CS:4420 Artificial Intelligence

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First-Order Logic

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Readings

- Chap. 8 of [Russell and Norvig, 3rd Edition]

Knowledge Representation and Logic

Recall:

The field of Mathematical Logic provides powerful, formal knowledge representation languages and inference systems to build reasoning agents

We will consider two languages, and associated inference systems, from mathematical logic:

- Propositional Logic
- First-order Logic

Pros and cons of Propositional Logic

+ PL is declarative: pieces of syntax correspond to facts
+ PL allows partial/disjunctive/negated information (unlike most data structures and databases)
+ Propositional logic is compositional: meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
+ Meaning in propositional logic is context-independent (unlike natural language, where meaning depends on context)

- Propositional logic has very limited expressive power (unlike natural language)
  E.g., cannot say "pits cause breezes in adjacent squares" except by writing one sentence for each square

First-order Logic

Whereas propositional logic assumes world contains facts,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries, . . .
- Relations: red, round, bogus, prime, brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- Functions: father of, best friend, third inning of, one more than, end of, . . .

Syntax of FOL: Basic elements

Constant symbols	$KingJohn,\ 2,\ Potus,\ [],\ \ldots$
Relation symbols	$Brothers(\_,\_),\ \_ > \_,\ Red(\_),\ \ldots$
Function symbols	$Sqrt(\_),\ LeftLegOf(\_),\ \_ + \_,\ \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge\ \vee\ \neg\ \Rightarrow\ \ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Atomic sentences

Atomic sentence $=$ $relation(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $=$ $function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$,

$Length(LeftLegOf(RobinHood)) > Length(LeftLegOf(KingJohn)))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \;\Leftrightarrow\; S_2$$

E.g. $Siblings(KingJohn, Richard) \Rightarrow Siblings(Richard, KingJohn)$

$x > 2 \;\vee\; 1 < x$

$1 > 2 \;\wedge\; \neg y > 2$

Language of FOL: Grammar

Sentence	$::=$	AtomicS $\mid$ ComplexS
AtomicS	$::=$	**True** $\mid$ **False** $\mid$ RelSymb(Term, . . .) $\mid$ Term $=$ Term
ComplexS	$::=$	(Sentence) $\mid$ Sentence Connective Sentence $\mid$ $\neg$Sentence
	$\mid$	Quantifier Sentence
Term	$::=$	FunSymb(Term, . . .) $\mid$ ConstSymb $\mid$ Variable
Connective	$::=$	$\wedge \mid \vee \mid \Rightarrow \mid \Leftrightarrow$
Quantifier	$::=$	$\forall$ Variable $\mid$ $\exists$ Variable
Variable	$::=$	$a \mid b \mid \cdots \mid x \mid y \mid \cdots$
ConstSymb	$::=$	$A \mid B \mid \cdots \mid John \mid 0 \mid 1 \mid \cdots \mid \pi \mid \ldots$
FunSymb	$::=$	$F \mid G \mid \cdots \mid Cosine \mid Height \mid FatherOf \mid + \mid \ldots$
RelSymb	$::=$	$P \mid Q \mid \cdots \mid Red \mid Brother \mid Apple \mid > \mid \cdots$

Truth in FOL

Sentences are true with respect to a model and an interpretation

A model contains ≥ 1 objects (domain elements) and relations and functions over them them

