CS:5810
Formal Methods in Software Engineering

Sets and Relations

These Notes

- review the concepts of sets and relations required for working with the Alloy language
- focus on the kind of set operation and definitions used in specifications
- give some small examples of how we will use sets in specifications

Set

- Collection of distinct objects
- Each set's objects are drawn from a larger *domain* of objects all of which have the same type --- sets are homogeneous
- Examples:

{2,4,5,6,…}	set of integers	*domain*
{red, yellow, blue}	set of colors	
{true, false}	set of boolean values	
{red, true, 2}	for us, not a set!	

Value of a Set

- Is the collection of its members

- Two sets A and B are equal iff
  - every member of A is a member of B
  - every member of B is a member of A

- $x \in S$ denotes "x is a member of S"
- Ø denotes the empty set

Defining Sets

- We can define a set by *enumeration*
  - PrimaryColors == {red, yellow, blue}
  - Boolean == {true, false}
  - Evens == {..., -4, -2, 0, 2, 4, ...}

- This works fine for finite sets, but
  - what do we mean by "..." ?
  - remember, we want to be precise

Defining Sets

- We can define a set by *comprehension*, that is, by describing a property that its elements must share
- Notation: $\{ x : D \mid P(x) \}$
  - Form a new set of elements drawn from domain D by including exactly the elements that satisfy predicate (i.e., Boolean function) P
- Examples:

$\{ x : \mathrm{N} \mid x < 10\}$ *Naturals less than 10*

$\{ x : \mathrm{Z} \mid (\exists\, y : \mathrm{Z} \mid x = 2y) \}$ *Even integers*

$\{ x : \mathrm{N} \mid x > x\}$ *Empty set of natural numbers*

Cardinality

- The *cardinality* (#) of a finite set is the number of its elements
- Examples:
  - – # {red, yellow, blue} = 3
  - – # {1, 23} = 2
  - – # $\mathbb{Z}$ = ?
- Cardinalities are defined for infinite sets too, but we'll be most concerned with the cardinality of finite sets

Set Operations

- Union (X, Y sets over domain D):
  - $X \cup Y \equiv \{e: D \mid e \in X \text{ or } e \in Y\}$
  - $\{red\} \cup \{blue\} = \{red, blue\}$
- Intersection
  - $X \cap Y \equiv \{e: D \mid e \in X \text{ and } e \in Y\}$
  - $\{red, blue\} \cap \{blue, yellow\} = \{blue\}$
- Difference
  - $X \setminus Y \equiv \{e: D \mid e \in X \text{ and } e \notin Y\}$
  - $\{red, yellow, blue\} \setminus \{blue, yellow\} = \{red\}$

Subsets

- A *subset* holds elements drawn from another set
  - $X \subseteq Y$ iff every element of X is in Y
  - $\{1, 7, 17, 24\} \subseteq Z$
- A *proper subset* is a non-equal subset
- Another view of set equality
  - $A = B$ iff ($A \subseteq B$ and $B \subseteq A$)

Power Sets

- The power set of set S (denoted *Pow*(S)) is the set of all subsets of S, i.e.,

$$Pow(S) \equiv \{e \mid e \subseteq S\}$$

- Example:
  - *Pow*({a,b,c}) = {∅, {a}, {b}, {c},
    {a,b}, {a,c}, {b,c},
    {a,b,c}}

Note: for any S, $\emptyset \subseteq S$ and thus $\emptyset \in Pow(S)$

Exercises

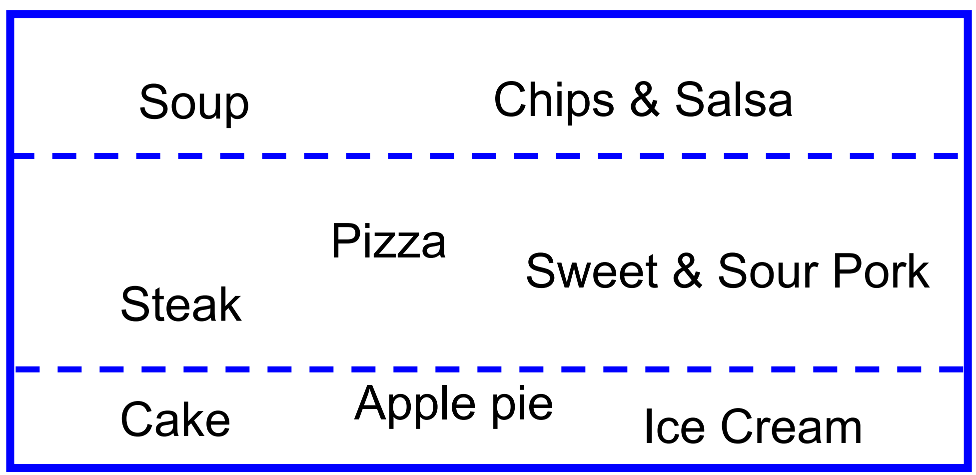
- These slides include questions that you should be able to solve at this point
- They may require you to think some
- You should spend some effort in solving them
  - … and may in fact appear on exams

