# CS:4350 Logic in Computer Science

## Derivation Systems for Propositional Logic

Cesare Tinelli

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The University of Iowa

# Outline

Derivation Systems for Propositional Logic

- Semantic consequence/entailment
- Derivability

# Logics, formally

A logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{L}$, the language, is
  a class of sentences described by a formal grammar
- $\mathcal{S}$, the semantics, is
  a formal specification for assigning meaning to sentences in $\mathcal{L}$
- $\mathcal{R}$, the derivation (or inference) system, is
  a set of axioms and derivation rules to *derive* (i.e., generate) sentences of $\mathcal{L}$ from given sentences of $\mathcal{L}$

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Propositional logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{L}$ is the set of all formulas built from Boolean variables and the propositional connectives $(\neg, \wedge, \vee, \ldots)$
- $\mathcal{S}$ is provided by interpretations of the variables as $0, 1$ and the connectives as certain Boolean functions
- $\mathcal{R}$ is ??

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We will study a few of them

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# Formal properties of derivation systems

A derivation system is defined by a set of derivation rules that allow one to derive formulas from given formulas

We will focus on these properties of our derivation systems:

**Soundness** Every derived formula is a semantic consequence of the given ones

**Completeness** Only semantic consequences are derivable

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# Semantic consequence (or entailment)

Given

- a set $\mathbf{S} = \{A_1, \ldots, A_n\}$ of formulas and
- a formula $B$

we write

$$\{A_1, \ldots, A_n\} \models B$$

iff every interpretation that satisfies every formula in $\mathbf{S}$
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$\mathbf{S} \models B$ is read as $B$ *is a semantic/logical consequence of* $\mathbf{S}$*, or*
$B$ *logically follows from* $\mathbf{S}$*, or*
$\mathbf{S}$ *entails* $B$

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> $\mathbf{S} \models A$ formally captures the notion of a fact $A$ following from assumptions $\mathbf{S}$

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**Note 2:** Do not confuse this use of $\models$ with that in $\mathcal{I} \models B$ where $\mathcal{I}$ is an interpretation

# Entailment, Examples

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\begin{array}{lll}
\{p\} & \models & p \vee q \\
\{p,\, p \rightarrow q\} & \models & q \\
\{p,\, q\} & \models & p \wedge q \\
\{\} & \models & r \rightarrow r \\
\{p,\, \neg r\} & \models & (p \vee q) \wedge (q \vee \neg r) \\
\{q\} & \models & (p \vee q) \wedge (q \vee \neg r) \\
\{p,\, q \vee \neg r\} & \not\models & p \wedge q \\
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| | $p$ | $q$ | $r$ | $\neg r$ | $p \rightarrow q$ | $p \vee q$ | $p \wedge q$ | $r \rightarrow r$ | $q \vee \neg r$ | $(p \vee q) \wedge (q \vee \neg r)$ |
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# Exercise

Determine which of the following entailments hold

$$
\begin{array}{lcl}
p \wedge q,\ r & \overset{?}{\models} & q \wedge r \\
p,\ \neg\neg(q \wedge r) & \overset{?}{\models} & \neg\neg p \wedge r \\
p,\ p \rightarrow q,\ q \rightarrow r & \overset{?}{\models} & r \\
p \vee q,\ p \rightarrow q,\ q \rightarrow r & \overset{?}{\models} & r \\
p \vee q,\ p \rightarrow r,\ q \rightarrow r & \overset{?}{\models} & r \\
p \rightarrow q & \overset{?}{\models} & \neg q \rightarrow \neg p \\
p \rightarrow q & \overset{?}{\models} & \neg p \rightarrow \neg q \\
p \vee (q \wedge r) & \overset{?}{\models} & (p \vee q) \wedge (p \vee r) \\
 & \overset{?}{\models} & p \rightarrow (q \rightarrow p) \\
p \rightarrow q,\ p \rightarrow \neg q & \overset{?}{\models} & \neg p
\end{array}
$$

