\neg p_1 \vdash \neg p_1$.

If A is $\bot$ then $n = 0$ and $\mathcal{I} \not\models A$. We can prove $\neg\bot$ from no premises by $\neg i$.

(Inductive Step) If A is not a variable or $\bot$, assume the result holds for all proper subformulas of A.
We reason by cases on the form of A.

(cont.)

Towards a proof of Theorem 5

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Towards a proof of Theorem 5

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Let A *be a formula over variables* $p_1, \ldots, p_n$ *with* $n \geq 0$ *and let* $\mathcal{I}$ *be an interpretation. Let* $\hat{p}_i = p$ *if* $\mathcal{I} \models p$ *and* $\hat{p}_i = \neg p$ *otherwise. Then,*

$$\hat{p}_1, \ldots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \quad \text{and} \quad \hat{p}_1, \ldots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A\,.$$

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Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = \neg B$) (that is, suppose A has the form $\neg B$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B$. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg B$.
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B$. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B$.
  Take a proof of B from $\hat{p}_1, \ldots, \hat{p}_n$ and apply $\neg\neg i$ to B.
  The resulting proof is a proof of $\neg A$.

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

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Towards a proof of Theorem 5

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Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \land B_2$)

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \wedge B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \models B_2$.

  By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B_1$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash B_2$.

  A proof of A from $\hat{p}_1, \ldots, \hat{p}_n$ is obtained by chaining a proof of B_1 and a proof of B_2 and applying $\wedge\mathrm{i}$ to B_1 and B_2.

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \wedge B_2$)

- If $\mathcal{I} \not\models A$ then $\mathcal{I} \not\models B_k$ for some $k \in \{1, 2\}$. Say $k = 1$ (the other case is similar).
  By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B_1$.
  A proof of $\neg B_1$ can be extended to a proof of $\neg A$ as follows:

1	$\vdots$	
2	$\neg B_1$	
3	$B_1 \wedge B_2$	assumption
4	B_1	$\wedge e_1$ 3
5	$\bot$	$\bot i$ 4, 2
6	$\neg(B_1 \wedge B_2)$	$\bot i$ 3, 5

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \vee B_2$)

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$)
(continued)

($A = B_1 \vee B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \models B_k$ for some $k \in \{1, 2\}$.

  A proof of A from $\hat{p}_1, \ldots, \hat{p}_n$ is obtained from a proof of B_k by applying $\vee\mathrm{i}_k$ to B_k to get $B_1 \vee B_2$.

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$)
(continued)

($A = B_1 \vee B_2$)

- If $\mathcal{I} \not\models A$ then $\mathcal{I} \not\models B_1$ and $\mathcal{I} \not\models B_2$.
  A proof of $\neg A$ from $\hat{p}_1, \ldots, \hat{p}_n$ is obtained by chaining a proof of $\neg B_1$ and a proof of $\neg B_2$ and continuing as follows:

1	$\vdots$			
2	$B_1 \vee B_2$			assumption
3	B_1	assumption	B_2	assumption
4	$\bot$	$\bot$i (with $\neg B_1$)	$\bot$	$\bot$i (with $\neg B_2$)
5	$\bot$			$\vee$e 2, 3 − −4
6	$\neg(B_1 \vee B_2)$			$\bot$i 2 − −5

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \rightarrow B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B_1$ or $\mathcal{I} \models B_2$.
  (exercise)
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \not\models B_2$.
  (exercise)

□

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \rightarrow B_2$)

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$\square$

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$)
(continued)

($A = B_1 \rightarrow B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B_1$ or $\mathcal{I} \models B_2$.
  (exercise)
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \not\models B_2$.
  (exercise)

□

Towards a proof of Theorem 5

Lemma 7

Let $L_2, \ldots, L_n, A$ *be formulas and let* p *be one of* $A's$ *variables.*
If $p, L_2, \ldots, L_n \vdash A$ *and* $\neg p, L_2, \ldots, L_n \vdash A$ *then* $L_2, \ldots, L_n \vdash A$.

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise	and	1	$\neg p$	premise
2	L_2	premise		2	L_2	premise
3	$\vdots$			3	$\vdots$	
4	A	$\ldots$		4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee$e

□

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

and

1	$\neg p$	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee$e

□

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise	and	1	$\neg p$	premise
2	L_2	premise		2	L_2	premise
3	$\vdots$			3	$\vdots$	
4	A	$\ldots$		4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee e$

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

and

1	$\neg p$	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee$e

□

Proof of Theorem 5 ($\models A$ implies $\vdash A$).

Let $p_1, \ldots, p_n$ be all of A's variables and consider the set

$$\mathbf{S} = \{ p_1, \neg p_1 \} \times \cdots \times \{ p_n, \neg p_n \} ,$$

of all tuples $(\hat{p}_1, \ldots, \hat{p}_n)$ where each $\hat{p}_i$ is either p_i or $\neg p_i$.
We prove by induction on $i = 1, \ldots, n+1$ that

$$\hat{p}_i, \ldots, \hat{p}_n \vdash A \quad \text{for every } (\hat{p}_1, \ldots, \hat{p}_n) \in \mathbf{S} . \tag{1}$$

The theorem then follows from Property (1) for $i = n + 1$.

($i = 1$) Property (1) holds by Lemma 6 since every $(\hat{p}_1, \ldots, \hat{p}_n) \in \mathbf{S}$ corresponds to an interpretation of A and all interpretations satisfy A (by def. of validity).

($i > 1$) Suppose $\hat{p}_i, \ldots, \hat{p}_n \vdash A$ for all $(\hat{p}_1, \ldots, \hat{p}_n) \in \mathbf{S}$.
We prove that $\hat{p}_{i+1}, \ldots, \hat{p}_n \vdash A$ for all $(\hat{p}_1, \ldots, \hat{p}_n) \in \mathbf{S}$.
Let $(\hat{p}_1, \ldots, p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n), (\hat{p}_1, \ldots, \neg p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n) \in \mathbf{S}$.
By induction hypothesis, $p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n \vdash A$ and $\neg p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n \vdash A$
Then $\hat{p}_{i+1}, \ldots, \hat{p}_n \vdash A$ by Lemma 7. □

Proof of Theorem 5 ($\models A$ implies $\vdash A$).

Let $p_1, \dots, p_n$ be all of A's variables and consider the set

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Let $(\hat{p}_1, \dots, p_i, \hat{p}_{i+1}, \dots, \hat{p}_n), (\hat{p}_1, \dots, \neg p_i, \hat{p}_{i+1}, \dots, \hat{p}_n) \in \mathbf{S}$.
By induction hypothesis, $p_i, \hat{p}_{i+1}, \dots, \hat{p}_n \vdash A$ and $\neg p_i, \hat{p}_{i+1}, \dots, \hat{p}_n \vdash A$
Then $\hat{p}_{i+1}, \dots, \hat{p}_n \vdash A$ by Lemma 7. □

Proof of Theorem 5 ($\models A$ implies $\vdash A$).

Let $p_1, \ldots, p_n$ be all of A's variables and consider the set

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