An interpretation specifies referents for

variables $\rightarrow$ objects

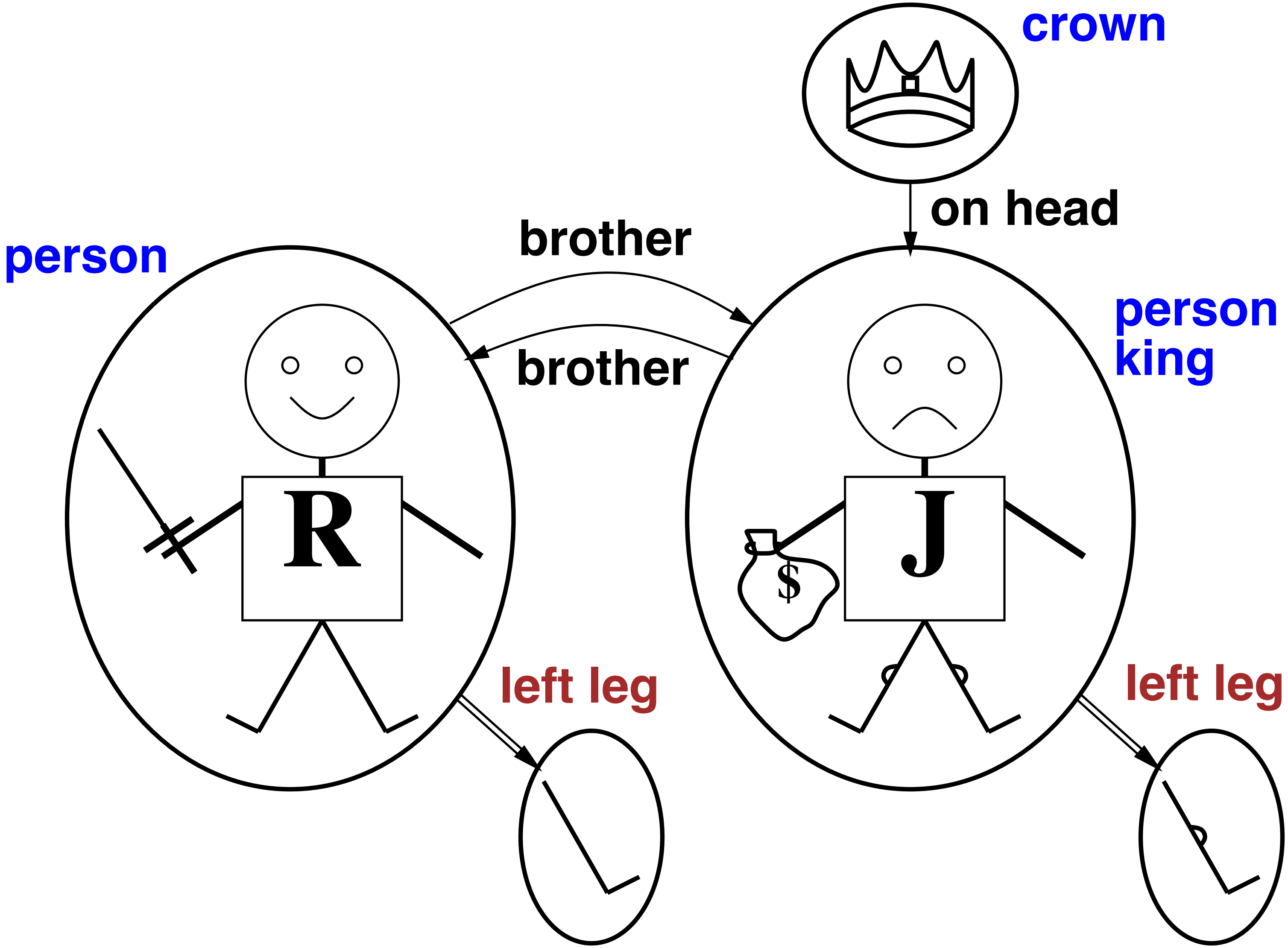
constant symbols $\rightarrow$ objects

predicate symbols $\rightarrow$ relations

function symbols $\rightarrow$ functional relations

An atomic sentence $P(t_1, \ldots, t_n)$ is true in an interpretation iff the objects referred to by $t_1, \ldots, t_n$ are in the relation referred to by P

Models for FOL: Example



Truth example

Consider the interpretation in which

$Richard$ $\rightarrow$ Richard the Lionheart
$John$ $\rightarrow$ the evil King John
$Brother$ $\rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Semantics of First-Order Logic

(A little) more formally:

An *interpretation* $\mathcal{I}$ is a pair $(\mathcal{D}, \sigma)$ where

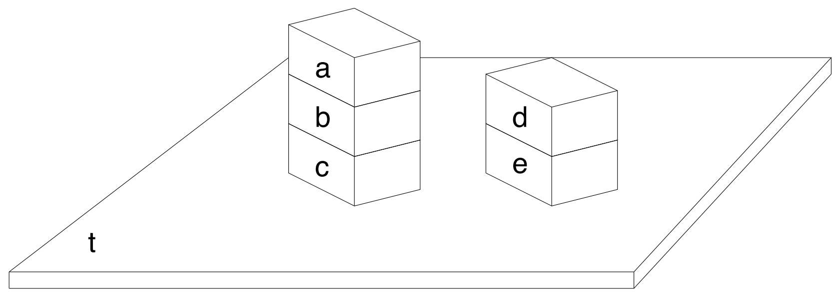
- $\mathcal{D}$ is a set of objects, the universe (or *domain*)
- σ is mapping from variables to objects in $\mathcal{D}$
- $c^{\mathcal{I}}$ is an object in $\mathcal{D}$ for every constant symbol c
- $f^{\mathcal{I}}$ is a function from $\mathcal{D}^n$ to $\mathcal{D}$ for every function symbol f of arity n
- $r^{\mathcal{I}}$ is a relation over $\mathcal{D}^n$ for every relation symbol r of arity n

An Interpretation $\mathcal{I}$ in the Blocks World

Constant Symbols: A, B, C, D, E, T

Function Symbols: $Support$

Relation Symbols: $On, Above, Clear$



$A^{\mathcal{I}} = \mathsf{a},\ B^{\mathcal{I}} = \mathsf{b},\ C^{\mathcal{I}} = \mathsf{c},\ D^{\mathcal{I}} = \mathsf{d},\ E^{\mathcal{I}} = \mathsf{e},\ T^{\mathcal{I}} = \mathsf{t}$

$$Support^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{b}\rangle, \langle \mathsf{b}, \mathsf{c}\rangle, \langle \mathsf{c}, \mathsf{t}\rangle, \langle \mathsf{d}, \mathsf{e}\rangle, \langle \mathsf{e}, \mathsf{t}\rangle, \langle \mathsf{t}, \mathsf{t}\rangle\}$$

$$On^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{b}\rangle, \langle \mathsf{b}, \mathsf{c}\rangle, \langle \mathsf{c}, \mathsf{t}\rangle, \langle \mathsf{d}, \mathsf{e}\rangle, \langle \mathsf{e}, \mathsf{t}\rangle\}$$

$$Above^{\mathcal{I}} = \{\langle \mathsf{a}, \mathsf{b}\rangle, \langle \mathsf{a}, \mathsf{c}\rangle, \langle \mathsf{a}, \mathsf{t}\rangle, \ldots\}$$

$$Clear^{\mathcal{I}} = \{\langle \mathsf{a}\rangle, \langle \mathsf{d}\rangle\}$$

Semantics of First-Order Logic

Let $\mathcal{I} = (\mathcal{D}, \sigma)$ be an interpretation and E an expression of FOL

We write $[\![e]\!]^{\mathcal{I}}$ to denote the *meaning of e in $\mathcal{I}$*

The meaning $[\![t]\!]^{\mathcal{I}}$ of a term t is an object of $\mathcal{D}$, inductively defined as follows:

$[\![x]\!]^{\mathcal{I}}$	$:=$	$\sigma(x)$	for all variables x
$[\![c]\!]^{\mathcal{I}}$	$:=$	$c^{\mathcal{I}}$	for all constant symbols c
$[\![f(t_1, \ldots, t_n)]\!]^{\mathcal{I}}$	$:=$	$f^{\mathcal{I}}([\![t_1]\!]^{\mathcal{I}}, \ldots, [\![t_n]\!]^{\mathcal{I}})$	for all n-ary function symbols f

