CS:4350 Logic in Computer Science

Semantic Tableaux

Cesare Tinelli

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The University of Iowa

Credits

These slides are largely based on slides originally developed by **Andrei Voronkov** at the University of Manchester. Adapted by permission.

Outline

Signed Formula

- *Signed formula*: an expression A^b, where A is a formula and b a boolean value
- A signed formula A^b is *satisfied* by an interpretation $\mathcal{I}$, written $\mathcal{I} \models A^b$, if $\mathcal{I}(A) = b$; it is *falsified* otherwise
- If $\mathcal{I} \models A^b$, we also say that $\mathcal{I}$ *is a model of* A^b
- A signed formula is *satisfiable* if it has a model

Note:

1. For every formula A and interpretation $\mathcal{I}$ exactly one of the signed formulas A^1 and A^0 is satisfied by $\mathcal{I}$
2. A formula A is satisfiable iff the signed formula A^1 is satisfiable
3. A formula A is falsifiable iff the signed formula A^0 is satisfiable

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How to find a model of a signed formula?

Example: $A \wedge B$

		B	B
	$\wedge$	0	1
A	0	0	0
A	1	0	1

$(A \wedge B)^1$: We can make $A \wedge B$ true iff we make A true (A^1) and B true (B^1)

$(A \wedge B)^0$: We can make $A \wedge B$ false iff we make A false (A^0) or B false (B^0)

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Tableau

The search for a model of a formula can be expressed by an AND-OR tree

Tableau: a tree having signed formulas at nodes (plural: tableaux)

A tableau for a signed formula A^b has A^b as a root

Alternatively, we can regard a tableau as a collection of branches; each branch is a set of signed formulas

Notation for branches: $A_1^{b_1} \mid \cdots \mid A_n^{b_n}$

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Constructing a semantic tableau

$$(\neg(q \vee p \to p \vee q))^1$$

$$(q \vee p \to p \vee q)^0$$

$$(q \vee p)^1$$
$$(p \vee q)^0$$

$$p^0$$
$$q^0$$

q^1 closed | p^1 closed

Rules to grow a tree branch:

$$(A_1 \vee A_2)^0 \quad \rightsquigarrow \quad A_1^0, A_2^0$$
$$(A_1 \vee A_2)^1 \quad \rightsquigarrow \quad A_1^1 \mid A_2^1$$
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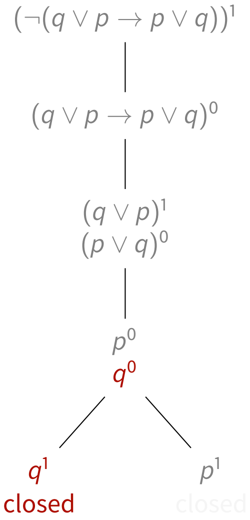

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Branch Expansion Rules

$$
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(A_1 \wedge \ldots \wedge A_n)^0 &\rightsquigarrow A_1^0 \mid \ldots \mid A_n^0 \\
(A_1 \wedge \ldots \wedge A_n)^1 &\rightsquigarrow A_1^1, \ldots, A_n^1 \\
\\
(A_1 \vee \ldots \vee A_n)^0 &\rightsquigarrow A_1^0, \ldots, A_n^0 \\
(A_1 \vee \ldots \vee A_n)^1 &\rightsquigarrow A_1^1 \mid \ldots \mid A_n^1 \\
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(A_1 \rightarrow A_2)^0 &\rightsquigarrow A_1^1, A_2^0 \\
(A_1 \rightarrow A_2)^1 &\rightsquigarrow A_1^0 \mid A_2^1 \\
\\
(\neg A)^0 &\rightsquigarrow A^1 \\
(\neg A)^1 &\rightsquigarrow A^0 \\
\\
(A_1 \leftrightarrow A_2)^0 &\rightsquigarrow A_1^0, A_2^1 \mid A_1^1, A_2^0 \\
(A_1 \leftrightarrow A_2)^1 &\rightsquigarrow A_1^0, A_2^0 \mid A_1^1, A_2^1
\end{aligned}
$$

Open and closed branches

A branch is *closed* in any of the following cases:

- it contains both p^0 and p^1 for some atom p
- it contains $\top^0$
- it contains $\bot^1$

Note: The formulas on a closed branch are jointly unsatisfiable

A branch is *complete* (or *saturated*) if it cannot be expanded further without adding a formula already in it

Note: From the signed atoms of a complete branch it is possible to construct a model of the root formula

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Example 2

$(\neg((p \to q) \land (p \land q \to r) \to (\neg p \to r)))^1$

$$\begin{aligned}
(A_1 \land A_2)^0 &\quad \rightsquigarrow \quad A_1^0 \mid A_2^0 \\
(A_1 \land A_2)^1 &\quad \rightsquigarrow \quad A_1^1, A_2^1 \\[1ex]
(A_1 \to A_2)^0 &\quad \rightsquigarrow \quad A_1^1, A_2^0 \\
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$$|$$

$$((p \to q) \land (p \land q \to r) \to (\neg p \to r))^0$$

$$|$$

$$((p \to q) \land (p \land q \to r))^1$$
$$(\neg p \to r)^0$$

$$(A_1 \land A_2)^0 \quad \rightsquigarrow \quad A_1^0 \mid A_2^0$$
$$(A_1 \land A_2)^1 \quad \rightsquigarrow \quad A_1^1, A_2^1$$

$$(A_1 \to A_2)^0 \quad \rightsquigarrow \quad A_1^1, A_2^0$$
$$(A_1 \to A_2)^1 \quad \rightsquigarrow \quad A_1^0 \mid A_2^1$$

$$(\neg A_1)^1 \quad \rightsquigarrow \quad A_1^0$$

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$$((p \to q) \land (p \land q \to r) \to (\neg p \to r))^0$$

$$|$$

$$((p \to q) \land (p \land q \to r))^1$$
$$(\neg p \to r)^0$$

$$|$$

$$(p \to q)^1$$
$$(p \land q \to r)^1$$

$$(A_1 \land A_2)^0 \quad \rightsquigarrow \quad A_1^0 \mid A_2^0$$
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$$(p \to q)^1$$
$$(p \land q \to r)^1$$

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|

$$((p \to q) \land (p \land q \to r))^1$$
$$(\neg p \to r)^0$$

|

$$(p \to q)^1$$
$$(p \land q \to r)^1$$

|

$$(\neg p)^1$$
$$r^0$$

$$\begin{array}{rcl} (A_1 \land A_2)^0 & \rightsquigarrow & A_1^0 \mid A_2^0 \\ (A_1 \land A_2)^1 & \rightsquigarrow & A_1^1, A_2^1 \\ (A_1 \to A_2)^0 & \rightsquigarrow & A_1^1, A_2^0 \\ (A_1 \to A_2)^1 & \rightsquigarrow & A_1^0 \mid A_2^1 \\ (\neg A_1)^1 & \rightsquigarrow & A_1^0 \end{array}$$

Example 2

$$(\neg((p \to q) \wedge (p \wedge q \to r) \to (\neg p \to r)))^1$$

$$((p \to q) \wedge (p \wedge q \to r) \to (\neg p \to r))^0$$

$$((p \to q) \wedge (p \wedge q \to r))^1$$
$$(\neg p \to r)^0$$

$$(p \to q)^1$$
$$(p \wedge q \to r)^1$$

$$(\neg p)^1$$
$$r^0$$

$$(A_1 \wedge A_2)^0 \quad \rightsquigarrow \quad A_1^0 \mid A_2^0$$
$$(A_1 \wedge A_2)^1 \quad \rightsquigarrow \quad A_1^1, A_2^1$$

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$$((p \to q) \land (p \land q \to r))^1$$
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$$(p \to q)^1$$
$$(p \land q \to r)^1$$

$$(\neg p)^1$$
$$r^0$$

$$p^0 \qquad q^1$$

$$(A_1 \land A_2)^0 \quad \rightsquigarrow \quad A_1^0 \mid A_2^0$$
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$$(p \to q)^1$$
$$(p \wedge q \to r)^1$$

