22C:296 Seminar on Randomization
Homework 2

Due: December 8

1. At the end of the Alon-McDiarmid-Molloy paper (*Journal of Graph Theory, 22(3), 231–237, 1996*) on edge disjoint cycles in regular directed graphs, there is the following theorem.

   > If G is a k-regular directed graph with no parallel edges and with $k \geq 2$, then G contains a collection of at least $k^2/8\ln(k)$ edge disjoint cycles.

   Note that this is weaker than the main result of the paper that proves a $\Omega(k^2)$ bound on the number of edge disjoint cycles. This weaker result can be obtained by a more straightforward application of the Lovasz Local Lemma to repeatedly find vertex disjoint cycles as before, but without the iterated splitting procedure. Prove this weaker result.

2. Consider a *1-dimensional random walk with a reflecting barrier*, which is defined as follows. For each natural number i, there is a state i. At state 0, with probability 1 the walk will move to state 1. At every other state $i > 0$, the walk will move to state $i+1$ with probability ρ and to state $i-1$ with probability $1-\rho$. Prove the following for the resulting Markov chain:

   (a) For $\rho > 1/2$, each state is transient.

   (b) For $\rho = 1/2$, each state is null-persistent.

   (c) For $\rho < 1/2$, each state is non-null persistent.

3. Let G be a 3-colorable graph. Consider the following algorithm for coloring the vertices of G with 2 colors such that no triangle of G is monochromatic. The algorithm begins with an arbitrary 2-coloring of G. While there is a monochromatic triangle in G, it chooses one such triangle, and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds a 2-coloring with the desired property.

4. Let G be a d-regular grapth. Show that the cover time of G is $O(n^2 \log n)$. This shows that the cover times of graphs that are regular is smaller than cover times of graphs in which the vertex degrees exhibit a disparity (for example, the lollipop graph).

5. Let $\#F$ be the number of distinct satisfying truth assignments corresponding to a given DNF formula F. Consider the following alternate approach to devising an (ε, δ)-FPRAS for estimating $\#F$. The t^{th} trial of this algorithm consists of picking a random clause C_t, where the probability of choosing C_j is proportional to the number of satisfying truth assignments for it. Next it selects a random satisfying truth assignment a for C_t. Let $\text{cov}(a)$ be the set of

clauses that are satisfied by a. Define the random variable $X_t = 1/|\mathrm{cov}(a)|$. The estimator for $\#F$ is the random variable

$$Y = \eta \times \sum_{i=1}^{N} \frac{X_t}{N}$$

where $\eta = \sum_a \mathrm{cov}(a)$, where the sum is taken over all possible truth assignments a. Prove that Y is a (ε, δ)-approximation for $\#F$, when

$$N = \frac{cm}{\varepsilon} \ln \frac{1}{\delta}.$$
