CS:4350 Logic in Computer Science

First-Order Logic

Cesare Tinelli

Spring 2021

THE UNIVERSITY OF IOWA

Credits

Part of these slides are based on Chap. 2 of *Logic in Computer Science* by M. Huth and M. Ryan, Cambridge University Press, 2nd edition, 2004.

Outline

First-order Logic

Propositional logic talks about facts, statements that can be true or false

First-order logic (FOL), like natural language, can talk about

- *Objects*: people, houses, numbers, theories, colors, baseball games, wars, centuries, …
- *Relations*: red, round, bogus, prime, brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, …
- *Functions*: father of, best friend, successor of, one more than, end of, …

Syntax of FOL: Basic elements

Constant symbols	$\mathit{kingJohn},\ 2,\ \mathit{potus},\ 0,1,2,\ \ldots$
Predicate symbols	$\mathit{Brothers}(\_,\_),\ \_ > \_,\ \mathit{Red}(\_),\ \ldots$
Function symbols	$\mathit{sqrt}(\_),\ \mathit{leftLeg}(\_),\ \_ + \_,\ \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge,\ \vee,\ \neg,\ \rightarrow,\ \leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Atomic formulas

Atomic formula = $\mathit{predicate}(\mathit{term}_1, \ldots, \mathit{term}_n)$
or $\mathit{term}_1 = \mathit{term}_2$

Term = $\mathit{function}(\mathit{term}_1, \ldots, \mathit{term}_n)$
or *constant* or *variable*

Example $\mathit{Brothers}(\mathit{kingJohn}, \mathit{richardTheLionheart})$,
$\mathit{length}(\mathit{leftLeg}(\mathit{robinHood})) > \mathit{length}(\mathit{leftLegOf}(\mathit{kingJohn}))$

Complex Formulas

Complex formulas are made from atomic formulas as with QBFs, using connectives and quantifiers with the same precedence rules as with QBFs

$$\neg F, \quad F_1 \wedge F_2, \quad F_1 \vee F_2, \quad F_1 \rightarrow F_2, \quad F_1 \leftrightarrow F_2, \quad \exists x F, \quad \forall x F$$

Example $\forall x \forall y (Siblings(x, y) \rightarrow Siblings(y, x))$

$x > 2 \vee 1 < x$

$1 > 2 \wedge \neg y > 2$

Truth in FOL

Formulas are true with respect to a *domain* (of discourse) and an *interpretation* of the constant, function and predicate symbols

- A domain is a set containing ≥ 1 objects (*domain elements*)
- An interpretation maps

variables	$\mapsto$	objects
constant symbols	$\mapsto$	objects
predicate symbols	$\mapsto$	relations
function symbols	$\mapsto$	functional relations

An atomic formula $P(t_1, \ldots, t_n)$ is true in an interpretation

iff

the objects denoted to by terms $t_1, \ldots, t_n$ are in the relation denoted by P

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Truth example

Consider the interpretation in which

$$\begin{array}{rcl} \textit{potus} & \mapsto & \text{Joe Biden} \\ \textit{fistLady} & \mapsto & \text{Jill Biden} \\ \textit{Married} & \mapsto & \text{the relation consisting of all pairs of married people} \end{array}$$

Under this interpretation,

- $\textit{Married}(\textit{potus}, \textit{firstLady})$ is true
- $\textit{Married}(\textit{potus}, \textit{potus})$ is false

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Semantics of First-Order Logic

Formally:

An *interpretation* $\mathcal{I}$ is a triple $(\mathcal{U}, (\_)^{\mathcal{I}}, \sigma)$ where

- $\mathcal{U}$ is a non-empty set of objects, the *universe or domain*
- σ is a mapping from variables to elements of $\mathcal{U}$, a *valuation* or *environment*
- $c^{\mathcal{I}}$ is an element in $\mathcal{U}$ for every constant symbol c
- $f^{\mathcal{I}}$ is a function from $\mathcal{U}^n$ to $\mathcal{U}$ (a subset of $\mathcal{U}^n \times \mathcal{U}$) for every function symbol f of arity n
- $r^{\mathcal{I}}$ is a relation over $\mathcal{U}^n$ (a subset of $\mathcal{U}^n$) for every predicate symbol r of arity n

Note

- An *interpretation* gives meaning to the *non-logical symbols* in formulas (constant, function, and predicate symbols and variables)
- The meaning of $=$, connectives and quantifiers is *fixed* for all interpretations

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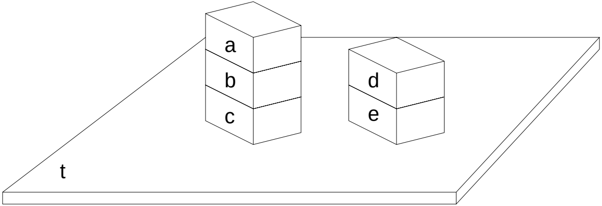
An Interpretation $\mathcal{I}$ in the Blocks World

constant symbols:	A, B, C, D, E, T
function symbols:	$support$
predicate symbols:	$On, Above, Clear$



