Equilibria for Economies with Production: Constant-Returns Technologies and Production Planning Constraints

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1 Introduction

We consider the computation of equilibria in two economic models that generalize the exchange model by including production. In the *constant returns model*, each producer has a convex, constant-returns-to-scale, technology. In particular, this means that if the technology can output a certain quantity of a good using as input certain quantities of other goods, then scaling all these quantities by a common, nonnegative, number also results in a technologically feasible plan. The technology also accomodates the *no-free-lunch* property, which says that it is not possible to produce something from nothing. At a given price, the producer picks a technologically feasible plan that maximizes her profit. Associated with each consumer is an initial endowment of goods and a utility function that describes her preferences between various bundles of goods. At a given price, the consumer sells her initial endowment, thus obtaining a certain income, and demands the bundle of goods maximizing her utility among all bundles that she can afford at the given price with her income.

An equilibrium for such an economy is a set of prices, one for each good, such that utility maximization is well-defined for each consumer, profit maximization is well-defined for each producer, and the optimal choices made by the consumers and producers are such that for any good, its total demand, over all consumer choices as well as input choices of producers, is at most the total supply, over all the initial endowments and the output choices of producers.

This model is widely used in applied general equilibrium [34]. Conceptually, it is best thought of as a generalization of the exchange model [21, 11, 10, 16, 19, 35, 17, 8, 6, 4] to include production of the no-free-lunch type (no outputs possible without inputs).

For economies where the utility functions of the consumers and the production functions of the producers are given by a range of CES (constant elasticity of substitution) functions (linear functions are included), as well as the more versatile nested CES functions, we show that the equilibria can be characterized by the solutions of convex feasibility problems. To obtain these results, we build on the ideas behind the convex programs of Nenakov-Primak [27], Jain [19], and Codenotti et al. [8] for the exchange model. In the process, we contribute significant ideas that are needed to handle not only production but also an enlarged class of functions.

In the *production planning model*, we do not assume that each agent has a fixed initial endowment of goods. Instead we assume that there are factories producing goods and each agent has a fixed initial endowment of shares in various factories. Each factory can produce a bundle of goods out of many possibilities e.g., a dairy can either produce ice-creams or produce butter and skimmed milk. The set of bundles of goods that can be produced by a factory is assumed to be a polytope in the positive orthant. (Thus the polytope is not be thought of as specifying a technology as in the constant returns case, but as specifying alternatives for the factory given its stock of resources.) The agents, or consumers, jointly own the factories according to specified shares. At a given price, each factory outputs the bundle that maximizes its revenue over all bundles in its polytope. This generates income for each of the agents. Each agent, who is equipped with her own utility function, asks for a bundle that maximizes her utility among all bundles that she can afford, at the given price, with her income.

An equilibrium is a price vector at which the actions (optimizations) of each factory and agent are well defined, and result in bundles that clear the market. This model has also received significant attention, see for instance [29, 27, 30, 28].

Our main result for this model is an explicit, polynomial-sized, convex program that characterizes the equilibria when the agents have linear utilities. The exchange model with linear utilities is a special case of this problem which is recently settled in [19].

Both models correspond to the ones considered by Arrow and Debreu [1] in their famous paper, where they show that an equilibrium exists under certain

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mild assumptions (See the textbook by Mas-Colell, Whinston, and Green [26] for a detailed and modern treatment).

The main products of our work on both models are convex programs. Polynomial time algorithms follow from standard convex programming methods. The equilibria can be irrational and these algorithms therefore compute approximate equilibria in time that is also polynomial in the number of bits needed to encode the rational number specifying the approximation parameter. The notions of approximate equilibria can be given a natural interpretation in the context of the models, as is the case for the exchange model [21].