Exercises

- Specifying using comprehension notation
  - Odd positive integers
  - The squares of integers, i.e. {1,4,9,16,…}
- Express the following logic properties on sets without using the # operator
  - Set has at least one element
  - Set has no elements
  - Set has exactly one element
  - Set has at least two elements
  - Set has exactly two elements

Set Partitioning

- Sets are *disjoint* if they share no elements
- Often when modeling, we will take some set S and divide its members into disjoint subsets called *blocks* or *parts*
- We call this division a *partition*
- Each member of S belongs to exactly one block of the partition



Example

Model residential scenarios

- Basic domains: *Person*, *Residence*
- Partitions:
  - Partition *Person* into *Child*, *Adult*
  - Partition *Residence* into *Home*, *DormRoom*, *Apartment*

Expressing Relationships

- It's useful to be able to refer to structured values
  - a group of values that are bound together
  - e.g., struct, record, object fields
- Alloy is a calculus of *relations*
- All of our Alloy models will be built using relations (sets of tuples)
- … but first some basic definitions

Product

- Given two sets A and B, the product of A and B, usually denoted A x B, is the set of all possible pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B \equiv \{ (a, b) \mid a \in A, b \in B \}$$

- Example: PrimaryColor x Boolean:

$$\left\{ \begin{array}{ll} \text{(red,true),} & \text{(red, false),} \\ \text{(blue,true),} & \text{(blue, false),} \\ \text{(yellow, true),} & \text{(yellow, false)} \end{array} \right\}$$

Relation

- A binary relation R between A and B is an element of *Pow* (A x B), i.e., $R \subseteq A \times B$

- Examples:
  – Parent : Person x Person
    - Parent = { (John, Autumn), (John, Sam) }
  – Square : Z x N
    - Square = {(1,1), (-1,1), (-2,4)}
  – ClassGrades : Person x {A, B, C, D, F}
    - ClassGrades = { (Todd,A), (Jane,B) }