Example

Consider the symbols $\mathit{MotherOf}$, $\mathit{SpouseOf}$ and the interpretation $\mathcal{I} = (\mathcal{D}, \sigma)$ where

$$\begin{array}{rl} \mathit{MotherOf}^{\mathcal{I}} & \text{is a unary fn mapping people to their mother} \\ \mathit{SpouseOf}^{\mathcal{I}} & \text{is a unary fn mapping people to their spouse} \\ \sigma & := \{x \mapsto \text{Bart},\ y \mapsto \text{Homer}, \ldots\} \end{array}$$

What is the meaning of $\mathit{SpouseOf}(\mathit{MotherOf}(x))$ in $\mathcal{I}$?

$$\begin{array}{rcl} [\![\mathit{SpouseOf}(\mathit{MotherOf}(x))]\!]^{\mathcal{I}} & = & \mathit{SpouseOf}^{\mathcal{I}}([\![\mathit{MotherOf}(x)]\!]^{\mathcal{I}}) \\ & = & \mathit{SpouseOf}^{\mathcal{I}}(\mathit{MotherOf}^{\mathcal{I}}([\![x]\!]^{\mathcal{I}})) \\ & = & \mathit{SpouseOf}^{\mathcal{I}}(\mathit{MotherOf}^{\mathcal{I}}(\sigma(x))) \\ & = & \mathit{SpouseOf}^{\mathcal{I}}(\mathit{MotherOf}^{\mathcal{I}}(\text{Bart})) \\ & = & \mathit{SpouseOf}^{\mathcal{I}}(\text{Marge}) \\ & = & \text{Homer} \end{array}$$

Semantics of First-Order Logic

Let $\mathcal{I} = (\mathcal{D}, \sigma)$ be an interpretation

The meaning $[\![\varphi]\!]^{\mathcal{I}}$ of a formula φ is either True or False

It is inductively defined as follows:

$[\![t_1 = t_2]\!]^{\mathcal{I}}$	$:=$	True	iff	$[\![t_1]\!]^{\mathcal{I}}$ is the same as $[\![t_2]\!]^{\mathcal{I}}$
$[\![r(t_1, \ldots, t_n)]\!]^{\mathcal{I}}$	$:=$	True	iff	$\langle [\![t_1]\!]^{\mathcal{I}}, \ldots, [\![t_n]\!]^{\mathcal{I}} \rangle \in r^{\mathcal{I}}$
$[\![\neg\varphi]\!]^{\mathcal{I}}$	$:=$	$\mathit{True}/\mathit{False}$	iff	$[\![\varphi]\!]^{\mathcal{I}} = \mathit{False}/\mathit{True}$
$[\![\varphi_1 \vee \varphi_2]\!]^{\mathcal{I}}$	$:=$	True	iff	$[\![\varphi_1]\!]^{\mathcal{I}} = \mathit{True}$ or $[\![\varphi_2]\!]^{\mathcal{I}} = \mathit{True}$
$[\![\exists x\ \varphi]\!]^{\mathcal{I}}$	$:=$	True	iff	$[\![\varphi]\!]^{\mathcal{I}}_{\sigma'} = \mathit{True}$ for some σ' that disagrees with σ at most on x

Semantics of First-Order Logic

Let $\mathcal{I} = (\mathcal{D}, \sigma)$ be an interpretation

The meaning of formulas built with the other logical symbols:

$$
\begin{array}{lcl}
[\![\varphi_1 \land \varphi_2]\!]^{\mathcal{I}} & := & [\![\neg(\neg\varphi_1 \lor \neg\varphi_2)]\!]^{\mathcal{I}} \\
[\![\varphi_1 \Rightarrow \varphi_2]\!]^{\mathcal{I}} & := & [\![\neg\varphi_1 \lor \varphi_2]\!]^{\mathcal{I}} \\
[\![\varphi_1 \Leftrightarrow \varphi_2]\!]^{\mathcal{I}} & := & [\![(\varphi_1 \Rightarrow \varphi_2) \land (\varphi_2 \Rightarrow \varphi_1)]\!]^{\mathcal{I}} \\
[\![\forall x \varphi]\!]^{\mathcal{I}} & := & [\![\neg\exists x \, \neg\varphi]\!]^{\mathcal{I}}
\end{array}
$$