$A^{\mathcal{I}} = \text{a},\ B^{\mathcal{I}} = \text{b},\ C^{\mathcal{I}} = \text{c},\ D^{\mathcal{I}} = \text{d},\ E^{\mathcal{I}} = \text{e},\ T^{\mathcal{I}} = \text{t}$

$$\begin{aligned}
support^{\mathcal{I}} &= \{(\text{a},\text{b}), (\text{b},\text{c}), (\text{c},\text{t}), (\text{d},\text{e}), (\text{e},\text{t}), (\text{t},\text{t})\} \\
On^{\mathcal{I}} &= \{(\text{a},\text{b}), (\text{b},\text{c}), (\text{c},\text{t}), (\text{d},\text{e}), (\text{e},\text{t})\} \\
Above^{\mathcal{I}} &= \{(\text{a},\text{b}), (\text{a},\text{c}), (\text{a},\text{t}), \ldots\} \\
Clear^{\mathcal{I}} &= \{(\text{a}), (\text{d})\}
\end{aligned}$$

Semantics of FOL Terms

Let $\mathcal{I}$ be an interpretation with universe $\mathcal{U}$ and valuation σ

If e is an FOL expression, we write $[\![e]\!]^{\mathcal{I}}$ to denote the *meaning of e in $\mathcal{I}$*

The meaning $[\![t]\!]^{\mathcal{I}}$ of a term t is an element of $\mathcal{U}$, inductively defined as follows:

$[\![x]\!]^{\mathcal{I}}$	$:=$	$\sigma(x)$	for all variables x
$[\![c]\!]^{\mathcal{I}}$	$:=$	$c^{\mathcal{I}}$	for all constant symbols c
$[\![f(t_1, \ldots, t_n)]\!]^{\mathcal{I}}$	$:=$	$f^{\mathcal{I}}([\![t_1]\!]^{\mathcal{I}}, \ldots, [\![t_n]\!]^{\mathcal{I}})$	for all n-ary function symbols f

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Example

Consider the symbols *mother*, *spouse* and the interpretation $\mathcal{I}$ with valuation σ where

$mother^{\mathcal{I}}$	is a unary function mapping people to their mother
$spouse^{\mathcal{I}}$	is a unary function mapping people to their spouse
σ	is $\{x \mapsto \text{Bart Simpson},\ y \mapsto \text{Homer Simpson}, \ldots\}$

What is the meaning of $spouse(mother(x))$ in $\mathcal{I}$?

$$
\begin{aligned}
[\![spouse(mother(x))]\!]^{\mathcal{I}} &= \\
&= \\
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\end{aligned}$$

Semantics of FOL Formulas

Let $\mathcal{I}$ be an interpretation with universe $\mathcal{U}$ and valuation σ

The meaning $[\![F]\!]^{\mathcal{I}}$ of a formula F is either 1 (true) or 0 (false)

It is inductively defined as follows:

$$
\begin{array}{rclll}
[\![t_1 = t_2]\!]^{\mathcal{I}} & := & 1 & \text{iff} & [\![t_1]\!]^{\mathcal{I}} \text{ is the same as } [\![t_2]\!]^{\mathcal{I}} \\
[\![r(t_1, \dots, t_n)]\!]^{\mathcal{I}} & := & 1 & \text{iff} & ([\![t_1]\!]^{\mathcal{I}}, \dots, [\![t_n]\!]^{\mathcal{I}}) \in r^{\mathcal{I}} \\
[\![\neg F]\!]^{\mathcal{I}} & := & 1 & \text{iff} & [\![F]\!]^{\mathcal{I}} = 0 \\
[\![F_1 \land \cdots \land F_n]\!]^{\mathcal{I}} & := & 1 & \text{iff} & [\![F_i]\!]^{\mathcal{I}} = 1 \text{ for all } i = 1, \dots, n \\
[\![F_1 \lor \cdots \lor F_n]\!]^{\mathcal{I}} & := & 1 & \text{iff} & [\![F_i]\!]^{\mathcal{I}} = 1 \text{ for some } i = 1, \dots, n \\
[\![F_1 \to F_2]\!]^{\mathcal{I}} & := & 1 & \text{iff} & [\![\neg F_1 \lor F_2]\!]^{\mathcal{I}} = 1 \\
[\![\exists x\, F]\!]^{\mathcal{I}} & := & 1 & \text{iff} & [\![F]\!]^{\mathcal{I}'} = 1 \text{ for some } \mathcal{I}' \text{ that disagrees} \\
 & & & & \text{with } \mathcal{I}' \text{ at most on } x \\
[\![\forall x\, F]\!]^{\mathcal{I}} & := & 1 & \text{iff} & [\![F]\!]^{\mathcal{I}'} = 1 \text{ for all } \mathcal{I}' \text{ that disagree} \\
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Models, Validity, etc. for formulas

