

CS:4350 Logic in Computer Science

Conjunctive Normal Form and DPLL

Cesare Tinelli

Spring 2021



Credits

These slides are largely based on slides originally developed by **Andrei Voronkov** at the University of Manchester. Adapted by permission.

Outline

DPLL

- Conjunctive Normal Form

- Clausal Form and Definitional Transformation

- Unit Propagation

- DPLL

- Expressing Counting

- Sudoku

- Loop the Loop

Satisfiability of clauses

The efficiency of splitting algorithms for satisfiability

- can be **massively** improved in practice
- by first putting the input formula in **normal form**

A popular satisfiability procedure called DPLL requires formulas in conjunctive normal form

We will see this next

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Literal: an atom p (**positive literal**) or a negated atom $\neg p$ (**negative literal**)

The *complement* \bar{L} of a literal L :

$$\bar{L} \stackrel{\text{def}}{=} \begin{cases} \neg L & \text{if } L \text{ is positive} \\ p & \text{if } L \text{ has the form } \neg p \end{cases}$$

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Clauses

Clause: a disjunction $L_1 \vee \dots \vee L_n$ of literals with $n \geq 0$

- *Empty clause* (\square) : when $n = 0$
- *Unit clause*: when $n = 1$
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Conjunctive Normal Form

A formula A is in *conjunctive normal form*, or simply **CNF**, if it is either \top or a conjunction of clauses:

$$A = \bigwedge_i \bigvee_j L_{i,j}$$

A formula B is *a conjunctive normal form of* a formula A if $B \equiv A$ and B is in CNF

Satisfiability on CNF

Note: An interpretation \mathcal{I}

1. satisfies a formula in CNF

$$A = \bigwedge_i \bigvee_j L_{i,j}$$

iff it satisfies **every** clause $C_i = \bigvee_j L_{i,j}$ in A

2. satisfies a clause

$$C = L_1 \vee \dots \vee L_n$$

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CNF transformation

Any propositional formula can be converted to CNF by the repeated applications of these rewrite rules:

1. $A \leftrightarrow B \Rightarrow (\neg A \vee B) \wedge (\neg B \vee A)$
2. $A \rightarrow B \Rightarrow \neg A \vee B$
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A formula to which no rewrite rule is applicable

- contains no \leftrightarrow
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$$\begin{aligned} & \neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \\ & \neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow \\ & \neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\ & (p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\ & (\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \end{aligned}$$

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Theorem

If A' is obtained from A by one of more applications of the CNF conversion rules, then $A' \equiv A$.

CNF and satisfiability

$$A = \neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$

...

$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

Note: Formula A above has the same models as the set consisting of these four clauses

$$\begin{aligned} &\neg p \vee q \\ &\neg p \vee \neg q \vee r \\ &p \\ &\neg r \end{aligned}$$

(An interpretation *satisfies*, or is a model of, a set S of formulas if it satisfies every formula in S)

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The CNF transformation reduces the sat problem for formulas to the sat problem for sets of clauses

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Problem

Let's compute the CNF of

$$F = p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

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If we continue, the formula will grow exponentially large!

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CNF is exponential

There are formulas whose **shortest CNF** has an **exponential size**

Is there any way to avoid exponential blowup? Yes!

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A space-efficient CNF transformation

Using so-called *naming* or *definition introduction*

1. Take a non-trivial subformula A of formula F
2. Introduce a new *name* n for it, i.e., a fresh propositional variable
3. Add a *definition for n* , i.e., a formula stating that n is equivalent to A

$$F = \begin{array}{l} p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \\ n \leftrightarrow (p_5 \leftrightarrow p_6) \end{array}$$

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4. Replace the subformula by its name:

$$S = \left\{ \begin{array}{l} p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n))) \\ n \leftrightarrow (p_5 \leftrightarrow p_6) \end{array} \right\}$$

A space-efficient CNF transformation

The new set S of formulas and the original formula F are not equivalent but they are *equisatisfiable*:

1. every model of S is a model of F and
2. every model of F can be extended to a model of S
(by assigning to n the value of $p_5 \leftrightarrow p_6$)

$$F = p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \\ n \leftrightarrow (p_5 \leftrightarrow p_6)$$

$$S = \left\{ \begin{array}{l} p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n))) \\ n \leftrightarrow (p_5 \leftrightarrow p_6) \end{array} \right\}$$