If a sentence is *closed*, i.e., it has no *free* variables, its meaning does not depend on the the variable assignment—although it may depend on the domain:

$$[\![\forall x \, \exists y \, R(x, y)]\!]^{\mathcal{I}} = [\![\forall x \, \exists y \, R(x, y)]\!]^{\mathcal{I}'} \quad \text{for any} \quad \mathcal{I}' = (\mathcal{D}, \sigma')$$

Models, Validity, etc. for Sentences

An interpretation $\mathcal{I} = (\mathcal{D}, \sigma)$ *satisfies* a sentence φ, or is a *model of* φ, if $[\![\varphi]\!]^{\mathcal{I}} = True$

A sentence is *satisfiable* if it has at least one model

Ex: $\forall x\, x \geq y$, $P(x)$

A sentence is unsatisfiable if it has no models

Ex: $P(x) \wedge \neg P(x)$, $\neg(x = x)$, $(\forall x\, Q(x, y)) \Rightarrow \neg Q(a, b)$

A sentence φ is *valid* if every interpretation is a model of it

Ex: $P(x) \Rightarrow P(x)$, $x = x$, $(\forall x\, P(x)) \Rightarrow \exists x\, P(x)$

Note: φ *is valid/unsatisfiable iff* $\neg\varphi$ *is unsatisfiable/valid*

Models, Validity, etc. for Sets of Sentences

An interpretation $(\mathcal{D}, \sigma)$ *satisfies* a set Γ of sentences, or is a *model of* Γ, if it is a model for *every* sentence in Γ

A set Γ of sentences is *satisfiable* if it has at least one model

Ex: $\{\forall x\, x \geq 0,\ \forall x\, x + 1 > x\}$

Γ is *unsatisfiable*, or *inconsistent*, if it has no models

Ex: $\{P(x),\ \neg P(x)\}$

Γ *entails* a sentence φ ($\Gamma \models \varphi$), if every model for Γ is also a model for φ

Ex: $\{\forall x\, P(x) \Rightarrow Q(x),\ P(A_{10})\} \models Q(A_{10})$

Note: As in propositional logic, $\Gamma \models \varphi$ iff $\Gamma \wedge \neg\varphi$ is unsatisfiable

Possible Interpretations Semantics

Sentences can be seen as constraints on the set S of all possible interpretations.

A sentence denotes all the possible interpretations that satisfy it (the models of φ):

If φ_1 denotes a set of interpretations S_1 and φ_2 denotes a set S_2, then

- $\varphi_1 \vee \varphi_2$ denotes $S_1 \cup S_2$,
- $\varphi_1 \wedge \varphi_2$ denotes $S_1 \cap S_2$,
- $\neg\varphi_1$ denotes $S \setminus S_1$,
- $\varphi_1 \models \varphi_2$ iff $S_1 \subseteq S_2$.

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Note 1: A sentence denotes either no interpretations or an infinite number of them!

Note 2: Valid sentences do not tell us anything about the world. They are satisfied by every possible interpretation!

Models for FOL: Lots!

We can enumerate the models for a given FOL sentence:

For each number of universe elements n from 1 to ∞
    For each k-ary predicate P_k in the sentence
        For each possible k-ary relation on n objects
            For each constant symbol C in the sentence
                For each one of n objects mapped to C
                ...

Enumerating models is not going to be easy!

