CS:4350 Logic in Computer Science

Introduction

Cesare Tinelli

Spring 2022

The University of Iowa

Credits

These slides are largely based on slides originally developed by **Andrei Voronkov** at the University of Manchester. Adapted by permission.

What is mathematical logic?

A branch of science that

- formalizes valid methods of reasoning
- was originally developed to formalize mathematics
- provides the mathematical foundations of CS
- drives several applications in CS and beyond

What is mathematical logic?

A branch of science that

- formalizes valid methods of reasoning
- was originally developed to formalize mathematics
- provides the mathematical foundations of CS
- drives several applications in CS and beyond

What is mathematical logic?

A branch of science that

- formalizes valid methods of reasoning
- was originally developed to formalize mathematics
- provides the mathematical foundations of CS
- drives several applications in CS and beyond

What is mathematical logic?

A branch of science that

- formalizes valid methods of reasoning
- was originally developed to formalize mathematics
- provides the mathematical foundations of CS
- drives several applications in CS and beyond

What is mathematical logic?

A branch of science that

- formalizes valid methods of reasoning
- was originally developed to formalize mathematics
- provides the mathematical foundations of CS
- drives several applications in CS and beyond

A bit of history

Logic as a discipline in Western thought dates back to the ancient Greeks

Aristotle and Stoic philosophers formulated systems of reasoning in the 4th century BCE

Independent studies were done in China (MoZi or Micius, 4th c. BCE) and India (Dignaga, 6th c. CE)

Christian (Boethius 6th c.; Ockham, 14th c.) and Islamic (Ibn Sina or Avicenna, 10th c.) philosophers advanced Aristotle’s idea in the middle ages

A major leap occurred in the 19th century in Europe with the goal of formalizing mathematics (Boole, Frege, Peano, …)

In the 20th century, European (Göedel, Turing, …) and American (Tarski, Church, Scott, …) logicians effectively laid down the foundations of CS, largely before computers were invented!

Presently, *mathematical logic has become to CS what calculus is to physics*

A bit of history

Logic as a discipline in Western thought dates back to the ancient Greeks

Aristotle and Stoic philosophers formulated systems of reasoning in the 4th century BCE

Independent studies were done in China (MoZi or Micius, 4th c. BCE) and India (Dignaga, 6th c. CE)

Christian (Boethius 6th c.; Ockham, 14th c.) and Islamic (Ibn Sina or Avicenna, 10th c.) philosophers advanced Aristotle's idea in the middle ages

A major leap occurred in the 19th century in Europe with the goal of formalizing mathematics (Boole, Frege, Peano, …)

In the 20th century, European (Göedel, Turing, …) and American (Tarski, Church, Scott, …) logicians effectively laid down the foundations of CS, largely before computers were invented!

Presently, *mathematical logic has become to CS what calculus is to physics*

A bit of history

Logic as a discipline in Western thought dates back to the ancient Greeks

Aristotle and Stoic philosophers formulated systems of reasoning in the 4th century BCE

Independent studies were done in China (MoZi or Micius, 4th c. BCE) and India (Dignaga, 6th c. CE)

Christian (Boethius 6th c.; Ockham, 14th c.) and Islamic (Ibn Sina or Avicenna, 10th c.) philosophers advanced Aristotle's idea in the middle ages

A major leap occurred in the 19th century in Europe with the goal of formalizing mathematics (Boole, Frege, Peano, …)

In the 20th century, European (Göedel, Turing, …) and American (Tarski, Church, Scott, …) logicians effectively laid down the foundations of CS, largely before computers were invented!

Presently, *mathematical logic has become to CS what calculus is to physics*

A bit of history

Logic as a discipline in Western thought dates back to the ancient Greeks

Aristotle and Stoic philosophers formulated systems of reasoning in the 4th century BCE

Independent studies were done in China (MoZi or Micius, 4th c. BCE) and India (Dignaga, 6th c. CE)

Christian (Boethius 6th c.; Ockham, 14th c.) and Islamic (Ibn Sina or Avicenna, 10th c.) philosophers advanced Aristotle's idea in the middle ages

A major leap occurred in the 19th century in Europe with the goal of formalizing mathematics (Boole, Frege, Peano, …)

In the 20th century, European (Göedel, Turing, …) and American (Tarski, Church, Scott, …) logicians effectively laid down the foundations of CS, largely before computers were invented!