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After several steps

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

$$p_1 \leftrightarrow (p_2 \leftrightarrow n_3)$$

$$n_3 \leftrightarrow (p_3 \leftrightarrow n_4)$$

$$n_4 \leftrightarrow (p_4 \leftrightarrow n_5)$$

$$n_5 \leftrightarrow (p_5 \leftrightarrow p_6)$$

The conversion of the original formula to CNF introduces 32 copies of p_6

The conversion of the new set of formulas to CNF introduces 4 copies of p_6

After several steps

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

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The conversion of the **original formula** to CNF introduces 32 **copies** of p_6

The conversion of the **new set of formulas** to CNF introduces 4 **copies** of p_6

Clausal Form

Clausal form of a formula A : a set S_A of clauses which is satisfiable iff A is satisfiable

Clausal form of a set S of formulas: a set S' of clauses which is satisfiable iff so is S

In fact, we can require something stronger:

1. A and S_A have the same models in the language of A
2. S and S' have the same models in the language of S

Big advantage of clausal normal form over CNF:
we can convert any formula to a set of clauses in almost linear time

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2. S and S' have the same models **in the language of S**

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Definitional Clause Form Transformation

How to convert a formula A into a set S of clauses that is **clausal normal form of A** :

1. If A has the form $C_1 \wedge \dots \wedge C_n$, where $n \geq 1$ and each C_i is a clause, then $S \stackrel{\text{def}}{=} \{C_1, \dots, C_n\}$.
2. Otherwise, introduce a name for each subformula B of A such that B is not a literal and use this name instead of the formula.

Converting a formula to clausal form, Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$n_2 \leftrightarrow (n_3 \rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$n_7 \leftrightarrow (p \rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Converting a formula to clausal form, Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$n_2 \leftrightarrow (n_3 \rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$n_7 \leftrightarrow (p \rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Consider all subformulas that are not literals

Converting a formula to clausal form, Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$n_2 \leftrightarrow (n_3 \rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$n_7 \leftrightarrow (p \rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Introduce names for these formulas

Converting a formula to clausal form, Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$n_2 \leftrightarrow (n_3 \rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$n_7 \leftrightarrow (p \rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Introduce definitions

Converting a formula to clausal form, Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$n_2 \leftrightarrow (n_3 \rightarrow n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \leftrightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \leftrightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$n_7 \leftrightarrow (p \rightarrow r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Convert the definition formulas to CNF in the standard way

Optimized Definitional Clause Form Transformation

If

- we introduce a name for a subformula and
- the occurrence of the subformula is positive or negative

then an **implication can be used instead of equivalence**

Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \rightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$(n_3 \rightarrow n_7) \rightarrow n_2$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \rightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \rightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \rightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \rightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$(p \rightarrow r) \rightarrow n_7$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \rightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$(n_3 \rightarrow n_7) \rightarrow n_2$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \rightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \rightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \rightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \rightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$(p \rightarrow r) \rightarrow n_7$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

The clauses in red are omitted by optimized transformation

Example

	subformula	definition	clauses
	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$		n_1
n_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$n_1 \rightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$(n_3 \rightarrow n_7) \rightarrow n_2$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$n_3 \rightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \rightarrow q$	$n_4 \rightarrow (p \rightarrow q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \rightarrow r$	$n_5 \rightarrow (n_6 \rightarrow r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \rightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \rightarrow r$	$(p \rightarrow r) \rightarrow n_7$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

The result is fewer clauses

Satisfiability-checking for sets of clauses

The CNF transformation of

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$$

gives the set of four clauses:

$$\begin{aligned} &\neg p \vee q \\ &\neg p \vee \neg q \vee r \\ &p \\ &\neg r \end{aligned}$$

To satisfy all these clauses we must assign 1 to p and 0 to r ,
so we do not have to guess values for them

In this case, we can do even better and establish unsatisfiability with no guessing

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To satisfy all these clauses we **must** assign 1 to p and 0 to r ,
so **we do not have to guess values for them**

In this case, we can do even better and establish unsatisfiability **with no guessing**

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

$$p$$

$$\neg r$$

This set of clauses is unsatisfiable

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

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$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$

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$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

$$p$$

$$\neg r$$

This set of clauses is unsatisfiable

Searching for satisfiability

$$\{p \mapsto 1, r \mapsto 0, q \mapsto 1\}$$

$$\begin{aligned} &\neg p \vee q \\ &\neg p \vee \neg q \vee r \quad \square \\ &p \\ &\neg r \end{aligned}$$

This set of clauses is unsatisfiable

Unit propagation

Let S be a set of clauses.