$$(\neg p)^1$$
$$r^0$$

$$p^0 \qquad\qquad q^1$$

$$
\begin{array}{rcl}
(A_1 \wedge A_2)^0 & \rightsquigarrow & A_1^0 \mid A_2^0 \\
(A_1 \wedge A_2)^1 & \rightsquigarrow & A_1^1, A_2^1 \\
 & & \\
(A_1 \to A_2)^0 & \rightsquigarrow & A_1^1, A_2^0 \\
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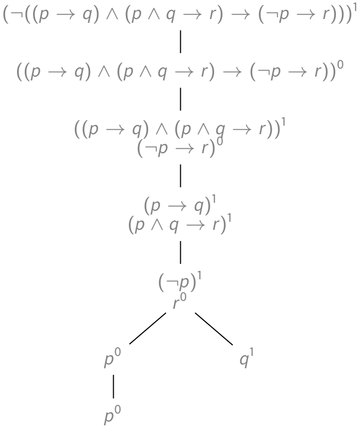
 & & \\
(\neg A_1)^1 & \rightsquigarrow & A_1^0
\end{array}
$$

Example 2



$$
\begin{array}{rcl}
(A_1 \wedge A_2)^0 & \rightsquigarrow & A_1^0 \mid A_2^0 \\
(A_1 \wedge A_2)^1 & \rightsquigarrow & A_1^1, A_2^1 \\
\\
(A_1 \rightarrow A_2)^0 & \rightsquigarrow & A_1^1, A_2^0 \\
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\\
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$$(p \to q)^1$$
$$(p \wedge q \to r)^1$$

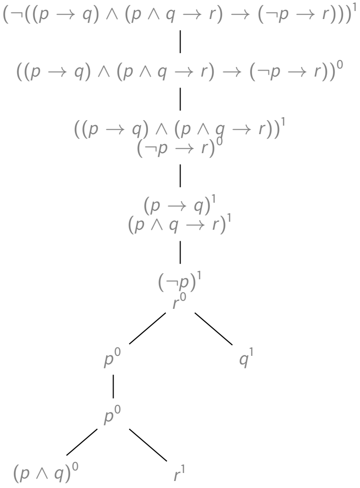
$$(\neg p)^1$$
$$r^0$$

$$p^0 \qquad\qquad q^1$$

$$p^0$$

$$\begin{aligned}
(A_1 \wedge A_2)^0 &\leadsto A_1^0 \mid A_2^0 \\
(A_1 \wedge A_2)^1 &\leadsto A_1^1, A_2^1 \\
(A_1 \to A_2)^0 &\leadsto A_1^1, A_2^0 \\
(A_1 \to A_2)^1 &\leadsto A_1^0 \mid A_2^1 \\
(\neg A_1)^1 &\leadsto A_1^0
\end{aligned}$$

Example 2



$$
\begin{aligned}
(A_1 \land A_2)^0 &\quad\rightsquigarrow\quad A_1^0 \mid A_2^0 \\
(A_1 \land A_2)^1 &\quad\rightsquigarrow\quad A_1^1, A_2^1 \\[1ex]
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$$

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$$p^0 \qquad q^1$$

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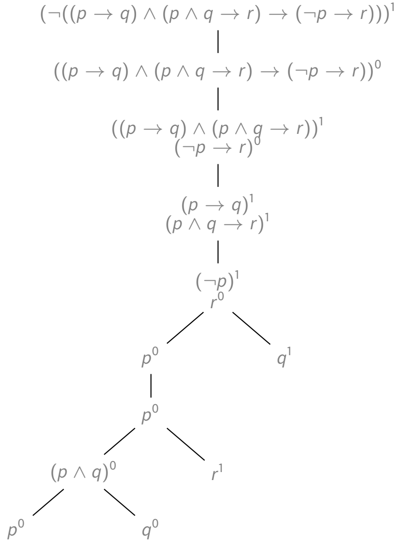

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\begin{array}{rcl}
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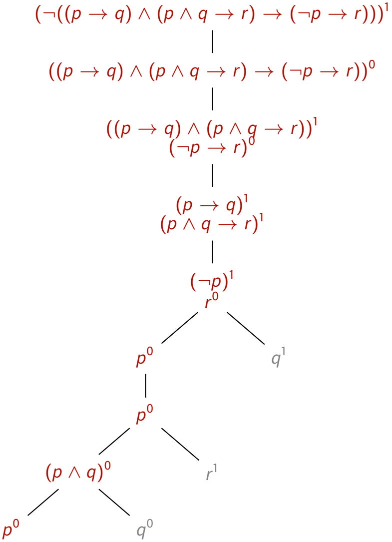
$$