Relation

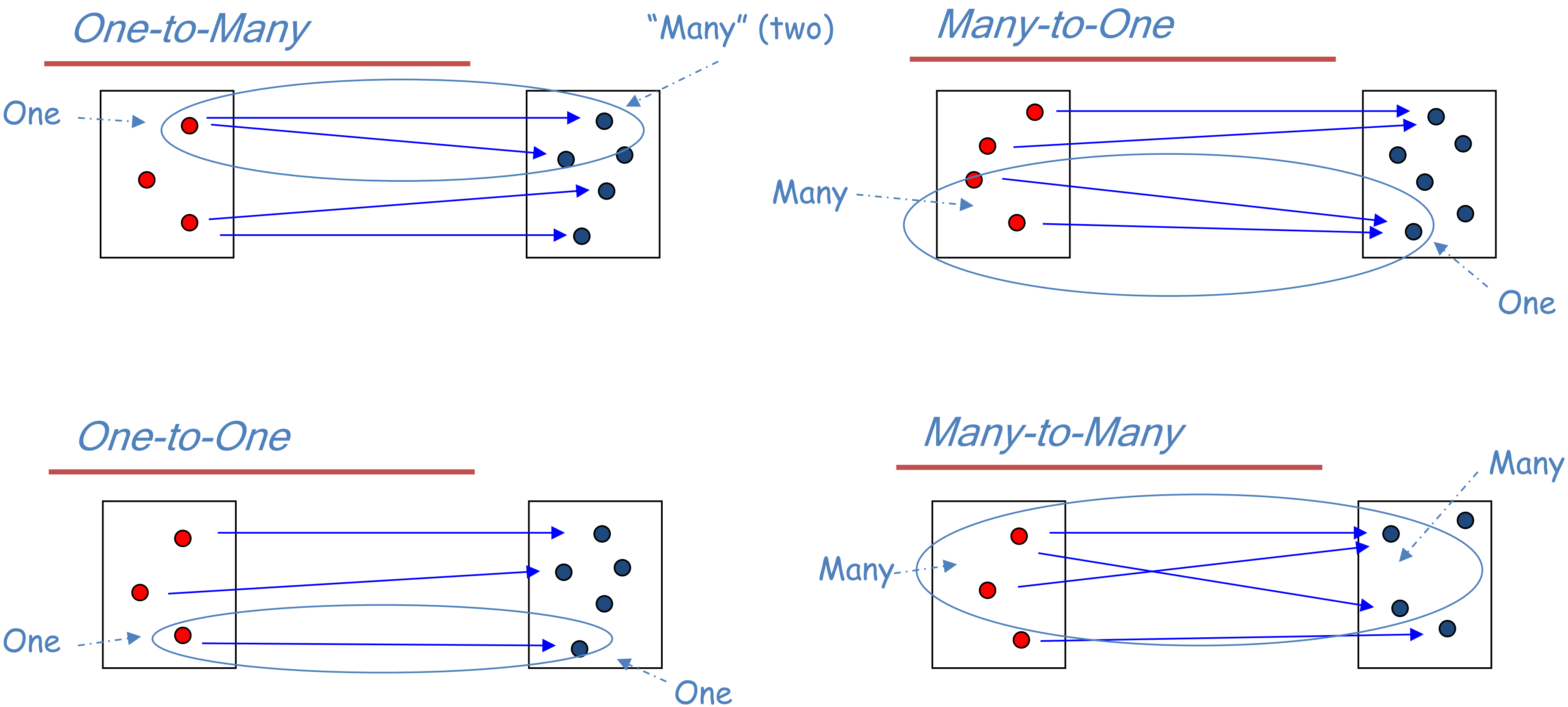
- A ternary relation R between A, B and C is an element of *Pow* (A x B x C)

- Example:
  - FavoriteBeer : Person x Beer x Price
    - FavoriteBeer = { (John, Miller, \$2), (Ted, Heineken, \$4), (Steve, Miller, \$2) }
- N-ary relations with n>3 are defined analogously (n is the arity of the relation)

Binary Relations

- The set of first elements is the *definition domain* of the relation
  - Parent = { (John, Autumn), (John, Sam) }
  - *domain* (Parent) = {John}   NOT Person!
- The set of second elements is the *image* of the relation
  - *image* (Square) = {1,4}   NOT $\mathbb{N}$!
- How about {(1,blue), (2,blue), (1,red)}
  - domain?   image?

Common Relation Structures



Functions

- A *function* is a relation F of arity n+1 containing no two distinct tuples with the same first n elements,
  - i.e., for n = 1,

    $\forall\ (a_1, b_1) \in F, \forall\ (a_2, b_2) \in F, (a_1 = a_2 \Rightarrow b_1 = b_2)$
- Examples:
  - { (2, red), (3, blue), (5, red) }
  - { (4, 2), (6,3), (8, 4) }
- Instead of F: A1 x A2 x … x An x B