Universal quantification

$\forall \langle variables \rangle \ \langle sentence \rangle$

Everyone at Berkeley is smart:
$\forall x \ At(x, Berkeley) \Rightarrow Smart(x)$

$\forall x \ P$ is true in an interpretation $\mathcal{I}$ iff P is true with x being each possible object in $\mathcal{I}$'s domain

Roughly speaking, equivalent to the conjunction of instantiations of P

$$
\begin{array}{ll}
 & (At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn)) \\
\wedge & (At(Richard, Berkeley) \Rightarrow Smart(Richard)) \\
\wedge & (At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley)) \\
\wedge & \ldots
\end{array}
$$

Existential quantification

$\exists \langle variables \rangle \; \langle sentence \rangle$

Someone at Stanford is smart:
$\exists x \;\; At(x, Stanford) \wedge Smart(x)$

$\exists x \;\; P$ is true in an interpretation $\mathcal{I}$ iff P is true with x being some possible object in $\mathcal{I}$'s domain

Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{array}{ll}
 & (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\
\vee & (At(Richard, Stanford) \wedge Smart(Richard)) \\
\vee & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\
\vee & \ldots
\end{array}
$$

Properties of quantifiers

$\forall x \ \forall y \ \varphi$ is equivalent to $\forall y \ \forall x \ \varphi$ (why?)

$\exists x \ \exists y \ \varphi$ is equivalent to $\exists y \ \exists x \ \varphi$ (why?)

$\exists x \ \forall y \ \varphi$ is not equivalent to $\forall y \ \exists x \ \varphi$

Ex.

$\exists x \ \forall y \ \ Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y \ \exists x \ \ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x \ \ Likes(x, IceCream) \qquad \neg\exists x \ \ \neg Likes(x, IceCream)$

$\exists x \ \ Likes(x, Broccoli) \qquad \neg\forall x \ \ \neg Likes(x, Broccoli)$

From English prepositions to FOL connectives

<table>
<tr><th>English</th><th>Logic</th></tr>
<tr><td>A and B | A but B</td><td>$A \wedge B$</td></tr>
<tr><td>A if B | A when B | A whenever B</td><td>$B \Rightarrow A$</td></tr>
<tr><td>if A, then B | A implies B | A forces B</td><td>$A \Rightarrow B$</td></tr>
<tr><td>only if A, B | B only if A |</td><td>$B \Rightarrow A$</td></tr>
<tr><td>A precisely when B | A if and only if B</td><td>$B \Leftrightarrow A$ | $A \Leftrightarrow B$</td></tr>
<tr><td>A or B (or both) | A unless B</td><td>$A \vee B$ (logical or)</td></tr>
<tr><td>either A or B (but not both)</td><td>$A \oplus B$ (exclusive or)</td></tr>
</table>

A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall\, x \;\; At(x, Berkeley) \wedge Smart(x)$$

means “Everyone is at Berkeley and everyone is smart”

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Compare with

$$\forall x \;\; At(x, Berkeley) \Rightarrow Smart(x)$$

"Everyone at Berkeley is smart"

Another common mistake to avoid

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Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

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Compare with

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"Someone at Stanford is smart"

Fun with sentences

Brothers are siblings

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Fun with sentences

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"Siblings" is symmetric

$\forall\, x, y \;\; Siblings(x, y) \;\Leftrightarrow\; Siblings(y, x)$

Fun with sentences

Brothers are siblings

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"Siblings" is symmetric

$\forall\, x, y \;\; Siblings(x, y) \;\Leftrightarrow\; Siblings(y, x)$

One's mother is one's female parent

Fun with sentences

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"Siblings" is symmetric

$\forall x, y \;\; Siblings(x, y) \;\Leftrightarrow\; Siblings(y, x)$

One's mother is one's female parent

$\forall x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$

Fun with sentences

Brothers are siblings

$\forall\, x, y\;\; Brothers(x, y) \Rightarrow Siblings(x, y)$

"Siblings" is symmetric

$\forall\, x, y\;\; Siblings(x, y) \;\Leftrightarrow\; Siblings(y, x)$

One's mother is one's female parent

$\forall\, x, y\;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$

A first cousin is a child of a parent's sibling

Fun with sentences

Brothers are siblings

$$\forall x, y \;\; Brothers(x, y) \Rightarrow Siblings(x, y)$$

"Siblings" is symmetric

$$\forall x, y \;\; Siblings(x, y) \;\Leftrightarrow\; Siblings(y, x)$$

One's mother is one's female parent

$$\forall x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$$

A first cousin is a child of a parent's sibling

$$\forall x_1, x_2 \;\; FirstCousin(x_1, x_2) \;\Leftrightarrow\; \exists p_1, p_2 \;\; Siblings(p_1, p_2) \wedge Parent(p_1, x_1) \wedge Parent(p_2, x_2)$$