Presently, *mathematical logic has become to CS what calculus is to physics*

A bit of history

Logic as a discipline in Western thought dates back to the ancient Greeks

Aristotle and Stoic philosophers formulated systems of reasoning in the 4th century BCE

Independent studies were done in China (MoZi or Micius, 4th c. BCE) and India (Dignaga, 6th c. CE)

Christian (Boethius 6th c.; Ockham, 14th c.) and Islamic (Ibn Sina or Avicenna, 10th c.) philosophers advanced Aristotle's idea in the middle ages

A major leap occurred in the 19th century in Europe with the goal of formalizing mathematics (Boole, Frege, Peano, …)

In the 20th century, European (Göedel, Turing, …) and American (Tarski, Church, Scott, …) logicians effectively laid down the foundations of CS, largely before computers were invented!

Presently, *mathematical logic has become to CS what calculus is to physics*

A bit of history

Logic as a discipline in Western thought dates back to the ancient Greeks

Aristotle and Stoic philosophers formulated systems of reasoning in the 4th century BCE

Independent studies were done in China (MoZi or Micius, 4th c. BCE) and India (Dignaga, 6th c. CE)

Christian (Boethius 6th c.; Ockham, 14th c.) and Islamic (Ibn Sina or Avicenna, 10th c.) philosophers advanced Aristotle's idea in the middle ages

A major leap occurred in the 19th century in Europe with the goal of formalizing mathematics (Boole, Frege, Peano, ...)

In the 20th century, European (Göedel, Turing, ...) and American (Tarski, Church, Scott, ...) logicians effectively laid down the foundations of CS, largely before computers were invented!

Presently, *mathematical logic has become to CS what calculus is to physics*

A bit of history

Logic as a discipline in Western thought dates back to the ancient Greeks

Aristotle and Stoic philosophers formulated systems of reasoning in the 4th century BCE

Independent studies were done in China (MoZi or Micius, 4th c. BCE) and India (Dignaga, 6th c. CE)

Christian (Boethius 6th c.; Ockham, 14th c.) and Islamic (Ibn Sina or Avicenna, 10th c.) philosophers advanced Aristotle's idea in the middle ages

A major leap occurred in the 19th century in Europe with the goal of formalizing mathematics (Boole, Frege, Peano, …)

In the 20th century, European (Göedel, Turing, …) and American (Tarski, Church, Scott, …) logicians effectively laid down the foundations of CS, largely before computers were invented!

Presently, *mathematical logic has become to CS what calculus is to physics*

Logic as the foundation of Computer Science

Logical methods are used to define

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- ...

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- ...

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- …

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- ...

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- ...

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- ...

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- ...

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- ...

Logic as the foundation of Computer Science

Logical methods are used to define

- the very idea of computation and computability
- the semantics of logical gates and circuits
- the operational semantics of programming languages
- the computational complexity of algorithms
- types systems and type checking
- distributed systems and protocols
- database semantics
- knowledge representation in AI
- …

Logics

A logic is also a formal mathematical construct

Each logic is characterized by its own:

- syntax and semantics
- inference mechanisms
- proof theory and model theory

Many formal logics have been developed

We will study several logics used in CS

Logics

A logic is also a formal mathematical construct

Each logic is characterized by its own:

- syntax and semantics
- inference mechanisms
- proof theory and model theory

Many formal logics have been developed

We will study several logics used in CS

Logics

A logic is also a formal mathematical construct

Each logic is characterized by its own:

- syntax and semantics
- inference mechanisms
- proof theory and model theory

Many formal logics have been developed

We will study several logics used in CS

Logics

A logic is also a formal mathematical construct

Each logic is characterized by its own:

- syntax and semantics
- inference mechanisms
- proof theory and model theory

Many formal logics have been developed

We will study several logics used in CS

Logics

A logic is also a formal mathematical construct

Each logic is characterized by its own:

- syntax and semantics
- inference mechanisms
- proof theory and model theory

Many formal logics have been developed

We will study several logics used in CS

Motivation: computational systems and correctness

Suppose we design a complex system containing various components such as sensors, networks, computers

All of these components now use software

Motivation: computational systems and correctness

Suppose we design a complex system containing various components such as sensors, networks, computers

All of these components now use software

We have requirements on how the system should function, for example safety, reliability, security, availability, absence of deadlocks, etc.

Motivation: computational systems and correctness

Suppose we design a complex system containing various components such as sensors, networks, computers

All of these components now use software

We have requirements on how the system should function, for example safety, reliability, security, availability, absence of deadlocks, etc.

How can one ensure that the system satisfies these requirements?

Motivation: computational systems and correctness

Suppose we design a complex system containing various components such as sensors, networks, computers

All of these components now use software

We have requirements on how the system should function, for example safety, reliability, security, availability, absence of deadlocks, etc.

How can one ensure that the system satisfies these requirements?

Most modern computer systems are unreliable

Small Example: Software

Consider the following fragment of a C program:

```
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length);

  for (i = 0; i <= length; i++)
    array[i] = 0;
  return array;
}
```

Is this program correct?