An interpretation $\mathcal{I}$ *satisfies* a formula F, or is a *model of* F, written $\mathcal{I} \models F$, if $[\![F]\!]^{\mathcal{I}} = 1$

A formula is *satisfiable* if it has at least one model

Ex: $\forall x\, x \geq y$, $P(x)$

A formula is *unsatisfiable* if it has no models

Ex: $P(x) \wedge \neg P(x)$, $\neg(x = x)$, $\forall x\, Q(x, y) \rightarrow \neg Q(a, b)$

A formula F is *valid* if every interpretation is a model of it

Ex: $P(x) \rightarrow P(x)$, $x = x$, $\forall x\, P(x) \rightarrow \exists x\, P(x)$

Note: F is valid/unsatisfiable iff $\neg F$ is unsatisfiable/valid

Models, Validity, etc. for Sets of Formulas

An interpretation *satisfies* a set S of formulas, or is a *model of* S, written $\mathcal{I} \models S$, if it is a model for *every* formula in S

A set S of formulas is *satisfiable* if it has at least one model

Ex: $\{\forall x\, x \geq 0,\ \forall x\, x + 1 > x\}$

S is *unsatisfiable*, or *inconsistent*, if it has no models

Ex: $\{P(x),\ \neg P(x)\}$

S *entails* a formula F, written $S \models F$, if every model for S is also a model for F

Ex: $\{\forall x\, (P(x) \rightarrow Q(x)),\ P(A_{10})\} \models Q(A_{10})$

Note: As in propositional logic, $S \models F$ iff $S \cup \{\neg F\}$ is unsatisfiable

Free and bound variables

The notions of quantifier scope, free/bound occurrence of a variable in a formula, and closed formula are defined exactly as with QBFs

Theorem 1

Let F be a closed formula and let $\mathcal{I}$ and $\mathcal{I}'$ be two interpretations that differ only in their variable valuation. Then,

$$\mathcal{I} \models F \text{ iff } \mathcal{I}' \models F .$$

As with QBFs, the satisfiability of a closed formula by an interpretation $\mathcal{I}$ does not depend on how $\mathcal{I}$ interprets the variables

However, it does depend on how $\mathcal{I}$ interprets the non-logical symbols

Example $\exists x\,(2 < x \wedge x < 3)$

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Example $\exists x\, (2 < x \wedge x < 3)$ is true over the reals and false over the integers

Lots of Models

An FOL formula F can have either no models at all or *infinitely many*

Levels of freedom in constructing a model:

- Cardinality of universe: finite $1, 2, \ldots, n, \ldots$ or infinite
  - Interpretation of each predicate symbol
    - Interpretation of each function symbol
      - Interpretation of each constant symbol
        - Interpretation of each variable

Symbol	Interpretation choices in a universe U of cardinality n	
a	n	(# of elements of U)
$P(\_)$	2^n	(# of subsets of U)
$Q(\_,\_)$	2^{n^2}	(# of subsets of U^2)
$R(\_,\_,\_)$	2^{n^3}	(# of subsets of U^3)

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Equality

Recall that $t_1 = t_2$ is true in a given interpretation iff t_1 and t_2 denote the element of the universe

Examples

- $a = b$
- $t = t$
- $a \neq a$
- $1 = 25$
- $x * x = x$
- $a = b \rightarrow b = a$
- $a = b \wedge b = c \rightarrow a = c$
- $a = b \rightarrow f(a) = f(b)$
- $f(a) = f(b) \rightarrow a = b$
- $a = b \rightarrow P(a, c) \leftrightarrow P(b, c)$

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Recall that $t_1 = t_2$ is true in a given interpretation iff t_1 and t_2 denote the element of the universe

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- $f(a) = f(b) \rightarrow a = b$ is invalid (not all functions are injective)
- $a = b \rightarrow P(a, c) \leftrightarrow P(b, c)$ is valid

Qualifying Universal Quantification

How do we interpret this formula?

$$\forall x\, Smart(x)$$

This statement is too broad (everything is smart?)

We often want to qualify the quantification

Which set of elements are we saying are all smart?

People? Dogs? Students at Iowa? Students at Iowa taking this course?

$\forall x\, (Person(x) \to Smart(x))$

$\forall x\, (Dog(x) \to Smart(x))$

$\forall x\, (Student(x) \wedge At(x, UIowa) \to Smart(x))$

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$\exists x\ (Student(x) \land At(x, UIowa) \land Smart(x))$

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General Quantification Schemas

Universal quantification

$$\forall x \; (\text{Qualifier for } x \rightarrow \text{Statement involving } x)$$

Existential quantification

$$\exists x \; (\text{Qualifier for } x \wedge \text{Statement involving } x)$$

Incorrect Qualifications

$$\forall x\,(Person(x) \wedge Smart(x))$$

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This states that everything is a person and is smart!

