22C:44 Homework 7

Due by 5pm on Tuesday, 5/1

Each problem is worth 10 points.

1. The problem is to take a given set of activities (intervals) and schedule these in the *fewest* number of rooms so that activities assigned to each room are mutually compatible. More precisely, the input is a set $A = \{a_1, a_2, \ldots, a_n\}$ of intervals, where, for each i, $a_i = [\ell_i, r_i)$ such that $\ell_i < r_i$. The output that is sought is the smallest collection $\{C_1, C_2, \ldots, C_k\}$ of sets of intervals C_i such that $\cup_{i=1}^{k} C_i = A$ and for each i, C_i contains mutually compatible intervals. Consider the following greedy algorithm for this problem:

```
GreedyActivityScheduling(A){
        Sort the activities in A by increasing right endpoint and label the
        intervals a_1, a_2, ..., a_n in order.
        for i ← 1 to n do
                Find the smallest j such that a_i is compatible with every
                interval in C_j and add a_i to C_j;
}
```

   Prove the correctness of this algorithm.
   Hint: Proceed as follows. Suppose that the answer produced by the algorithm is $\{C_1, C_2, \ldots, C_k\}$. Show that there is a point x and k intervals $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $x \in a_{i_j}$ for each $j = 1, 2, \ldots, k$. This means that any pair of the intervals in $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ are mutually incompatible. This means that each of these has to be assigned to a distinct set C_i. That in turn means that any solution to the problem contains at least k sets of intervals. Since we have a solution with k sets, it is optimal.

2. Consider the problem of finding a maximum size independent set in an arbitrary graph. Prove or disprove the correctness of the following greedy algorithm.

```
GreedyMaximumIndependentSet(G){
        S ← ∅;
        while (G has at least one vertex) do {
                Find a vertex v with minimum degree;
                S ← S ∪{v};
                Delete from G the vertex v and all neighbors of v;
        }
        return S;
}
```

3. The *graph coloring* problem is to find a smallest set S of colors such that when each vertex of the given graph G is assigned a color from S, no two neighboring vertices are assigned the same color.

   A given graph can be greedily colored as follows. Suppose that the palette of colors we want to use is $\{1, 2, 3, \ldots\}$. Process the vertices in any order and to each vertex assign the smallest available color.

   (a) Prove that if the given graph G has maximum vertex degree Δ, the above algorithm will use at most $(\Delta + 1)$ colors.

   (b) Draw a tree that needs 3 or more colors if we color it using the above greedy algorithm. Briefly describe the running of the algorithm, with emphasis on why it needs more than 2 colors.

   (c) Show a coloring of the above tree that uses only two colors.

4. Problem 17.3-2 on page 344.
5. Problem 23.2-7 on page 476.
   Hint: The diameter of an arbitrary graph can be computed in $\Theta(|V|(|V|+|E|))$ time by performing $|V|$ breadth-first-search operations, one at each vertex. In a tree $|E| = |V| - 1$ and therefore this simplifies to $\Theta(|V|^2)$. However, by paying attention to the fact that the given graph is a tree the problem can be solved in $\Theta(|V|)$ time. In particular, you only need to do 2 breadth-first-search operations.