Small Example: Software

Consider the following fragment of a C program:

```
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length);

  for (i = 0; i <= length; i++)
    array[i] = 0;
  return array;
}
```

Is this program correct? Hardly: it writes into memory that has not been allocated

Small Example: Software

Consider the following fragment of a C program:

```
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length);

  for (i = 0; i < length; i++)
    array[i] = 0;
  return array;
}
```

Is this program correct?

Small Example: Software

Consider the following fragment of a C program:

```
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length); // may return 0!

  for (i = 0; i < length; i++)
    array[i] = 0;
  return array;
}
```

Is this program correct? No: it may write to the null address

Small Example: Software

Consider the following fragment of a C program:

```
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length);
  if (!array) return 0;
  for (i = 0; i < length; i++)
    array[i] = 0;
  return array;
}
```

Is this program correct?

Small Example: Software

Consider the following fragment of a C program:

```
/* Returns a new array of integers of a given
   length initialized by a non-zero value */
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length);
  if (!array) return 0;
  for (i = 0; i < length; i++)
    array[i] = 0;
  return array;
}
```

Is this program correct?

Small Example: Software

Consider the following fragment of a C program:

```
/* Returns a new array of integers of a given
   length initialized by a non-zero value */
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length);
  if (!array) return 0;
  for (i = 0; i < length; i++)
    array[i] = 0;
  return array;
}
```

Is this program correct? No: it initializes the array by zeros

Small Example: Software

Consider the following fragment of a C program:

```
/* Returns a new array of integers of a given
   length initialized by a non-zero value */
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int) * length);
  if (!array) return 0;
  for (i = 0; i < length; i++)
    array[i] = 0;
  return array;
}
```

We discussed correctness of a program without ever defining what it means

Small Example: Software

Consider the following fragment of a C program:

```c
/* Returns a new array of integers of a given
   length initialized by a non-zero value */
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(s
  if (!array) retu
  for (i = 0; i < length; i++)
    array[i] = 0;
  return array;
}
```

So, what is correctness?

We discussed correctness of a program without ever defining what it means

Note

- We could spot the *first two errors* without knowing anything about the intended purpose of the program
- However, we had to understand the meaning of C programs in general and some specific properties of programming in C
- To understand the *last error* we had to know something about the program's intended behavior

Note

- We could spot the *first two errors* without knowing anything about the intended purpose of the program

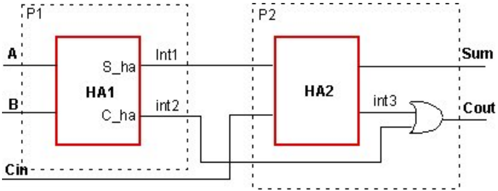
- However, we had to understand the meaning of C programs in general and some specific properties of programming in C

- To understand the *last error* we had to know something about the program's intended behavior

Note

- We could spot the *first two errors* without knowing anything about the intended purpose of the program
- However, we had to understand the meaning of C programs in general and some specific properties of programming in C
- To understand the *last error* we had to know something about the program's intended behavior

Another example: circuit design

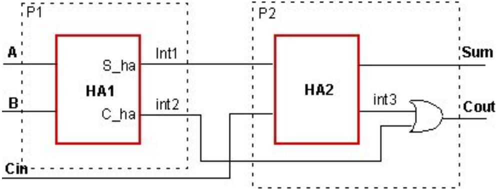


We would like to replace a circuit C_1 in a processor by another circuit C_2 (because, say, C_2 results in a lower energy consumption)

We want to be sure that C_2 is correct, that is, it will behave according to some specification

If we already know that C_1 is correct, it is sufficient to prove that C_2 is functionally equivalent to C_1

Another example: circuit design

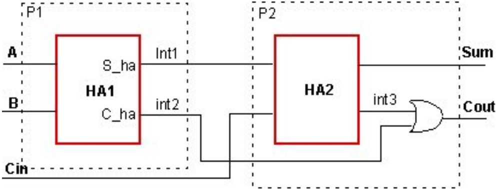


We would like to replace a circuit C_1 in a processor by another circuit C_2 (because, say, C_2 results in a lower energy consumption)

We want to be sure that C_2 is correct, that is, it will behave according to some specification

If we already know that C_1 is correct, it is sufficient to prove that C_2 is functionally equivalent to C_1

Another example: circuit design



We would like to replace a circuit C_1 in a processor by another circuit C_2 (because, say, C_2 results in a lower energy consumption)

We want to be sure that C_2 is correct, that is, it will behave according to some specification

If we already know that C_1 is correct, it is sufficient to prove that C_2 is functionally equivalent to C_1