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Incorrect Qualifications

$$\forall x\,(Person(x) \wedge Smart(x))$$

This states that everything is a person and is smart!

$$\exists x\,(Person(x) \rightarrow Smart(x))$$

This is satisfied by any interpretation where $Person(x)$ is always false!

Useful Quantifier Equivalences

$$\forall x \, \forall y \, F \equiv \forall y \, \forall x \, F \qquad \exists x \, \exists y \, F \equiv \exists y \, \exists x \, F$$

$$\neg \forall x \, F \equiv \exists x \, \neg F \qquad \neg \exists x \, F \equiv \forall x \, \neg F$$

$$\forall x \, (F \wedge G) \equiv \forall x \, F \wedge \forall x \, G \qquad \exists x \, (F \vee G) \equiv \exists x \, F \vee \exists x \, G$$

Conditional Quantifier Equivalences

$$\forall x\, G \equiv G \qquad \exists x\, G \equiv G$$

$$\forall x\,(F \lor G) \equiv \forall x\, F \lor G \qquad \exists x\,(F \land G) \equiv \exists x\, F \land G$$

$$\forall x\,(F \to G) \equiv \exists x\, F \to G \qquad \exists x\,(F \to G) \equiv \forall x\, F \to G$$

$$\forall x\,(G \to F) \equiv G \to \forall x\, F \qquad \exists x\,(G \to F) \equiv G \to \exists x\, F$$

if x is not free in G

From English to FOL

First step

Choose a set of constant, function and predicate symbols to represent specific individuals, functions, and relations, respectively

Example

Constant	Intended meaning	Function	Intended meaning
annie	some person named Annie	$mother(x)$	x's mother
jane	some person named Jane	$father(x)$	x's father

Predicate	Intended meaning	Predicate	Intended meaning
$Person(x)$	x is a person	$Brothers(x, y)$	x and y are brothers
$Married(x)$	x is married	$Sisters(x, y)$	x and y are sisters
$Dog(x)$	x is a dog	$Siblings(x, y)$	x and y are siblings
$Male(x)$	x is a male	$Cousin(x, y)$	x and y are first cousins
$Female(x)$	x is a female	$Spouse(x, y)$	y is x's spouse
$Mammal(x)$	x is a mammal	$Parent(x, y)$	x is a parent of y

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From English to FOL, Examples

Dogs are mammals

Brothers are siblings

"Siblings" is a symmetric relation

Jane is Annie’s mother

Annie’s mother and father are married

Jane is married

Annie is Jane’s only daughter

One’s mother is one’s female parent

Everybody is the child of somebody

First cousins are people who have parents who are siblings

From English to FOL, Examples

Dogs are mammals $\forall x\,(Dog(x) \rightarrow Mammal(x))$

Brothers are siblings $\forall x\,\forall y\,(Brothers(x,y) \rightarrow Siblings(x,y))$

"Siblings" is a symmetric relation $\forall x\,\forall y\,(Siblings(x,y) \rightarrow Siblings(y,x))$

Jane is Annie's mother $jane = mother(annie)$

Annie's mother and father are married $Married(mother(annie), father(annie))$

Jane is married $\exists x\, Married(Jane, x)$

Annie is Jane's only daughter $mother(annie) = jane \wedge$
$\forall x\,(mother(x) = jane \wedge Female(x) \rightarrow x = annie)$

One's mother is one's female parent
$\forall x\,\forall y\,(y = mother(x) \leftrightarrow Female(y) \wedge Parent(y,x))$

Everybody is the child of somebody $\forall x\,(Person(x) \rightarrow \exists y\,(Person(x) \wedge Parent(y,x)))$

First cousins are people who have parents who are siblings $\forall x_1\,\forall x_2\,(Cousins(x_1,x_2) \leftrightarrow$
$Person(x) \wedge Person(y) \wedge \exists p_1\,\exists p_2\,(Siblings(p_1,p_2) \wedge Parent(p_1,x_1) \wedge Parent(p_2,x_2)))$

From FOL to English, Examples

$\forall x \neg(Persont(x) \wedge Siblings(x, x))$

$\forall x \forall y\, (Brothers(x, y) \rightarrow Male(x) \wedge Male(y))$

$\forall x\, (Person(x) \rightarrow (Male(x) \vee Female(x)) \wedge \neg(Male(x) \wedge Female(x)))$

$\forall x\, (Person(x) \wedge Married(x) \rightarrow \exists y\, Spouse(x, y))$

$\forall x \forall y\, (Person(x) \wedge Spouse(x, y) \rightarrow Married(x))$

$\forall x \forall y\, (Person(x) \wedge Spouse(x, y) \rightarrow \neg Siblings(x, y))$