  we write F: A1 x A2 x … x An -> B

Exercises

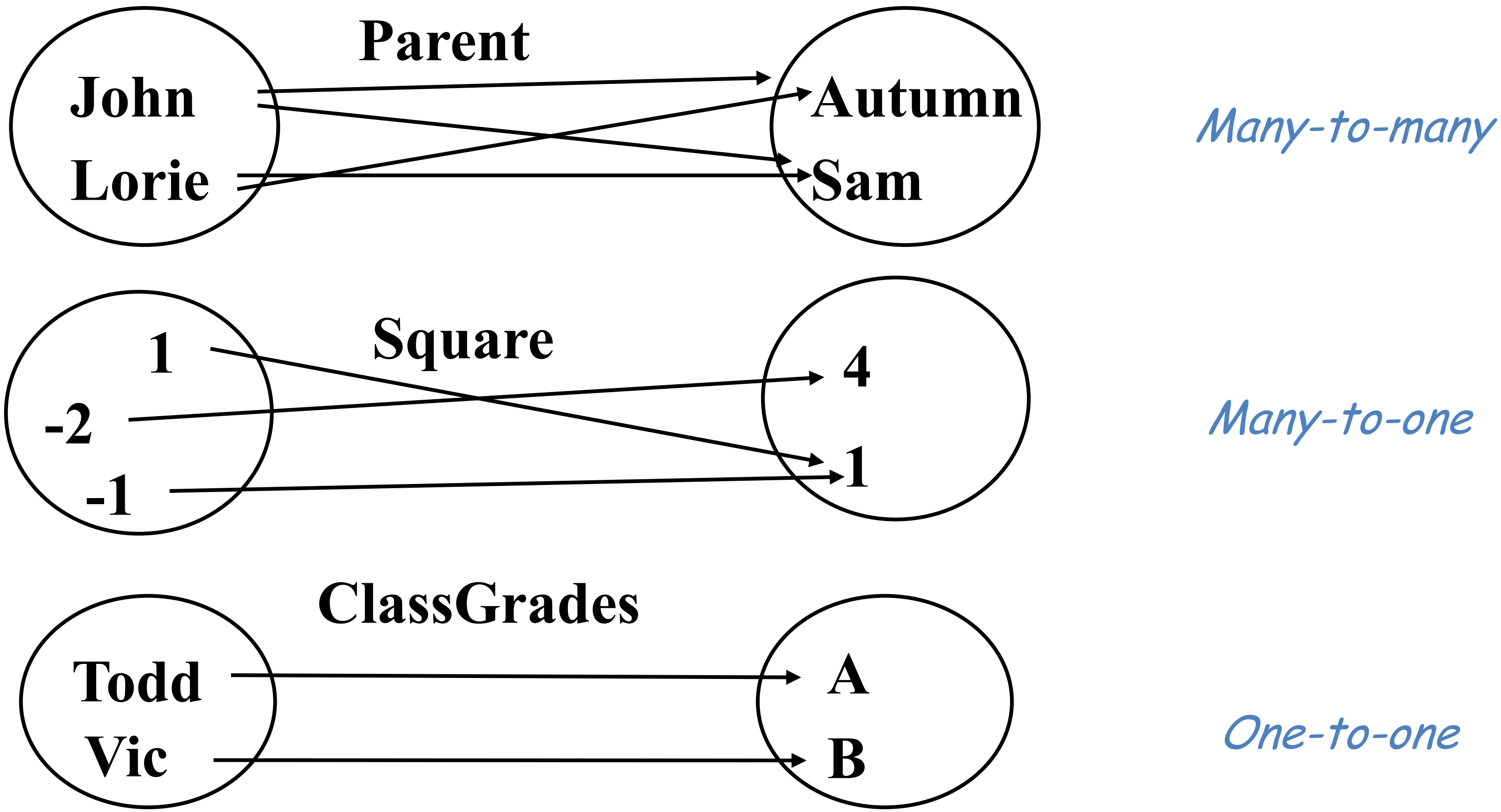
- Which of the following are functions?

1. Parent = { (John, Autumn), (John, Sam) }

2. Square = { (1, 1), (-1, 1), (-2, 4) }

3. ClassGrades = { (Todd, A), (Vic, B) }

Relations vs. Functions



In other words, a function is a relation that is X-to-one.

Special Kinds of Functions

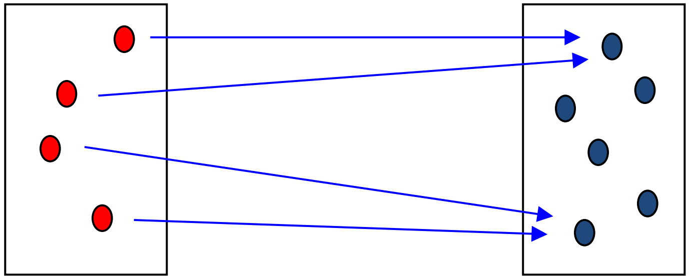
- Consider a function f from S to T

- f is *total* if defined for all values of S
- f is *partial* if undefined for some values of S

- Examples
  - Squares : $\mathrm{Z} \to \mathrm{N}$, Squares = {..., (-1,1), (0,0), (1, 1), (2,4), ...}
  - SquareRoot : $\mathrm{N} \to \mathrm{N} = \{ (x, y) : \mathrm{N} \times \mathrm{N} \mid y^2 = x \}$