Fun with sentences

Brothers are siblings

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$$\forall\, x_1, x_2 \;\; FirstCousin(x_1, x_2) \;\Leftrightarrow\; \exists\, p_1, p_2 \;\; Siblings(p_1, p_2) \wedge Parent(p_1, x_1) \wedge Parent(p_2, x_2)$$

Dogs are mammals

Fun with sentences

Brothers are siblings

$\forall x, y \;\; Brothers(x, y) \Rightarrow Siblings(x, y)$

"Siblings" is symmetric

$\forall x, y \;\; Siblings(x, y) \;\Leftrightarrow\; Siblings(y, x)$

One's mother is one's female parent

$\forall x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \land Parent(x, y))$

A first cousin is a child of a parent's sibling

$$\forall x_1, x_2 \;\; FirstCousin(x_1, x_2) \;\Leftrightarrow\; \exists p_1, p_2 \;\; Siblings(p_1, p_2) \land Parent(p_1, x_1) \land Parent(p_2, x_2)$$

Dogs are mammals

$\forall x \;\; Dog(x) \Rightarrow Mammal(x)$

Equality

Recall that $t_1 = t_2$ is true under a given interpretation if and only if t_1 and t_2 refer to the same object

E.g., $1 = 2$ and $x * x = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall\, x, y \;\; Siblings(x, y) \;\Leftrightarrow\; [\neg(x = y) \land \exists\, m, f \;\; \neg(m = f) \land Parent(m, x) \land Parent(f, x) \land Parent(m, y) \land Parent(f, y)]$$

More fun with sentences

1. No one is his/her own sibling
2. Sisters are female, brothers are male
3. Every one is male or female but not both
4. Every married person has a spouse
5. Married people have spouses
6. Only married people have spouses
7. People cannot be married to their siblings
8. Not everybody has a spouse
9. Everybody has a mother
10. Everybody has a mother and only one

More fun with sentences

1. $\forall x\ \neg Siblings(x, x)$
2. $$\forall x, y \quad (Sisters(x, y) \Rightarrow Female(x) \wedge Female(y)) \wedge$$ $$(Brothers(x, y) \Rightarrow Male(x) \wedge Male(y))$$
3. $$\forall x\ \ Person(x) \Rightarrow (Male(x) \vee Female(x)) \wedge$$ $$\neg(Male(x) \wedge Female(x))$$
4. $\forall x\ \ (Person(x) \wedge Married(x)) \Rightarrow \exists y\ \ Spouse(x, y)$
5. $\forall x\ \ (Person(x) \wedge Married(x)) \Rightarrow \exists y\ \ Spouse(x, y)$
6. $\forall x, y\ \ (Person(x) \wedge Person(y) \wedge Spouse(x, y)) \Rightarrow Married(x) \wedge Married(y)$
7. $\forall x, y\ \ Spouse(x, y) \Rightarrow \neg Siblings(x, y)$
8. $\neg\forall x\ \ Person(x) \Rightarrow \exists y\ \ Spouse(x, y)$
   Alter.: $\exists x\ \ Person(x) \wedge \neg\exists y\ \ Spouse(x, y)$
9. $\forall x\ \ Person(x) \Rightarrow \exists y\ \ Mother(y, x)$
10. $$\forall x\ \ Person(x) \Rightarrow \exists y\ \ Mother(y, x) \wedge$$ $$\neg\exists z\ \ \neg(y = z) \wedge Mother(z, x)$$