$\neg\forall x\, (Person(x) \wedge \exists y\, Parent(x, y) \rightarrow Married(x))$

$\forall x \forall y\, (Person(x) \wedge Parent(y, x) \rightarrow Person(x))$

$\forall x \exists y\, (Person(x) \rightarrow y = mother(x))$

$\exists y \forall x\, (Person(x) \rightarrow y = mother(x))$

From FOL to English, Examples

$\forall x\, \neg(Persont(x) \land Siblings(x, x))$ No one is his or her own sibling

$\forall x\, \forall y\, (Brothers(x, y) \to Male(x) \land Male(y))$ Brothers are male

$\forall x\, (Person(x) \to (Male(x) \lor Female(x)) \land \neg(Male(x) \land Female(x)))$ Every person is either male or female but not both

$\forall x\, (Person(x) \land Married(x) \to \exists y\, Spouse(x, y))$ Married people have spouses

$\forall x\, \forall y\, (Person(x) \land Spouse(x, y) \to Married(x))$ Only married people have spouses

$\forall x\, \forall y\, (Person(x) \land Spouse(x, y) \to \neg Siblings(x, y))$ People cannot be married to their own siblings

$\neg\forall x\, (Person(x) \land \exists y\, Parent(x, y) \to Married(x))$ Not everybody who has children is married

$\forall x\, \forall y\, (Person(x) \land Parent(y, x) \to Person(x))$ People's parents are people too

$\forall x\, \exists y\, (Person(x) \to y = mother(x))$ Everyone has a mother

$\exists y\, \forall x\, (Person(x) \to y = mother(x))$ Everyone has the *same* mother

Natural Deduction for FOL

The natural deduction inference system for propositional logic extends to FOL with the addition of rules for

- equality and
- the quantifiers

Freeness

Let x be a variable, t a term, and F a formula of FOL

Recall F_x^t denotes the result of replacing every free occurrence of x in F by t

t is *free for x in F* if no free occurrence of x in F occurs in the scope of $\exists\forall y$
for any variable y of t
iff every variable of t remains free in F_x^t

Example F: $S(x) \wedge \forall y\, (P(z) \rightarrow Q(y))$

$F_x^{f(y)}$: $S(f(y)) \wedge \forall y\, (P(z) \rightarrow Q(y))$ $\quad$ $F_z^{f(y)}$: $S(x) \wedge \forall y\, (P(f(y)) \rightarrow Q(y))$

Term $f(y)$ is free for x in F but not for z

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Let x be a variable, t a term, and F a formula of FOL

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Example F: $S(x) \land \forall y\, (P(z) \to Q(y))$

$F^{f(y)}_x$: $S(f(y)) \land \forall y\, (P(z) \to Q(y))$ $\quad$ $F^{f(y)}_z$: $S(x) \land \forall y\, (P(f(y)) \to Q(y))$

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Example $F\colon\ S(x) \land \forall y\,(P(z) \rightarrow Q(y))$

$F_x^{f(y)}\colon\ S(f(y)) \land \forall y\,(P(z) \rightarrow Q(y))$ $\qquad$ $F_z^{f(y)}\colon\ S(x) \land \forall y\,(P(f(y)) \rightarrow Q(y))$

Term $f(y)$ is free for x in F but not for z

$=$ introduction and elimination

$$\frac{}{t = t}\ {=}\mathrm{i} \qquad\qquad \frac{s = t \quad A^s_x \quad s, t \text{ free for } x \text{ in } A}{A^t_x}\ {=}\mathrm{e}$$

There rules are sufficient to derive all main properties of equality:

$\vdash\ a = a$

$a = b \vdash\ b = a$

$a = b, b = c \vdash\ a = c$

$a = b \vdash\ f(a) = f(b)$

$a = b \vdash\ P(a) \leftrightarrow P(b)$

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Example derivation

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Proof 1 $a = b$ premise

Example derivation

$$\frac{}{t = t} =\text{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} =\text{e}$$

$$a = b \vdash b = a$$

Proof

1	$a = b$	premise
2	$a = a$	$=$i

Example derivation

$$\frac{}{t = t} =\text{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} =\text{e}$$

$$a = b \vdash b = a$$

Proof

1	$a = b$	premise	
2	$a = a$	$=$i	
3	$b = a$	$=$e	1 applied to left-hand side of 2

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} =\mathrm{e}$$

$$a = b,\ b = c \vdash a = c$$

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Proof

1 $a = b$ premise

2 $b = c$ premise

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} =\mathrm{e}$$

$$a = b,\ b = c \vdash a = c$$

Proof

1	$a = b$	premise	
2	$b = c$	premise	
3	$a = c$	$=\mathrm{e}$	2 applied to right-hand side of 1