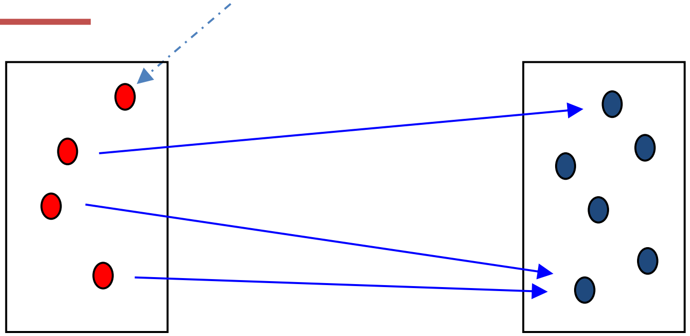
Function Structures

Total Function



Partial Function

Undefined for this input



Note: the empty relation over an non-empty domain is a partial function

Special Kinds of Functions

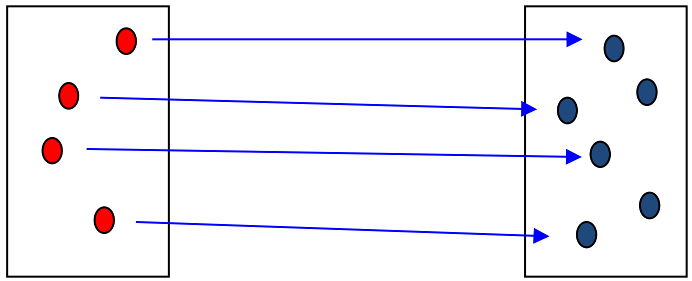
A function f: S -> T is

- *injective* (*one-to-one*) if no image element is associated with multiple domain elements
- *surjective* (*onto*) if its image is T
- *bijective* if it is both injective and surjective

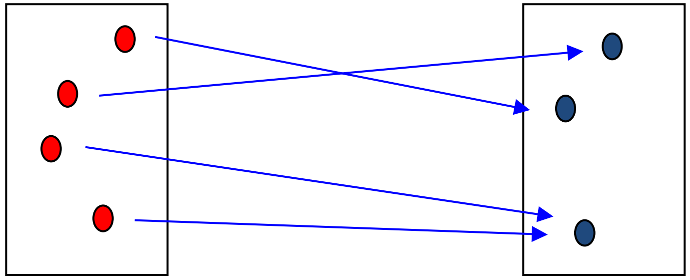
We'll see that these come up frequently
- can be used to define properties concisely

