Lecture Notes CS:5360 Randomized Algorithms
Week 9: October 22, 2018
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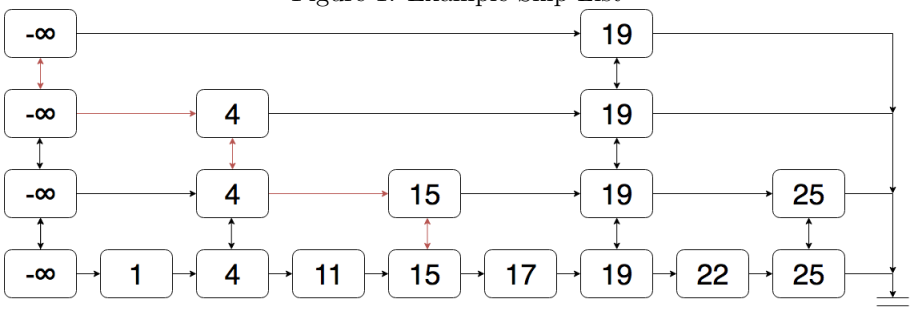
1 Skip Lists

GOAL: Implement a simple data structure that supports search, insert, and delete in $O(\log n)$ time each.

1.1 Initial Idea: "Deterministic" Skip Lists

The data structure contains some number of levels $(L_0, L_1, ..., L_N)$. Each level is an ordered list of elements. L_0 contains all elements. The set of elements in $L_{i+1} \subseteq$ set of elements in L_i.

Figure 1: Example Skip List



We used the deterministic rule "keep every alternate element" in going from L_i to L_{i+1}. **Search 16** is represented by the red arrows in the above picture. Elements in the search path are $-\infty, -\infty, 4, 4, 15, 15$. Search returns the largest element smaller than the searched number (15 in this case).
-Deterministic rule is hard to maintain for insert and delete so we will try to use randomization.

1.2 Probabilistic Rule

Each element in L_i is chosen independently with probability $\frac{1}{2}$ to be copied to L_{i+1} Search works as before.

1.2.1 Insert

Insert k works as follows:

- First search for k and find search path $e_t, e_{t-1}, ..., e_0$
- Insert k immediately after e_0
- We toss a fair coin to determine if k should be inserted into L_1

- If we determine k should be inserted, we walk back the search path to the first element e_i in Level L_1
- We insert k immediately after e_i in L_1
- We continue coin tossing until tails

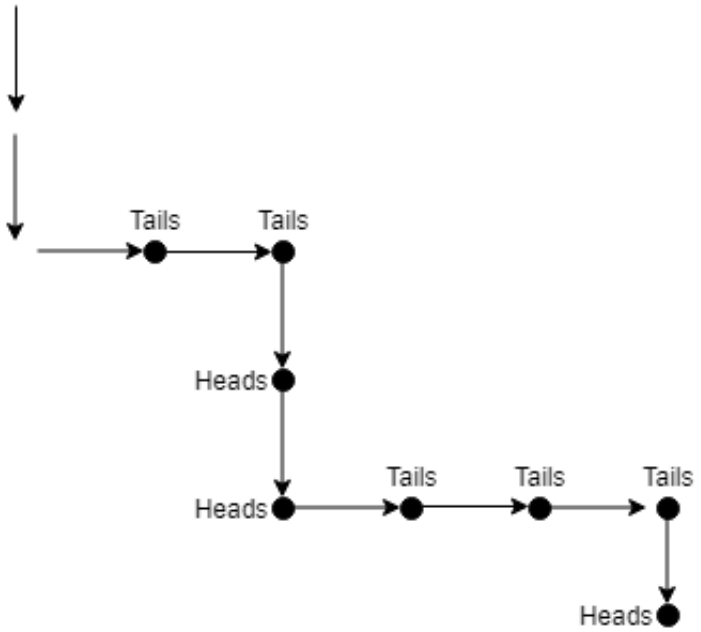
1.2.2 Delete

Delete k also involves doing search on k first and then removing k from all levels it occurs in

1.2.3 Runtime

What is the running time of search, insert, and delete? Search can be viewed as a sequence of down and right arrow traversals.