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} =\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

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Proof

1	$a = b$	premise
2	$P(a)$	assumption
3	$P(b)$	$=$e 1 applied to 2

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1	$a = b$	premise
2	$P(a)$	assumption
3	$P(b)$	$=$e 1 applied to 2
4	$P(a) \to P(b)$	$\to$i 2–3

Example derivation

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Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=$e	1 applied to 2
4	$P(a) \to P(b)$	$\to$i	2–3
5	$a = a$	$=$i	

Example derivation

$$\frac{}{t = t} = \text{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} = \text{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=$e	1 applied to 2
4	$P(a) \rightarrow P(b)$	$\rightarrow$i	2–3
5	$a = a$	$=$i	
6	$b = a$	$=$e	1 applied to 5

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} =\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=\mathrm{e}$	1 applied to 2
4	$P(a) \rightarrow P(b)$	$\rightarrow\mathrm{i}$ 2–3	
5	$a = a$	$=\mathrm{i}$	
6	$b = a$	$=\mathrm{e}$	1 applied to 5
7	$P(b) \rightarrow P(b)$	$=\mathrm{e}$	1 applied to 4

Example derivation

$$\frac{}{t = t} =\text{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} =\text{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
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5	$a = a$	$=$i	
6	$b = a$	$=$e	1 applied to 5
7	$P(b) \to P(b)$	$=$e	1 applied to 4
8	$P(b) \to P(a)$	$=$e	6 applied to 7

Example derivation

$$\frac{}{t = t} {=}\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} {=}\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=$e	1 applied to 2
4	$P(a) \to P(b)$	$\to$i	2–3
5	$a = a$	$=$i	
6	$b = a$	$=$e	1 applied to 5
7	$P(b) \to P(b)$	$=$e	1 applied to 4
8	$P(b) \to P(a)$	$=$e	6 applied to 7
9	$P(a) \leftrightarrow P(b)$	$\leftrightarrow$i	1, 2

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

$\forall$ introduction and elimination

$$\cfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A^{x_0}_x \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \cfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A^t_x}\ \forall\text{e}$$

Example 1 Prove $\forall z\, P(z) \vdash P(a)$

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\text{e}$$

Example 1 Prove $\forall z\, P(z) \vdash P(a)$

1	$\forall z\, P(z)$	premise

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\;\forall\mathrm{i} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\;\forall\mathrm{e}$$

Example 1 Prove $\forall z\, P(z) \vdash P(a)$

1	$\forall z\, P(z)$	premise
2	$P(a)$	$\forall$e 1

$\forall$ introduction and elimination

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad\qquad \frac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\text{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

1 $\forall z\,(P(z) \wedge Q(z))$ premise

∀ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

	1	$\forall z\,(P(z) \wedge Q(z))$	premise
x_0	2		

$\forall$ introduction and elimination

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \frac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\text{e}$$

Example 2 Prove $\forall z\,(P(z) \land Q(z)) \vdash \forall y\, Q(y)$

	1	$\forall z\,(P(z) \land Q(z))$	premise
x_0	2		
	3	$P(x_0) \land Q(x_0)$	$\forall$e 1

$\forall$ introduction and elimination

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad \vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \frac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\text{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

	1	$\forall z\,(P(z) \wedge Q(z))$	premise
x_0	2		
	3	$P(x_0) \wedge Q(x_0)$	$\forall$e 1
	4	$Q(x_0)$	$\wedge$e$_2$ 2

$\forall$ introduction and elimination

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \frac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\text{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

	1	$\forall z\,(P(z) \wedge Q(z))$	premise	
x_0	2			
	3	$P(x_0) \wedge Q(x_0)$	$\forall$e	1
	4	$Q(x_0)$	$\wedge$e$_2$	2
	5	$\forall y\, Q(y)$	$\forall$i	2–5

$\forall$ introduction and elimination

$$\cfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \cfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\text{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad \vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

x_0 1

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\text{i} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\text{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

$$\begin{array}{lll} x_0 & 1 & \\ & 2 & x_0 = x_0 \quad =\text{i} \end{array}$$

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

x_0	1			
	2	$x_0 = x_0$	$=$i	
	3	$\forall x\, x = x$	$\forall$i	1–2

Example derivation

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A^{x_0}_x \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad \frac{\forall x\, A}{A^t_x}\ \forall\mathrm{e}$$

$$\vdash\ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

Example derivation

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad \dfrac{\forall x\, A}{A_x^t}\ \forall\mathrm{e}$$

$$\vdash\ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0 1

Example derivation

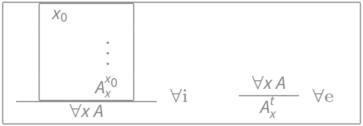
$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad \dfrac{\forall x\, A}{A_x^t}\ \forall\mathrm{e}$$

$$\vdash \ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0 1

y_0 2

Example derivation



$$\vdash \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0	1		
y_0	2		
	3	$x_0 = y_0$	assumption