CES and Nested CES Functions. Before presenting our results and their significance, we describe the utility and production functions that we are concerned with. A popular family of functions is given by CES (constant elasticity of substitution) functions. Let $\mathbf{R}_+^n$ denote the non-negative orthant of $\mathbf{R}^n$, the set of points with non-negative coordinates. A function $f : \mathbf{R}_+^n \rightarrow R$ is a CES function if it has the form $f(x_1, \dots, x_n) = \left(\sum_{j=1}^{n} c_j x_j^{\rho}\right)^{\frac{1}{\rho}}$, where $c_j \geq 0$, and $-\infty < \rho < 1$, but $\rho \neq 0$. The limiting case $\rho = 1$ corresponds to a linear function $f(x_1, \dots, x_n) = \sum_j a_j x_j$, where $a_j \geq 0$. The limiting case $\rho = 0$ corresponds to the Cobb-Douglas function $f(x_1, \dots, x_n) = \prod_j x_j^{\alpha_j}$, where $\alpha_j \geq 0$ and $\sum_j \alpha_j = 1$. The limiting case $\rho = -\infty$ corresponds to the Leontief function $f(x_1, \dots, x_n) = \min_j a_j x_j$, where $a_j \geq 0$. For ease of exposition, we sometimes treat these limiting cases as special cases of CES functions. The quantity $1/(1-\rho)$ is called the *elasticity of substitution* of the CES function f. CES functions are concave and homogeneous, that is, $f(\alpha x) = \alpha f(x)$ for any scalar $\alpha \geq 0$.

The CES functions model a wide variety of preferences. The demand of a consumer with a CES utility function with $\rho \geq 0$ satisfies *gross substitutability (GS)* : increasing the price of a good cannot result in a decrease in her demand for the other goods. The demand of a consumer with a CES utility function with $\rho < 0$ need not satisfies GS. We refer the reader to [8] for a more detailed exposition of the usefulness of CES functions.

A nested CES function from $\mathbf{R}_+^n \rightarrow R$ is defined recursively: (1) Any CES function is a nested CES function; and (2) if $g : \mathbf{R}_+^t \rightarrow R$ is a t-variate CES function, and $h_1, \dots, h_t$ are n-variate nested CES functions, then $f(x) = \max g(h_1(x^1), \dots, h_t(x^t))$, over all $x^1, \dots, x^t \in \mathbf{R}_+^n$ such that $x^1 + \cdots x^t = x$, is a nested CES function. In the special case where for each good j, at most one of the h_i's depends on its j-th argument, we in fact have $f(x) = g(h_1(x), \dots, h_t(x))$. In what follows, we assume the special case for ease of exposition. Such a nested CES function may be visualized using a tree-like structure, where at each node of the tree we have a CES function.

Consider for example a consumer with the nested CES utility function $u(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/2} + x_3$. At any price vector, she will spend her income entirely on either a bundle that consists exclusively of goods 1 and 2, or entirely on a bundle that consists entirely of good 3. She makes the choice depending on the 'bang for the buck' for each choice. (If the prices are such that the two choices offer the same bang for the buck, she can divide her income arbitrarily between the two choices.) This is a feature of the linear function at the top nest. If the choice consisting of just goods 1 and 2 is the better one, she will spend half her income on good 1, and half on good 2, no matter what the relative prices of these goods: this is a feature of the Cobb-Douglas function $x_1^{1/2} x_2^{1/2}$. It is also easy to check that gross substitutability does not hold for this utility function.

As an introduction to the constant-returns model of production considered in this paper, consider a producer who outputs good 4 using goods 1,2, and 3 as inputs using the nested CES production function $f(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/2} + x_3$. The production function says that the producer's technology can output $f(x_1, x_2, x_3)$ units of good 4 given x_1 units of good 1, x_2 units of good 2, and x_3 units of good 3. Reasoning as above, we see that at a given (non-degenerate) price vector, a profit maximizing plan will either use only goods 1 and 2 as inputs, or only good 3 as input.

Nested CES functions are used extensively to model both production and consumption in applied general equilibrium: We refer the reader to the book by Shoven and Whalley [34] for a sense of their pervasiveness. The popoular modeling language MPSGE [32] uses nested CES functions to model production and consumption.