Figure 2: Example skip list traversal



Any elements visited when moving right are visited because they are smaller than k. If the element was on previous levels, we would have visited it.
How many down traversals can we make? Let H_e denote the height of element e. H_e is a geometrically distributed random variable.

$$P(H_e \geq k) = \frac{1}{2^{k-1}}$$

$\therefore$ Setting $k = 3log_2 n$ gives us $P(H_e \geq 3log_2 n) = \frac{2}{n^3}$.

Let H denote $\max_e H_e$.

$$P(H \geq 3log_2 n) = P(\exists_e e : H_e \geq 3log_2 n) \tag{1}$$
$$\leq \sum eP(H_e \geq 3log_2 n) \tag{2}$$
$$= \frac{2}{n^2} \tag{3}$$

(2) is established by union bound. $\therefore$ With probability $\geq 1 - \frac{2}{n^2}$, $H < 3log_2 n$.
Let L be the length of the search path. We will upper bound $P(L \geq clog_2 n)$ for appropriately chosen c.

$$P(L \geq clog_2 n) = P(L \geq clog_2 nH \geq 3log_2 n)P(H \geq 3log_2 n) \tag{4}$$
$$+ P(L \leq clog_2 nH < 3log_2 n)P(H < 3log_2 n) \tag{5}$$
$$\leq \frac{2}{n^2} + P(L \geq clog_2 nH < 3log_2 n) \tag{6}$$

We want to focus on upper bounding $P(L \geq clog_2 nH < 3log_2 n)$.
Now view L as being obtained via a sequence of coin tosses. We are given that the number of heads $< 3log_2 n$.
$\therefore$ If $L = clog_2 n$, the number of tails $\geq (c-3)\log_2 n$. The expected number of tails is $\frac{c}{2}\log n$
Let T = number of tails. We want to use Chernoff bounds to upper bound the $P(T \geq (c-3)\log n)$.

$$(1+\delta)\mu = (1+\delta)\frac{c}{2}\log n = (c-3)log$$
$$\delta = 1 - \frac{6}{c}$$

If we pick $c > 6$, δ will satisfy $0 < \delta < 1$. Using form (b) of the Chernoff bounds gives us:

$$P(t \geq (c-3)\log n) \leq e^{\frac{-(1-\frac{6}{c})^2}{3}\frac{c}{2}\log n} \tag{7}$$
$$= e^{\frac{-(c-6)^2}{6c}\log n} \tag{8}$$

We can pick c large enough to make $\frac{(c-6)^2}{6c}$ as large as we want to ensure $P(T \geq (c-3)\log n) \leq \frac{1}{n^2}$. Using this, we can plug into equation (6) to get the following:

$$P(L \geq c\log n) \leq \frac{2}{n^2} + \frac{1}{n^2} \tag{9}$$
$$= \frac{3}{n^2} \tag{10}$$

Insert time = time to search + height of element being inserted $\therefore$ Insert time is $O(\log n)$ with high probability. The time analysis of delete is similar.

2 Extensions to Chernoff Bounds

2.1 Negative Dependence

Let $X_1, X_2, ..., X_n$ be a random variable such that for any $I, J \subseteq 1, 2, ..., n$ with $I \cap J = \emptyset$ given that $X_i's$ where $i \in I$ take on "high values", the probability that $X_j's$ where $j \in J$ take on "low values" increases.

2.1.1 Example: Balls in Bins

For $i = 1, 2, ..., n$ let

$$B_i = \begin{cases} 1 & \text{if bin is empty} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

$B_1, B_2, ...B_n$ exhibit negative dependence.
Implications of negative dependence

1. If X_1 and X_2 exhibit negative dependence then

   $$E[X_1 X_2] \leq E[X_1]E[X_2]$$

   This makes sense because

   $$P(X_1 = high \cap X_2 = high) \leq P(X_1 = high)P(X_2 = high)$$

2. If $X_1, X_2, ..., X_n$ exhibit negative dependence then

   $$E[\prod_{i=1}^{n} f_i(X_i)] \leq \prod_{i=1}^{n} E[f_i(X_i)]$$

   for non-decreasing function f_i. A special case of 2. is

   $$E[\prod_{i=1}^{n} e^{tx_i}] \leq \prod_{i=1}^{n} E[e^{tx_i}]$$

   Mutual independence gave us $=$ in place of the $\leq$.
   Let $B = \sum_{i=1}^{n} B_i$ so B denotes the number of empty bins.