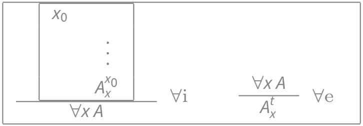
Example derivation

x_0 ... $A_x^{x_0}$ / $\forall x\, A$ ($\forall$i)	$\dfrac{\forall x\, A}{A_x^t}$ ($\forall$e)

$$\vdash \forall x\, \forall y\, (x = y \rightarrow f(x) = f(y))$$

x_0	1		
y_0	2		
	3	$x_0 = y_0$	assumption
	4	$f(x_0) = f(x_0)$	$=$i

Example derivation



$$\vdash \forall x\, \forall y\, (x = y \rightarrow f(x) = f(y))$$

x_0	1			
y_0	2			
	3	$x_0 = y_0$	assumption	
	4	$f(x_0) = f(x_0)$	$=$i	
	5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4

Example derivation

$$\frac{\boxed{\begin{array}{l} x_0 \\ \vdots \\ A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad \frac{\forall x\, A}{A_x^t}\ \forall\mathrm{e}$$

$$\vdash \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0	1			
y_0	2			
	3	$x_0 = y_0$	assumption	
	4	$f(x_0) = f(x_0)$	$=$i	
	5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4
	6	$x_0 = y_0 \to f(x_0) = f(y_0)$	$\to$i	3–5

Example derivation

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad\qquad \dfrac{\forall x\, A}{A_x^t}\ \forall\mathrm{e}$$

$$\vdash\ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0 1			
y_0 2			
3	$x_0 = y_0$	assumption	
4	$f(x_0) = f(x_0)$	$=$i	
5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4
6	$x_0 = y_0 \to f(x_0) = f(y_0)$	$\to$i	3–5
7	$\forall y\, (x_0 = y \to f(x_0) = f(y))$	$\forall$i	2–6

Example derivation

$$\frac{\boxed{\begin{matrix} x_0 \\ \vdots \\ A_x^{x_0} \end{matrix}}}{\forall x\, A}\ \forall\mathrm{i} \qquad \frac{\forall x\, A}{A_x^t}\ \forall\mathrm{e}$$

$$\vdash\ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0	1			
y_0	2			
	3	$x_0 = y_0$	assumption	
	4	$f(x_0) = f(x_0)$	$=$i	
	5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4
	6	$x_0 = y_0 \to f(x_0) = f(y_0)$	$\to$i	3–5
	7	$\forall y\, (x_0 = y \to f(x_0) = f(y))$	$\forall$i	2–6
	8	$\forall x\, \forall y\, (x = y \to f(x) = f(y))$	$\forall$i	1–7

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{cc} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 1 Prove $P(a) \vdash \exists z\, P(z)$

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 1 Prove $P(a) \vdash \exists z\, P(z)$

1	$P(a)$	premise

∃ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 1 Prove $P(a) \vdash \exists z\, P(z)$

1	$P(a)$	premise	
2	$\exists z\, P(z)$	$\exists$i	1

$\exists$ introduction and elimination

$$\frac{A^t_x \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

Example 2 Prove $\exists x\, P(x),\ \forall x\, \neg P(x) \vdash \bot$

$\exists$ introduction and elimination

$$\frac{A^t_x \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{cc} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 2 Prove $\exists x\, P(x),\ \forall x\, \neg P(x) \vdash \bot$

1	$\exists x\, P(x)$	premise

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

Example 2 Prove $\exists x\, P(x), \forall x\, \neg P(x) \vdash \bot$

1	$\exists x\, P(x)$	premise
2	$\forall x\, \neg P(x)$	premise

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 2 Prove $\exists x\, P(x), \forall x\, \neg P(x) \vdash \bot$

	1	$\exists x\, P(x)$	premise
	2	$\forall x\, \neg P(x)$	premise
x_0	3	$P(x_0)$	assumption

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 2 Prove $\exists x\, P(x), \forall x\, \neg P(x) \vdash \bot$

	1	$\exists x\, P(x)$	premise
	2	$\forall x\, \neg P(x)$	premise
x_0	3	$P(x_0)$	assumption
	4	$\neg P(x_0)$	$\forall$e 3

$\exists$ introduction and elimination

$$\frac{A^t_x \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 2 Prove $\exists x\, P(x), \forall x\, \neg P(x) \vdash \bot$

	1	$\exists x\, P(x)$	premise
	2	$\forall x\, \neg P(x)$	premise
x_0	3	$P(x_0)$	assumption
	4	$\neg P(x_0)$	$\forall$e 3
	5	$\bot$	$\neg$e 3, 4

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\text{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