Unit propagation. Repeatedly apply the following transformation:

if S contains a unit clause, i.e. a clause consisting of one literal L , then

1. remove from S every clause of the form $L \vee C'$
2. replace in S every clause of the form $\bar{L} \vee C'$ by the clause C'

Unit Propagation, Example

 n_1 $\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$ $\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$ $\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$ $\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$ $\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$ $\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$ $\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$

Unit Propagation, Example

n_1

$\neg n_1 \vee \neg n_2$

$n_1 \vee n_2$

$\neg n_2 \vee \neg n_3 \vee n_7$

$n_3 \vee n_2$

$\neg n_7 \vee n_2$

$\neg n_3 \vee n_4$

$\neg n_3 \vee n_5$

$\neg n_4 \vee \neg n_5 \vee n_3$

$\neg n_4 \vee \neg p \vee q$

$p \vee n_4$

$\neg q \vee n_4$

$\neg n_5 \vee \neg n_6 \vee r$

$n_6 \vee n_5$

$\neg r \vee n_5$

$\neg n_6 \vee p$

$\neg n_6 \vee q$

$\neg p \vee \neg q \vee n_6$

$\neg n_7 \vee \neg p \vee r$

$p \vee n_7$

$\neg r \vee n_7$

Unit Propagation, Example

$$\begin{array}{l} \neg n_2 \\ \neg n_2 \vee \neg n_3 \vee n_7 \\ n_3 \vee n_2 \\ \neg n_7 \vee n_2 \\ \neg n_3 \vee n_4 \\ \neg n_3 \vee n_5 \\ \neg n_4 \vee \neg n_5 \vee n_3 \\ \neg n_4 \vee \neg p \vee q \\ p \vee n_4 \\ \neg q \vee n_4 \\ \neg n_5 \vee \neg n_6 \vee r \\ n_6 \vee n_5 \\ \neg r \vee n_5 \\ \neg n_6 \vee p \\ \neg n_6 \vee q \\ \neg p \vee \neg q \vee n_6 \\ \neg n_7 \vee \neg p \vee r \\ p \vee n_7 \\ \neg r \vee n_7 \end{array}$$

Unit Propagation, Example

$$\neg n_2$$

$$\neg n_2 \vee \neg n_3 \vee n_7$$

$$n_3 \vee n_2$$

$$\neg n_7 \vee n_2$$

$$\neg n_3 \vee n_4$$

$$\neg n_3 \vee n_5$$

$$\neg n_4 \vee \neg n_5 \vee n_3$$

$$\neg n_4 \vee \neg p \vee q$$

$$p \vee n_4$$

$$\neg q \vee n_4$$

$$\neg n_5 \vee \neg n_6 \vee r$$

$$n_6 \vee n_5$$

$$\neg r \vee n_5$$

$$\neg n_6 \vee p$$

$$\neg n_6 \vee q$$

$$\neg p \vee \neg q \vee n_6$$

$$\neg n_7 \vee \neg p \vee r$$

$$p \vee n_7$$

$$\neg r \vee n_7$$

Unit Propagation, Example

n_3

$\neg n_7$

$\neg n_3 \vee n_4$

$\neg n_3 \vee n_5$

$\neg n_4 \vee \neg n_5 \vee n_3$

$\neg n_4 \vee \neg p \vee q$

$p \vee n_4$

$\neg q \vee n_4$

$\neg n_5 \vee \neg n_6 \vee r$

$n_6 \vee n_5$

$\neg r \vee n_5$

$\neg n_6 \vee p$

$\neg n_6 \vee q$

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Unit Propagation, Example

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 $\neg n_7$
 $\neg n_3 \vee n_4$
 $\neg n_3 \vee n_5$
 $\neg n_4 \vee \neg n_5 \vee n_3$
 $\neg n_4 \vee \neg p \vee q$
 $p \vee n_4$

$\neg q \vee n_4$
 $\neg n_5 \vee \neg n_6 \vee r$
 $n_6 \vee n_5$
 $\neg r \vee n_5$
 $\neg n_6 \vee p$
 $\neg n_6 \vee q$
 $\neg p \vee \neg q \vee n_6$
 $\neg n_7 \vee \neg p \vee r$
 $p \vee n_7$
 $\neg r \vee n_7$