Our Results and their Significance: The constant-returns model. We present explicit convex programs that characterize (and hence polynomial-time algorithms that compute) the equilibria in economies where consumers have nested CES utility functions, producers have nested CES production functions, and the ρ's at each of the nests of the functions are between 0 and 1. We also obtain convex programs and polynomial time algorithms when the ρ's at each of the nests is between -1 and 0, provided an additional technical condition holds. For example, this condition, specialized to CES functions, says, loosely speaking, that there is a $-1 \le \rho < 0$ such that for each good, the elasticity of any consumer who demands the good or of any producer who demands it as input is at least $1/(1-\rho)$, while the elasticity of any producer who supplies the good as output is at most $1/(1-\rho)$. We give significant examples of economies that satisfy this condition. We note that convex programs for $\rho < -1$ are ruled out even for exchange economies with CES functions by a result of Gjerstad [18].

To our knowledge, these are the first convex programs and polynomial-time algorithms for economies that generalize the exchange model by including no-free-lunch production technologies. Convex programs are also known for generalizations of the Fisher model [15, 12, 9, 7, 35, 3, 17, 4] that include production, see Polterovich [29], and recently, Jain et al. [22]. In these models, however, the relative incomes of the consumers is fixed, and so they do not generalize the exchange model. Moreover, both the model that we consider and the functions that we address are widely employed in applied general equilibrium [23, 34].

If obtaining polynomial time algorithms is significantly more difficult for the exchange model in comparison to the Fisher model, the addition of constant returns to scale production technologies makes the problem even harder. To appreciate this, let us consider first the exchange model, in which there is no production. Here it is well known that if the demand of the consumers satisfies GS, then the set of equilibrium prices is a convex set [2]. The GS property holds in the exchange model for example when consumers have CES utility functions with ρ from the range $[0, 1]$.

When we add to the model production with constant returns to scale technologies, the demand of each consumer is identical to the demand of the consumer in an exchange model. Thus it is natural to ask if the equilibrium prices form a convex set in the production context when consumer demands satisfy GS. It is well known that the answer is quite decisively in the negative. The survey of Kehoe [23] gives an example with two consumers with linear utility functions, four goods, and a simple constant-returns technology for which there are multiple disconnected equilibria. In fact, a necessary condition for ruling out multiple disconnected equilibria is that the aggregate (excess) demands of the consumers satisy the *weak axiom of revealed preference* property [26]. This is a rather stringent necessary condition – to ensure this with the kind of utility functions we consider here, the initial endowments of the consumers would have to be proportional, thus fixing the relative incomes of the consumers.

We are led to the conclusion that the only reasonable way to ensure uniqueness or connectedness of equilibria is to restrict the production side as well [23]. One such restriction is the notion of a *Super Cobb-Douglas* economy, due to Mas-Colell [25]. Using techniques originating from mathematical index theory, Mas-Colell was able to show that a (non-degenerate) Super Cobb-Douglas economy has a unique equilibrium. We do not recite the defintion of a Super Cobb-Douglas economy here, but note that an economy with nested CES utility and production functions with the ρ's of the nest in the range $[0, 1]$ is a pre-eminent example of a Super Cobb-Douglas economy. Mas-Colell's proof of uniqueness, however, does not tell us how to efficiently compute the equilibrium.

Our first result therefore gives convex programs and polynomial time algorithms for an important instance of an economy where uniqueness was already known. On the other hand, our second result, which addresses the case where the ρ's are between -1 and 0, yields new results on uniqueness[1]. The uniqueness of equilibrium for even for the exchange model with CES functions was established only recently [8].

We also point out that our work gives the first convex programs that characterize equilibria for exchange economies with nested CES utility functions. As we argue later, previous methods [27, 19, 8] do not yield a convex program in this setting. And as we have already pointed out, GS need not hold, and therefore methods that exploit this property [6] do not apply. Quite significantly, our approach for obtaining convex programs for these exchange economies requires (for conceptual ease at least) the machinery of production.

The production planning model. Recall that in this model, the production polytopes lie in the positive orthant. Primak [30] gave an infinite linear program, based on the revealed preference inequality [2], that characterizes the equilibria of this model when agents have linear utilities. Recently, Codenotti et al. [4] extended this to the case where agents have utilities that satisfy gross substitutability, and showed that an ellipsoid-based algorithm computes an approximate equilibrium in polynomial time. An ellipsoid algorithm is also suggested by Newman and Primak [28], but it does not guarantee an approximate equilibrium in polynomial time (see [6]).