$$E[B] = \sum_{i=1}^{n} E[B_i] \tag{12}$$

$$= \sum_{i=1}^{n} P(B_i = 1) \tag{13}$$

$$= \sum_{i=1}^{n} (1 - \frac{1}{n})^n \tag{14}$$

$$\approx \frac{n}{e} \tag{15}$$

So let μ denote $\frac{n}{e}$. Since $B_i's$ exhibit negative dependence, we can use Chernoff Bounds on B.

$$\therefore P(B \geq (1+\delta)\frac{n}{e}) \leq e^{-\frac{\delta^2}{3}\mu} \tag{16}$$

$$\leq \frac{1}{\exp(n)} \tag{17}$$

for any small constant δ where $0 < \delta < 1$

2.2 K-Wise Independence

Recall that $X_1, X_2, ..., X_n$ are k-wise independent if for any $I \subseteq 1, 2, ..., n$, $|I| \leq k$ and any $i \in I$

$$P(X_i = x_i | \wedge_{j \in I} X_j = x_j) = P(X_i = x_i)$$

The setting k = 2 is called pair-wise independence. There are Chernoff bounds for k-wise independent random variables (Srinivasan, Schmidt, Siegel).

2.3 Bounded Dependence

Each random variable is independent with respect to all other random variables except for at most d. This is usually defined with a dependency graph.

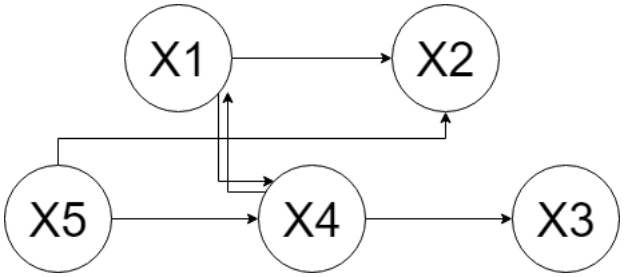


Figure 3: Example of bounded dependency graph. X1 is independent of X2 and X5. The outdegree of each node $\leq$ d

Chernoff Bounds worsen an exponential "$\frac{1}{d}$" factor

2.3.1 Example

$$P(X \geq (1+\delta)\mu) \leq (e^{\frac{-\delta^2\mu}{3}})^{\frac{1}{d}}$$

2.4 Chernoff-Hoeffding bounds

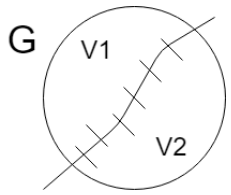
There are extensions to Chernoff bounds that allow $X_i's$ to take on values other than 0 and 1. THese are called Chernoff-Hoeffding bounds.

3 Probabilistic Method

Refers to technique for proving that an object with certain properties exist. Sometimes probabilistic method proofs can be turned into algorithms. These algorithms may be slightly weaker and not hold all the properties that the proof does.

3.1 Example: MaxCut

INPUT: A graph $G = (V, E)$
OUTPUT: A partition of V into (V_1, V_2) such that the number of edges between V_1 and V_2 is maximized.



This is NP-hard. Could we get an efficient approximation algorithm? i.e. could we design an algorithm such that the number of edges crossing the partition it produces $\geq 90\%$ of the size of the optimal cut.

Theorem 1. *Let $G = (V, E)$ be a graph with m edges. Then there exists a partition (V_1, V_2) of V such that $\geq \frac{m}{2}$ edges cross that (V_1, V_2) partition.*

Proof. Take each vertex $v \in V$ and place it in A or in B independently with probability $= \frac{1}{2}$. Let the edges be $e_1, e_2, ..., e_m$. For $i = 1, 2, ..., m$ let

$$X_i = \begin{cases} 1 & \text{if } e_i \text{ crosses the } (A, B) \text{ partition} \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

$$P(X_i = 1) = \frac{1}{2}$$

$$E[X_i] = \frac{1}{2}$$

$$E[X] = \frac{m}{2}$$

where

$$X = \sum_{i=1}^{m} X_i$$

X = number of edges crossing (A,B) partition. Since $\frac{m}{2}$ is the average, there must be some cut that is at least $\frac{m}{2}$. □