Function Structures

Injective Function



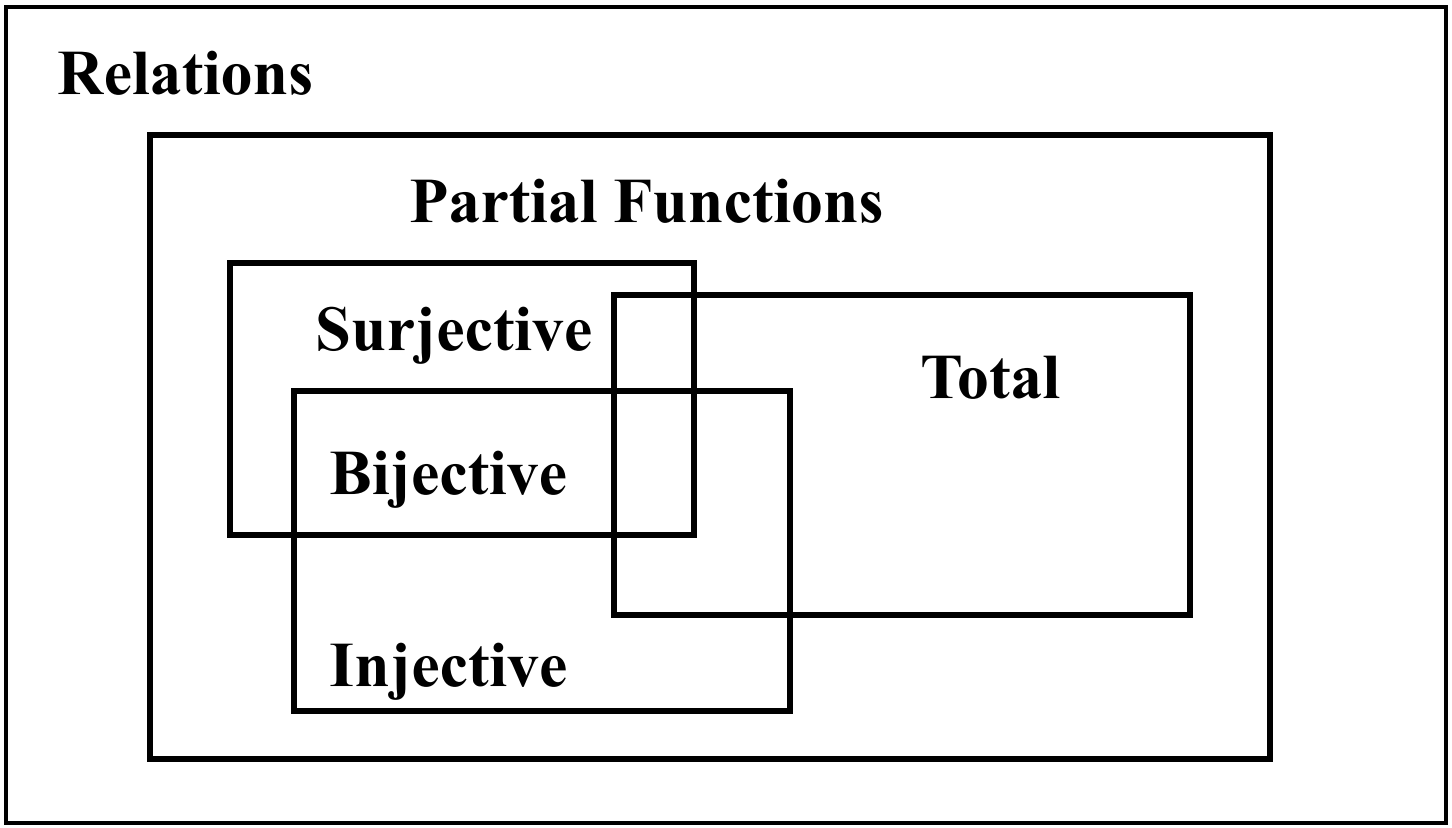
Surjective Function



Exercises

- What kind of function/relation is Abs?
  - Abs = { (x, y) : $\mathbb{Z}$ x $\mathbb{N}$ | (x < 0 and y = -x) or (x ≥ 0 and y = x) }
- How about Squares?
  - Squares : $\mathbb{Z}$ x $\mathbb{N}$, Squares = { (x, y) : $\mathbb{Z}$ x $\mathbb{N}$ | y = x*x }

Special Cases



Functions as Sets

- Functions are relations and hence sets
- We can apply to them all the usual operators
  - ClassGrades = { (Todd, A), (Jane, B) }
  - #(ClassGrades U { (Matt, C) }) = 3

Exercises

- In the following if an operator fails to preserve a property give an example
- What operators preserve function-ness?
  - $\cap$?
  - $\cup$?
  - $\setminus$?
- What operators preserve surjectivity?
- What operators preserve injectivity?

Relation Composition

- Use two relations to produce a new one
  - map domain of first to image of second
  - Given s: A x B and r: B x C then s;r : A x C

$$s;r \equiv \{ (a,c) \mid (a,b) \in s \text{ and } (b,c) \in r \}$$

- For example
  - s = { (red,1), (blue,2) }
  - r = { (1,2), (2,4), (3,6) }
  - s;r = { (red,2), (blue,4) }

Not limited to binary relations

Relation Transitive Closure

- Intuitively, the transitive closure of a binary relation $r: S \times S$, written r^+, is what you get when you keep navigating through r until you can't go any farther.

$$r^+ \equiv r \cup (r;r) \cup (r;r;r) \cup \ldots$$

- Formally, $r^+ \equiv$ smallest transitive relation containing r

- For example
  - GrandParent = Parent;Parent
  - Ancestor = Parent^+

Relation Transpose

- Intuitively, the transpose of a relation r: $S \times T$, written $\sim r$, is what you get when you reverse all the pairs in r

$$\sim r \equiv \{ (b,a) \mid (a,b) \in r \}$$

- For example
  - ChildOf = ~Parent
  - DescendantOf = $(\sim\text{Parent})^{+}$

Exercises

- What properties, i.e., function-ness, onto-ness, 1-1-ness, are preserved by these relation operators?
  - composition (;)
  - closure ($^+$)
  - transpose (~)
- If an operator fails to preserve a property give an example

Acknowledgements

Some of these slides are adapted from

David Garlan's slides from Lecture 3 of his course of Software Models entitled "Sets, Relations, and Functions" (http://www.cs.cmu.edu/afs/cs/academic/class/15671-f97/www/)