Unit Propagation, Example

$$\begin{array}{l} n_4 \\ n_5 \\ \neg n_4 \vee \neg p \vee q \\ p \vee n_4 \end{array}$$

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Unit Propagation, Example

$$\neg n_6$$

$$\neg n_6 \vee q$$

$$\neg q \vee n_6$$

$$q$$

Unit Propagation, Example

$$\neg n_6$$

$$\neg n_6 \vee q$$

$$\neg q \vee n_6$$

$$q$$

Unit Propagation, Example



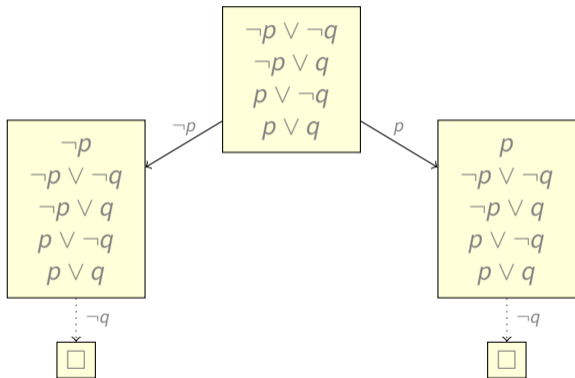
We established unsatisfiability of this set of clauses **in a completely deterministic way**, by unit propagation.

DPLL = splitting + unit propagation

```
procedure DPLL(S)
input: set of clauses S
output: satisfiable or unsatisfiable
parameters: function select_literal
begin
  S := propagate(S)
  if S is empty then return satisfiable
  if S contains  $\square$  then return unsatisfiable
  L := select_literal(S)
  if DPLL(S  $\cup$  { L }) = satisfiable
    then return satisfiable
    else return DPLL(S  $\cup$  {  $\bar{L}$  })
end
```

DPLL, Example 1

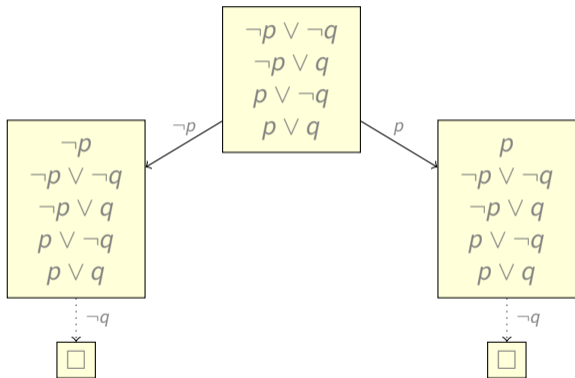
Can be illustrated using **DPLL trees** (similar to splitting trees)



Since all branches end up in a set containing the empty clause, the initial set of clauses is **unsatisfiable**.

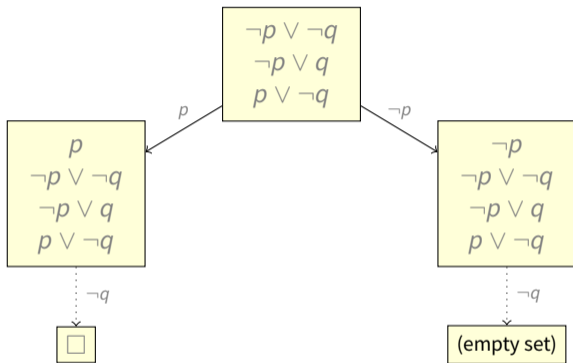
DPLL, Example 1

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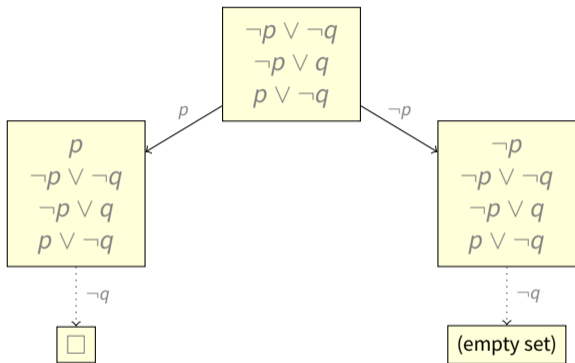
Since all branches end up in a set containing the **empty clause**, the initial set of clauses is **unsatisfiable**.