Convex programs based on the revealed preference inequality do not share some of the advantages

[1] In this section, we sometimes take uniqueness and the lack of multiple disconnected equilibria to mean the same thing. The notions coincide for economies that are regular, that is, non-degenerate in a technical sense [26].

of explicit convex programs, such as those of Eisenberg and Gale's for the Fisher setting [13, 15], or Jain's for the exchange setting [19]. First, these programs are infinite and necessarily require the use of the Ellipsoid algorithm. Second, the variables used in the programs are only for the prices. Prices are the means not the ends in the study of market equilibria. Ideally, we want to get the information about the assignment variables - i.e., who gets what and who derives how much utility. Eisenberg and Gale's convex program reveals many theorems about these variables.

Nenakov and Primak [27] give an explicit convex program for the production planning model with linear utilities. Unfortunately, the number of constraints in their program is infinite. Our main contribution here is an explicit, polynomial sized convex program for this model. This allows the use of efficient interior point methods, such as those developed by Ye [35]. Our program also reveals many structural details about the market equilibria, for instance, convexity in the assignment variables as well as log-convexity in the price variables. The derivation of the program makes a subtle use of techniques developed in [19].

Finally, we wish to stress that our work on both models is a contribution to the important tradition of characterizing equilibria via convex programs. While combinatorial methods are also being explored for computing equilibria, and these are interesting in their own right and have their own advantages, convex characterizations have the advantage of shedding light on the structure of the equilibrium problem. We have already alluded to the uniqueness issue. In addition, convex characterizations can be used to establish revealed-preference type inequalities, which have consequences for the convergence properties of price adjustment mechanisms like tatonnement.

Organization In Section 2, we describe the two models formally. We address the constant returns model in Section 3. In Section 3.1, we describe a simple procedure that simplifies the consumer side of the economy at the expense of adding more producers. This step is straightforward, but seems crucial to obtain our results for even exchange economies. The step applies to any homogeneous utility function, and does not use properties of nested CES functions. Its effectiveness suggests that it may be of independent interest. In Section 3.2, we describe a system of inequalities that characterize equilibria for consumers with homogeneous utility functions and producers with homogeneous production functions. In Section 3.3, we consider nested CES functions for positive ρ's, and show how we derive our convex programs for this class from the system of inequalities in Section 3.2. For lack of space, we omit the more complicated derivation of our programs for negative ρ. Finally, in Section 4, we derive our convex program for the production planning model.

2 The Models

Constant Returns Technologies. We consider a model with n goods, m traders, and l producers. The k'th producer is equipped with a technology that is capable of producing some good, say o_k, using the n goods as input. The technology is specified by a concave function $f_k : \mathbf{R}_+^n \to \mathbf{R}_+$ that is assumed to be homogeneous of degree one. The interpretation is that given quantity $z_j \geq 0$ of good j, for $1 \leq j \leq n$, the technology can produce upto $f_k(z_1, \ldots, z_n)$ units of good o_k.

At a given price vector $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}_+^n$, the producer will choose a technologically feasible production plan that maximizes her profit. That is, she will choose $z_1, \ldots, z_n \geq 0$ that maximizes the profit $\pi_{o_k} f_k(z_1, \ldots, z_n) - \sum_{j=1}^n \pi_j z_j$. Now if there is a choice of $z_1, \ldots, z_n \geq 0$ such that $\pi_{o_k} f_k(z_1, \ldots, z_n) - \sum_{j=1}^n \pi_j z_j > 0$, then using inputs $\alpha z_1, \ldots, \alpha z_n$, for $\alpha > 1$, she can obtain a profit of

$$\pi_{o_k} f_k(\alpha z_1, \ldots, \alpha z_n) - \sum_{j=1}^n \pi_j \alpha z_j$$
$$= \alpha(\pi_{o_k} f_k(z_1, \ldots, z_n) - \sum_{j=1}^n \pi_j z_j).$$

Thus a profit maximizing plan is not defined in this case. A profit maximizing plan is defined if and only if no feasible plan can make a strictly positive profit. In such a case, a profit maximizing plan is one that makes zero profit. In particular, the trivial choice $z_j = 0$, for $1 \leq j \leq n$, for which $f_k(z_1, \ldots, z_n) = 0$ is always a profit maximizing plan whenever profit maximization is well defined.