How to establish correctness of a system

1. Consider the system as a mathematical object by building a formal model of the system

2. Find a formal language $\mathcal{L}$ for expressing intended properties

3. The language must have a formal semantics, defining the possible interpretations of the sentences of $\mathcal{L}$

   - The semantics is normally based on notions of truth and satisfiability

4. Write a specification, that is, intended properties of the system in this language

5. Prove formally that the system model satisfies the specification

How to establish correctness of a system

1. Consider the system as a mathematical object by building a formal model of the system

2. Find a formal language $\mathcal{L}$ for expressing intended properties

3. The language must have a formal semantics, defining the possible interpretations of the sentences of $\mathcal{L}$
   - The semantics is normally based on notions of truth and satisfiability

4. Write a specification, that is, intended properties of the system in this language

5. Prove formally that the system model satisfies the specification

How to establish correctness of a system

1. Consider the system as a mathematical object by building a formal model of the system

2. Find a formal language $\mathcal{L}$ for expressing intended properties

3. The language must have a formal semantics, defining the possible interpretations of the sentences of $\mathcal{L}$
   - The semantics is normally based on notions of truth and satisfiability

4. Write a specification, that is, intended properties of the system in this language

5. Prove formally that the system model satisfies the specification

How to establish correctness of a system

1. Consider the system as a mathematical object by building a formal model of the system

2. Find a formal language $\mathcal{L}$ for expressing intended properties

3. The language must have a formal semantics, defining the possible interpretations of the sentences of $\mathcal{L}$

   - The semantics is normally based on notions of truth and satisfiability

4. Write a specification, that is, intended properties of the system in this language

5. Prove formally that the system model satisfies the specification

How to establish correctness of a system

1. Consider the system as a mathematical object by building a formal model of the system

2. Find a formal language $\mathcal{L}$ for expressing intended properties

3. The language must have a formal semantics, defining the possible interpretations of the sentences of $\mathcal{L}$
   - The semantics is normally based on notions of truth and satisfiability

4. Write a specification, that is, intended properties of the system in this language

5. Prove formally that the system model satisfies the specification

How to establish correctness of a system

1. Consider the system as a mathematical object by building a formal model of the system

2. Find a formal language $\mathcal{L}$ for expressing intended properties

3. The lan… …e possibl…

   - The …

4. Write a specification, that is, intended properties of the system in this language

5. Prove formally that the system model satisfies the specification

All of this can be done with the proper logic

Logics, formally

A logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{L}$, the *language*, is
  a class of sentences described by a formal grammar
- $\mathcal{S}$, the *semantics*, is
  a formal specification for assigning meaning to sentences in $\mathcal{L}$
- $\mathcal{R}$, the *derivation (or, inference) system*, is
  a set of axioms (i.e., valid sentences) and derivation rules to *derive* (i.e., generate) sentences of $\mathcal{L}$ from given sentences of $\mathcal{L}$

Logics, formally

A logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{L}$, the *language*, is
  a class of sentences described by a formal grammar
- $\mathcal{S}$, the *semantics*, is
  a formal specification for assigning meaning to sentences in $\mathcal{L}$
- $\mathcal{R}$, the *derivation (or, inference) system*, is
  a set of axioms (i.e., valid sentences) and derivation rules to *derive* (i.e., generate) sentences of $\mathcal{L}$ from given sentences of $\mathcal{L}$

Logics, formally

A logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{L}$, the *language*, is
  a class of sentences described by a formal grammar
- $\mathcal{S}$, the *semantics*, is
  a formal specification for assigning meaning to sentences in $\mathcal{L}$
- $\mathcal{R}$, the *derivation (or, inference) system*, is
  a set of axioms (i.e., valid sentences) and derivation rules to *derive* (i.e., generate) sentences of $\mathcal{L}$ from given sentences of $\mathcal{L}$

Logics, formally

A logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{L}$, the *language*, is
  a class of sentences described by a formal grammar
- $\mathcal{S}$, the *semantics*, is
  a formal specification for assigning meaning to sentences in $\mathcal{L}$
- $\mathcal{R}$, the *derivation (or, inference) system*, is
  a set of axioms (i.e., valid sentences) and derivation rules to *derive* (i.e., generate) sentences of $\mathcal{L}$ from given sentences of $\mathcal{L}$

Derivations Systems

Important theoretical properties:

Consistency	It is impossible to derive both a sentence and its negation from the axioms
Independence	No axiom is derivable from the others
Soundness	All derivable sentences are valid (always true)
Completeness	All valid sentences are derivable

My God, it's full of logics!

There are many, many logics: propositional, first-order, higher-order, modal, temporal, intuitionistic, linear, non-monotonic, many-valued, …

We will concentrate on a few, starting with propositional logic

My God, it's full of logics!

There are many, many logics: propositional, first-order, higher-order, modal, temporal, intuitionistic, linear, non-monotonic, many-valued, …

We will concentrate on a few, starting with propositional logic