DPLL, Example 2



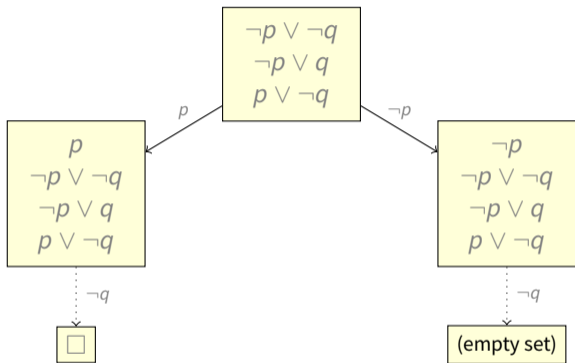
The set of clauses is **satisfiable**

DPLL, Example 2



A model is described by all **selected literals** and **unit-propagated literals** on the branch ending in the empty set

DPLL, Example 2



This DPLL tree gives us the model $\{p \mapsto 0, q \mapsto 0\}$

Two optimizations

1. Any clause $p \vee \neg p \vee C$ is a **tautology**

Tautologies can be removed from a set without affecting its satisfiability

2. A literal L in S is called **pure** if S contains no clauses of the form $\bar{L} \vee C$

All clauses containing a pure literal can be satisfied by making that literal true

Hence, clauses containing pure literals can be removed, too

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Pure literals: example

$$\begin{aligned} & \neg p_2 \vee \neg p_3 \\ & p_1 \vee \neg p_2 \\ & \neg p_1 \vee p_2 \vee \neg p_3 \\ & \neg p_1 \vee \neg p_3 \\ & p_1 \vee p_2 \\ & \neg p_1 \vee \neg p_2 \vee \neg p_3 \end{aligned}$$

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Literal $\neg p_3$ is pure in this clause set: we can remove all clauses containing it (by assigning 0 to p_3)

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We obtain the empty set of clauses

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Since r remained unconstrained, this gives us two models:

$$\begin{aligned} & \{ p_1 \mapsto 1, p_2 \mapsto 0, p_3 \mapsto 0 \} \\ & \{ p_1 \mapsto 1, p_2 \mapsto 1, p_3 \mapsto 0 \} \end{aligned}$$

Horn clauses

A clause is called **Horn** if it contains at most one positive literal. Examples:

$$\begin{aligned} & p_1 \\ & \neg p_1 \vee p_2 \\ & \neg p_1 \vee \neg p_2 \vee p_3 \\ & \neg p_3 \vee \neg p_4 \end{aligned}$$

The following clauses are non-Horn:

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Can be decided by unit propagation

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Satisfiability of Horn clauses