# Properties of entailment

- $\mathbf{S} \models A$ for all $A \in \mathbf{S}$ (*inclusion*)
- if $\mathbf{S} \models A$ then $\mathbf{T} \models A$ for all $\mathbf{T} \supseteq \mathbf{S}$ (*monotonicity*)
- $A$ is valid iff $\emptyset \models A$ (also written as $\models A$)
- $A$ is unsatisfiable iff $A \models \bot$
- $\mathbf{S} \models A$ iff $\mathbf{S} \cup \{\neg A\}$ is unsatisfiable
- $\{A_1, \ldots, A_n\} \models B$ iff $\{A_1, \ldots, A_{n-1}\} \models A_n \to B$ (*deduction*)
- $\{A_1, \ldots, A_n\} \models B$ iff $\{A_1 \wedge \cdots \wedge A_n\} \models B$ iff $\emptyset \models (A_1 \wedge \cdots \wedge A_n) \to B$
- $A \equiv B$ iff $\{A\} \models B$ and $\{B\} \models A$

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- $A \equiv B$ iff $\{A\} \models B$ and $\{B\} \models A$

# Properties of entailment

- $\mathbf{S} \models A$ for all $A \in \mathbf{S}$ (*inclusion*)
- if $\mathbf{S} \models A$ then $\mathbf{T} \models A$ for all $\mathbf{T} \supseteq \mathbf{S}$ (*monotonicity*)
- $A$ is valid iff $\emptyset \models A$ (also written as $\models A$)
- $A$ is unsatisfiable iff $A \models \bot$
- $\mathbf{S} \models A$ iff $\mathbf{S} \cup \{\neg A\}$ is unsatisfiable
- $\{A_1, \ldots, A_n\} \models B$ iff $\{A_1, \ldots, A_{n-1}\} \models A_n \rightarrow B$ (*deduction*)
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# Derivation systems for propositional logic

An *derivation system* $I$ is a collection of formal rules for inferring formulas from formulas

Given

- a set $\mathbf{S} = \{A_1, \ldots, A_n\}$ of formulas (*premises*) and
- a formula $B$ (*conclusion*)

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$$\{A_1, \ldots, A_n\} \vdash_I B$$

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$\mathbf{S} \vdash_I A$ is read as $\mathbf{S}$ *derives* $B$ *in* $I$, or
$B$ *derives from* $\mathbf{S}$ *in* $I$, or
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We write just $\mathbf{S} \vdash A$ when $I$ is clear from context

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Ideally, $I$ should also be such that $\mathbf{S} \vdash_I A$ if $\mathbf{S} \models A$

## So many symbols!

**Note:**

$A \wedge B \to C$ is a formula, a sequence of symbols manipulated by an derivation system $I$

$A \wedge B \models C$ is a mathematical abbreviation for the statement: *"every interpretation that satisfies $A \wedge B$, also satisfies $C$"*

$A \wedge B \vdash_I C$ is a mathematical abbreviation for the statement: *"$I$ derives $C$ from $A \wedge B$"*

# So many symbols!

In other words,

- $\rightarrow$ is a symbol of propositional logic, processed by derivation systems
- $\models$ denotes a relation from sets of formulas to formulas, based on their meaning in propositional logic
- $\vdash_I$ denotes a relation from sets of formulas to formulas, based on their derivability in $I$

# Implication vs. Entailment

The connective $\rightarrow$ and the relation $\models$ are related as follows:

$$A \rightarrow B \text{ is valid iff } A \models B$$

**Example:** $p \rightarrow (p \vee q)$ is valid and $p \models p \vee q$

| | $p$ | $q$ | $p \vee q$ | $p \rightarrow (p \vee q)$ |
|---|---|---|---|---|
| 1. | 0 | 0 | 0 | 1 |
| 2. | 0 | 1 | 1 | 1 |
| 3. | 1 | 0 | 1 | 1 |
| 4. | 1 | 1 | 1 | 1 |

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# Soundness and completeness

The relations $\models$ and $\vdash_I$ are related as by these two properties of derivation systems $I$

**Soundness** $I$ is *sound* if it can derive from a given set $\mathbf{S}$ of formulas only formulas entailed by $\mathbf{S}$:

$$\text{if } \mathbf{S} \vdash_I A \text{ then } \mathbf{S} \models A$$

**Completeness** $I$ is *complete* if it can derive from a given set $\mathbf{S}$ of formulas all formulas entailed by $\mathbf{S}$:

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