Example 2 Prove $\exists x\, P(x),\ \forall x\, \neg P(x) \vdash \bot$

	1	$\exists x\, P(x)$	premise
	2	$\forall x\, \neg P(x)$	premise
x_0	3	$P(x_0)$	assumption
	4	$\neg P(x_0)$	$\forall$e 3
	5	$\bot$	$\neg$e 3, 4
	6	$\bot$	$\exists$e 1, 3–5

Example derivation

$$\frac{A^t_x}{\exists x\, A}\ \exists\mathrm{i} \qquad \frac{\exists x\, A \quad \boxed{\begin{array}{cc} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y)\ \vdash\ \exists x\, Q(x)$$

Example derivation

$$\frac{A^t_x}{\exists x\, A}\ \exists\text{i} \qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \ \vdash\ \exists x\, Q(x)$$

1	$\forall z\,(P(z) \to Q(z))$	premise

Example derivation

$$\frac{A^t_x}{\exists x\, A}\ \exists\mathrm{i} \qquad \frac{\exists x\, A \quad \boxed{x_0 \quad A^{x_0}_x \ \vdots\ B}}{B}\ \exists\mathrm{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \vdash \exists x\, Q(x)$$

1	$\forall z\,(P(z) \to Q(z))$	premise
2	$\exists y\, P(y)$	premise

Example derivation

$$\frac{A_x^t}{\exists x\, A}\ \exists\text{i} \qquad \frac{\exists x\, A \quad \boxed{x_0 \quad A_x^{x_0} \ \vdots\ B}}{B}\ \exists\text{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \vdash \exists x\, Q(x)$$

	1	$\forall z\,(P(z) \to Q(z))$	premise
	2	$\exists y\, P(y)$	premise
x_0	3	$P(x_0)$	assumption

Example derivation

$$\frac{A^t_x}{\exists x\, A}\ \exists\text{i} \qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \ \vdash\ \exists x\, Q(x)$$

	1	$\forall z\,(P(z) \to Q(z))$	premise
	2	$\exists y\, P(y)$	premise
x_0	3	$P(x_0)$	assumption
	4	$P(x_0) \to Q(x_0)$	$\forall$e 1

Example derivation

$$\frac{A_x^t}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

$$\forall z\,(P(z) \rightarrow Q(z)),\ \exists y\, P(y) \ \vdash\ \exists x\, Q(x)$$

	1	$\forall z\,(P(z) \rightarrow Q(z))$	premise
	2	$\exists y\, P(y)$	premise
x_0	3	$P(x_0)$	assumption
	4	$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
	5	$Q(x_0)$	$\rightarrow$e 3, 4

Example derivation

$$\frac{A_x^t}{\exists x\, A}\ \exists\text{i} \qquad \frac{\exists x\, A \quad \boxed{x_0 \quad A_x^{x_0} \;\vdots\; B}}{B}\ \exists\text{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \vdash \exists x\, Q(x)$$

	1	$\forall z\,(P(z) \to Q(z))$	premise
	2	$\exists y\, P(y)$	premise
x_0	3	$P(x_0)$	assumption
	4	$P(x_0) \to Q(x_0)$	$\forall$e 1
	5	$Q(x_0)$	$\to$e 3, 4
	6	$\exists x\, Q(x)$	$\exists$i 5

Example derivation

$$\frac{A_x^t}{\exists x\, A}\ \exists\text{i} \qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\text{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \ \vdash\ \exists x\, Q(x)$$

	1	$\forall z\,(P(z) \to Q(z))$	premise
	2	$\exists y\, P(y)$	premise
x_0	3	$P(x_0)$	assumption
	4	$P(x_0) \to Q(x_0)$	$\forall$e 1
	5	$Q(x_0)$	$\to$e 3, 4
	6	$\exists x\, Q(x)$	$\exists$i 5
	7	$\exists x\, Q(x)$	$\exists$e 2, 3–6

(Lines 3–6 are enclosed in a box.)

Soundness and Completeness of Natural Deduction

Let $F, F_1, \ldots, F_n$ be FOL formulas

Theorem 2 (Soundness)

If $F_1, \ldots, F_n \vdash F$ *then* $F_1, \ldots, F_n \models F$*.*

Theorem 3 (Completeness)

If $F_1, \ldots, F_n \models F$ *then* $F_1, \ldots, F_n \vdash F$*.*

As in Propositional Logic, the proof of reduces to proving that

- formulas derivable from no premises are valid (soundness)
- valid formulas are derivable from no premises (completeness)

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- valid formulas are derivable from no premises (completeness)

Undecidability of FOL

The problem of determining the validity of FOL formulas is undecidable:

> *There is no general validity procedure guaranteed to determine in finite time that a given formula is invalid*

In fact, FOL is powerful enough to encode several undecidable problems

Several useful fragments of FOL are, however, decidable

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