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DIMACS input format:

```
p cnf 3 4
1 0
-1 2 0
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3 variables, 4 clauses.

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Expressing Properties “ k out of n variables are true”

Suppose we have variables v_1, \dots, v_n and we want to express that exactly k of them are true

We will write this property as a formula $T_k(v_1, \dots, v_n)$

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Expressing Properties “ k out of n variables are true”

To define T_k for $0 < k < n$, introduce two formulas:

$T_{\leq k}(v_1, \dots, v_n)$: **at most** k variables among v_1, \dots, v_n are true, where
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Sudoku

4		8	7	9			3	
		9	8	2		5		
	2							
9		2			6		1	7
		6	5	8	7	9		
7	8		2			4		6
							4	
		5		4	8	2		
	9			7	2	3		5

Enter digits from 1 to 9 into the blank spaces

Every row must contain one of each digit

So must every column

as must every 3x3 square

This instance has exactly one solution

Sudoku

4		8	7	9			3	
		9	8	2		5		
	2							
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9	5	2	4	3	6	8	1	7
1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
2	6	1	9	5	3	7	4	8
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1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
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This instance has exactly one solution

Sudoku

4	1	8	7	9	5	6	3	2
6	3	9	8	2	1	5	7	4
5	2	7	3	6	4	1	8	9
9	5	2	4	3	6	8	1	7
1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
2	6	1	9	5	3	7	4	8
3	7	5	6	4	8	2	9	1
8	9	4	1	7	2	3	6	5

Enter digits from 1 to 9 into the blank spaces

Every row must contain one of each digit

So must every column

as must every 3x3 square

This instance has exactly one solution

Sudoku

4	1	8	7	9	5	6	3	2
6	3	9	8	2	1	5	7	4
5	2	7	3	6	4	1	8	9
9	5	2	4	3	6	8	1	7
1	4	6	5	8	7	9	2	3
7	8	3	2	1	9	4	5	6
2	6	1	9	5	3	7	4	8
3	7	5	6	4	8	2	9	1
8	9	4	1	7	2	3	6	5

Enter digits from 1 to 9 into the blank spaces

Every row must contain one of each digit

So must every column

as must every 3x3 square

This instance has exactly one solution

Sudoku as an instance of SAT

9	4		8	7	9			3	
8			9	8	2		5		
7		2							
6	9		2			6		1	7
5			6	5	8	7	9		
4	7	8		2			4		6
3								4	
2			5		4	8	2		
1		9			7	2	3		5
	1	2	3	4	5	6	7	8	9

Sudoku as an instance of SAT

9	4		8	7	9			3	
8			9	8	2		5		
7		2							
6	9		2			6		1	7
5			6	5	8	7	9		
4	7	8		2			4		6
3								4	
2			5		4	8	2		
1		9			7	2	3		5
	1	2	3	4	5	6	7	8	9

Introduce 729 propositional variables $v_{r,c,d}$, where $r, c, d \in \{1, \dots, d\}$

The variable $v_{r,c,d}$ denotes that the cell in the row number r and column number c contains the digit d

Sudoku as an instance of SAT

9	4		8	7	9			3	
8			9	8	2		5		
7		2							
6	9		2			6		1	7
5			6	5	8	7	9		
4	7	8		2			4		6
3							4		
2			5		4	8	2		
1		9			7	2	3		5
	1	2	3	4	5	6	7	8	9

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For example, this configuration satisfies the formula $v_{1,2,9} \wedge v_{2,6,8} \wedge \neg v_{6,9,1}$

Sudoku as an instance of SAT

9	4		8	7	9			3	
8			9	8	2		5		
7		2							
6	9		2			6		1	7
5			6	5	8	7	9		
4	7	8		2			4		6
3							4		
2			5		4	8	2		
1		9			7	2	3		5
	1	2	3	4	5	6	7	8	9

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For example, this configuration satisfies the formula $v_{1,2,9} \wedge v_{2,6,8} \wedge \neg v_{6,9,1}$

We should express all rules of Sudoku using the variables $v_{r,c,d}$

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit

$$\{ v_{rc1} \vee v_{rc2} \vee \dots \vee v_{rc8} \vee v_{rc9} \mid r, c \in \{1, \dots, 9\} \}$$

$$\{ \neg v_{rc1} \vee \neg v_{rc2} \mid r, c \in \{1, \dots, 9\} \}$$

$$\{ \neg v_{rc1} \vee \neg v_{rc3} \mid r, c \in \{1, \dots, 9\} \}$$

⋮

$$\{ \neg v_{rc8} \vee \neg v_{rc9} \mid r, c \in \{1, \dots, 9\} \}$$

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit

$$\begin{aligned} & \{ v_{rc1} \vee v_{rc2} \vee \dots \vee v_{rc8} \vee v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc2} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc3} \mid r, c \in \{1, \dots, 9\} \} \\ & \quad \vdots \\ & \{ \neg v_{rc8} \vee \neg v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \end{aligned}$$

Every row must contain one of each digit:

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit

$$\begin{aligned} & \{ v_{rc1} \vee v_{rc2} \vee \dots \vee v_{rc8} \vee v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc2} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc3} \mid r, c \in \{1, \dots, 9\} \} \\ & \quad \vdots \\ & \{ \neg v_{rc8} \vee \neg v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \end{aligned}$$

Every row must contain one of each digit:

$$\{ \neg v_{r,c,d} \vee \neg v_{r,c',d} \mid r, c, c', d \in \{1, \dots, 9\}, c < c' \}$$

Encoding Sudoku in SAT

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$$\begin{aligned} & \{ v_{rc1} \vee v_{rc2} \vee \dots \vee v_{rc8} \vee v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc2} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc3} \mid r, c \in \{1, \dots, 9\} \} \\ & \quad \vdots \\ & \{ \neg v_{rc8} \vee \neg v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \end{aligned}$$

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Every column must contain one of each digit: similar

Every 3x3 square must contain one of each digit: similar

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit

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2,997 clauses
6,561 literals

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2,916 clauses
5,832 literals

Every column must contain one of each digit: similar

2,916 clauses
5,832 literals

Every 3x3 square must contain one of each digit: similar

2,916 clauses,
5,832 literals

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit

$$\begin{aligned} & \{ v_{rc1} \vee v_{rc2} \vee \dots \vee v_{rc8} \vee v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc2} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc3} \mid r, c \in \{1, \dots, 9\} \} \\ & \quad \vdots \\ & \{ \neg v_{rc8} \vee \neg v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \end{aligned}$$

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Every column must contain one of each digit: similar

2,916 clauses
5,832 literals

Every 3x3 square must contain one of each digit: similar

2,916 clauses,
5,832 literals

729 variables, 11,745 clauses, 24,057 literals, nearly all clauses are binary

Encoding Sudoku in SAT

We have to write down that each cell contains exactly one digit

$$\begin{aligned} & \{ v_{rc1} \vee v_{rc2} \vee \dots \vee v_{rc8} \vee v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc2} \mid r, c \in \{1, \dots, 9\} \} \\ & \{ \neg v_{rc1} \vee \neg v_{rc3} \mid r, c \in \{1, \dots, 9\} \} \\ & \quad \vdots \\ & \{ \neg v_{rc8} \vee \neg v_{rc9} \mid r, c \in \{1, \dots, 9\} \} \end{aligned}$$

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2,916 clauses
5,832 literals

Every column must contain one of each digit: similar