Example 2



$$
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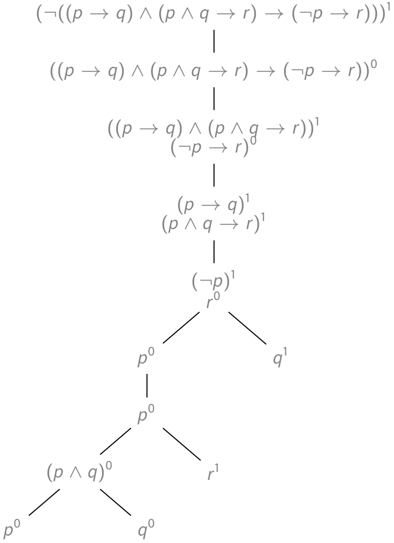
\end{array}
$$

The leftmost branch is complete (nothing new can be added)

Finding Models Using Tableaux



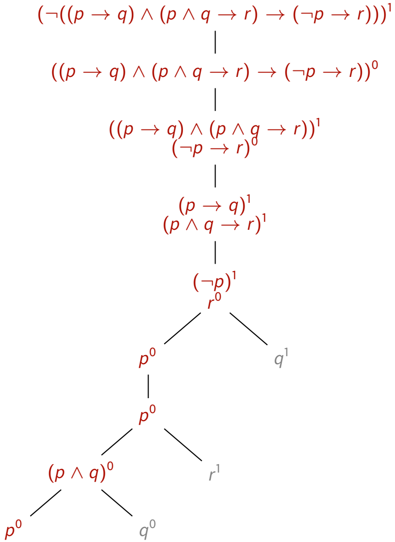
Build a complete branch

Select the signed atoms on it

They give us a (possibly partial) model of the root formula:

$$\{ r \mapsto 0, p \mapsto 0, q \mapsto \cdots \}$$

Finding Models Using Tableaux



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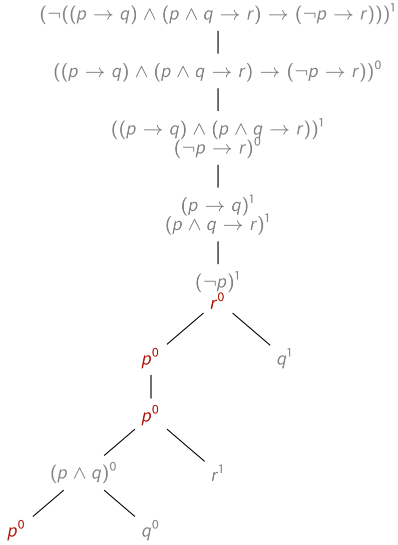

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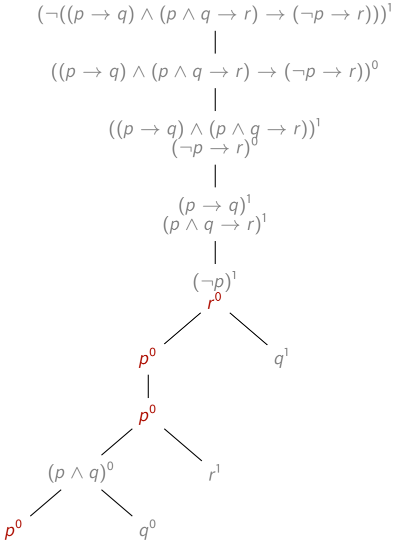

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Checking Other Properties with Tableaux

A formula A is satisfiable iff a tableau for A^1 contains a complete open branch (and iff every tableau for A^1 contains a complete open branch)

A formula A is valid iff there is a closed a tableau for A^0 (and iff every tableau for A^0 is closed)

Formulas A and B are equivalent iff there is a closed tableau for $(A \leftrightarrow B)^0$ (and iff every tableau for $(A \leftrightarrow B)^0$ is closed)

A fully expanded tableau for A^1 gives us all models of A

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