22C:296 Seminar on Randomization
Homework 1

Due: November 5

1. This is a result we needed in showing the correctness of Beck's algorithmic version of the Lovasz Local Lemma:

   > Let G be a d-regular graph on n vertices. Show that the number of connected induced subgraphs of G of size r is at most nd^{2r}.

   Prove this result.

2. This problem points to a generalization of Chernoff's bound.

   (a) A function $f : \Re \rightarrow \Re$ is said to be *convex* if for any x_1 and x_2 and $0 \leq \lambda \leq 1$ the following inequality is satisfied:

   $$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2).$$

   Show that the function e^{tx} is convex for any $t > 0$. What can you say when $t \leq 0$?

   (b) Let Z be a random variable that assumes values in the interval $[0, 1]$, and let $p = E[Z]$. Define a binary random variable X with $\text{Prob}[X = 1] = p$ and $\text{Prob}[X = 0] = 1 - p$. Show that for any convex function f, $E[f(Z)] \leq E[f(X)]$.

   (c) Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables over $[0, 1]$ and define $Y = \sum_{i=1}^{n} Y_i$. Using parts (a) and (b) derive upper and lower tail bounds on the random variable Y using the Chernoff bound technique. In particular, show that

   $$\text{Prob}[Y - E[Y] > \delta] \leq e^{-2\delta^2/n}.$$

   Note: If you are having trouble proving these results for with the assumption that the random variables $Z, Y_1, Y_2, \ldots, Y_n$ are continuous, you may assume that these take on a discrete set of values in the interval $[0, 1]$.

3. In the analysis of `LAZY_SELECT` we skipped over the proof of the claim:

   $$\text{Prob}[\ |P| > 4n^{3/4} + 2] = O(n^{-1/4}).$$

   Prove this claim.

4. The *Mycielski graph* is the graph we constructed as an example of a triangle-free graph with arbitrary chromatic number. As a function of k, calculate the number of vertices in a Mycielski graph with chromatic number k. This answer will be exponential in k. Using the probabilistic argument of Erdös, show that for any positive integer k, there exists a triangle-free graph with chromatic number k that has polynomially many (in k) vertices.

5. The first theorem we proved in this class was:

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$. This implies that $R(k,k) > 2^{k/2}$.

This was proved using some basic probability theory arguments. Using the Lovasz Local Lemma, the following stronger result can be proved.

If $e(\binom{k}{2}\binom{n}{k-2} + 1) \cdot 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. This implies that

$$R(k,k) > \frac{\sqrt{2}}{e}(1 + o(1)) \cdot k \cdot 2^{k/2}.$$

Prove this result.