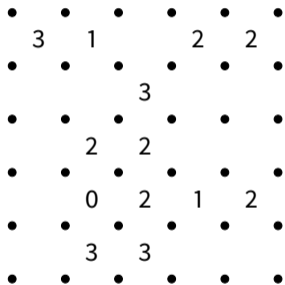
2,916 clauses
5,832 literals

Every 3x3 square must contain one of each digit: similar

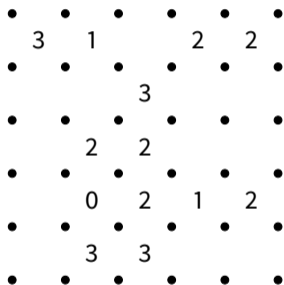
2,916 clauses,
5,832 literals

Finally, we add unit clauses (e.g., v_{129}) corresponding to the initial configuration

Loop the Loop



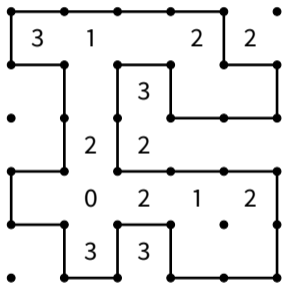
Loop the Loop



You have to draw lines between the dots to form a single loop without crossings or branches.

The numbers indicate how many lines surround it

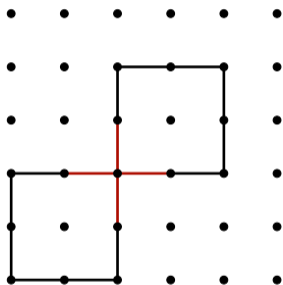
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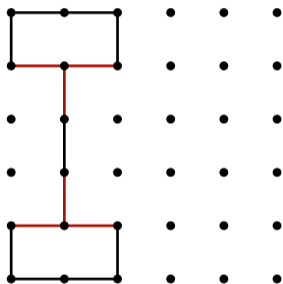
Loop the Loop



You have to draw lines between the dots to form a single loop **without crossings** or branches.

The numbers indicate how many lines surround it

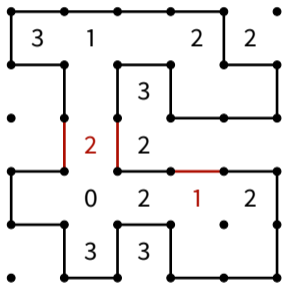
Loop the Loop



You have to draw lines between the dots to form a single loop without crossings or **branches**.

The numbers indicate how many lines surround it

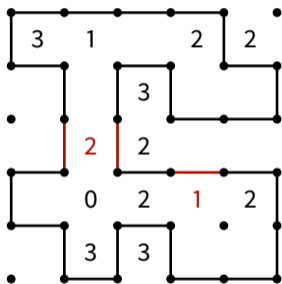
Loop the Loop



You have to draw lines between the dots to form a single loop without crossings or branches.

The numbers indicate **how many lines surround it**

Loop the Loop

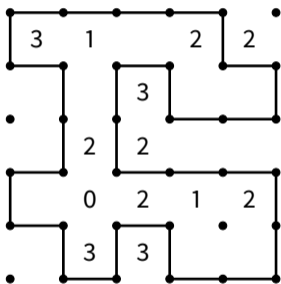


You have to draw lines between the dots to form a single loop without crossings or branches.

The numbers indicate **how many lines surround it**

If a cell **contains a number** m , then **there should be** m **arcs around this number**

Loop the Loop



You have to draw lines between the dots to form a single loop without crossings or branches.

The numbers indicate how many lines surround it

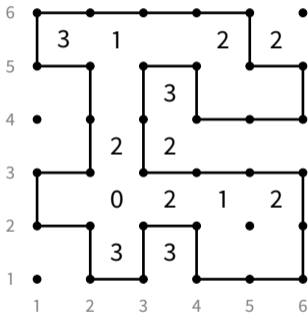
A crossing is a **node with four arcs attached to it**

A branch is a **node with three arcs attached to it**

If a cell contains a number m , then **there should be m arcs around this number**

All these properties are formulated in terms of (a number of) arcs

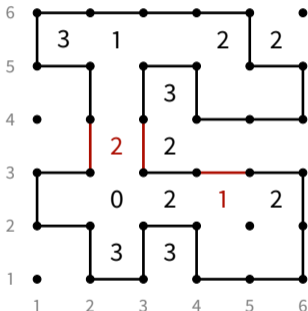
Formalization



Introduce variables denoting arcs:

- v_{ij} : there is a vertical arc between the nodes (i,j) and $(i,j+1)$
- h_{ij} : there is a horizontal arc between the nodes (i,j) and $(i+1,j)$

Formalization

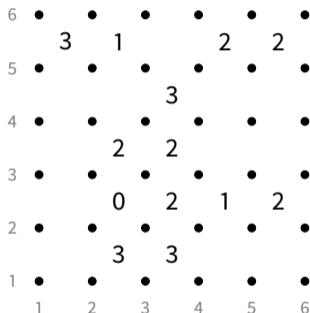


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Example: $v_{23} \wedge v_{33} \wedge h_{43}$

Formalization

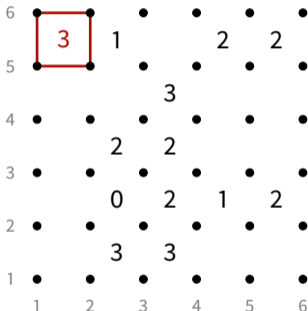


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Then almost all properties are formulated using the formulas T_k and these variables

Formalization



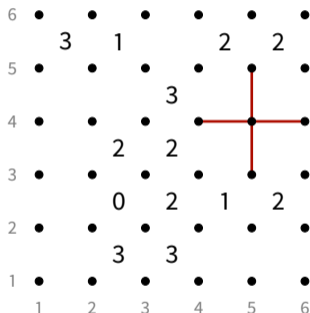
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$$T_3(v_{15}, v_{25}, h_{15}, h_{16})$$

Formalization



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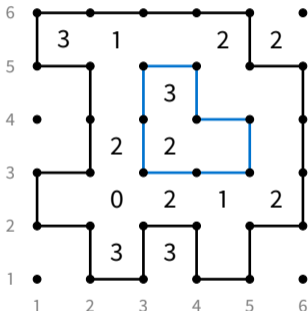
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$$T_3(v_{15}, v_{25}, h_{15}, h_{16})$$

$$T_0(v_{53}, v_{54}, h_{44}, h_{45}) \vee T_2(v_{53}, v_{54}, h_{44}, h_{45})$$

Formalization



Introduce variables denoting arcs:

- v_{ij} : there is a vertical arc between the nodes (i, j) and $(i, j + 1)$
- h_{ij} : there is a horizontal arc between the nodes (i, j) and $(i + 1, j)$

Then almost all properties are formulated using the formulas T_k and these variables For example,

$$T_3(v_{15}, v_{25}, h_{15}, h_{16}) \\ T_0(v_{53}, v_{54}, h_{44}, h_{45}) \vee T_2(v_{53}, v_{54}, h_{44}, h_{45})$$

What we cannot express is **the property to have a single loop**

There is no simple way of expressing this in PL