It is useful to restate the above in terms of the unit cost function $c_k : \mathbf{R}_+^n \to \mathbf{R}_+$. This is defined, at any given price vector $(\pi_1, \ldots, \pi_n) \in \mathbf{R}_+^n$, to be the minimum cost for producing one unit of good o_k. That is,

$$c_k(\pi) = \min\{\sum_{j=1}^n \pi_j z_j | z_j \geq 0, f_k(z_1, \ldots, z_n) \geq 1\}.$$

If $\pi_{o_k} > c_k(\pi)$, then profit maximization is undefined. If $\pi_{o_k} < c_k(\pi)$, then the only profit maximizing plan is the trivial plan. If $\pi_{o_k} = c_k(\pi)$, the trivial plan, as well as any $(x_1, \ldots, x_n)$ such that

$f_k(z_1,\dots,z_n)c_k(\pi) = \sum_{j=1}^n \pi_j z_j$, is a profit maximizing plan.

Each consumer i is equipped with an initial endowment of goods $w_i \in \mathbf{R}_+^n$ and a concave, homogeneous, nonsatiated, utility function $u_i : \mathbf{R}_+^n \to \mathbf{R}_+$. At a given price vector π, the consumer will choose an $x = (x_1,\dots,x_n) \in \mathbf{R}_+^n$ that maximizes $u_i(x)$ subject to the constraint $\sum_{j=1}^n \pi_j x_j \leq \sum_{j=1}^n \pi_j w_{ij}$. That is, she chooses a utility maximizing bundle among those bundles that cost no more than her income[2] $\sum_{j=1}^n \pi_j w_{ij}$. Such a bundle is said to be her demand at price π. Analogous to the unit cost function, the unit expenditure function $e_i : \mathbf{R}_+^n \to \mathbf{R}_+$ is defined, at any given price vector $(\pi_1,\dots,\pi_n) \in \mathbf{R}_+^n$, to be the minimum expenditure for purchasing one unit of utility. That is,

$$e_i(\pi) = \min\{\sum_{j=1}^n \pi_j x_j | x_j \geq 0, u_i(x_1,\dots,x_n) \geq 1\}.$$

Note that $(x_1,\dots,x_n)$ is the demand of trader i at price π if and only if $u_i(x_1,\dots,x_n)e_i(\pi) = \sum_j \pi_j w_{ij}$ and $\sum_j \pi_j x_{ij} = \sum_j \pi_j w_{ij}$.

An equilibrium is a vector prices $\pi = (\pi_1,\dots,\pi_n) \in \mathbf{R}_+^n$ at which there is a bundle $x_i = (x_{i1},\dots,x_{in}) \in \mathbf{R}_+^n$ of goods for each trader i and a bundle $z_k = (z_{k1},\dots,z_{kn}) \in \mathbf{R}_+^n$ for each producer k such that the following three conditions hold: (i) For each firm k, profit maximization is well-defined[3] at π and the inputs $z_k = (z_{k1},\dots,z_{kn})$ and output $q_{ko_k} = f_k(z_{k1},\dots,z_{kn})$ is a profit maximizing plan; (ii) For each consumer i, utility maximization is well-defined and the vector x_i is her demand at price π; and (iii) for each good j, the total demand is no more than the total supply; that is, the market clears:

$$\sum_i x_{ij} + \sum_k z_{kj} \leq \sum_i w_{ij} + \sum_{k:j=o_k} q_{kj}.$$

From the nonsatiation of the utility functions, it follows from standard arguments that a strict inequality can hold in the inequality (iii) corresponding to some good j only if $\pi_j = 0$.

[2]Arrow and Debreu [1] allow convex technologies that need not be of the constant-returns type. In such an economy, the consumers also derive income from the profit of producers in accordance with prescribed shares. In a constant-returns technology, the equilibrium profit is zero, hence there is no income that consumers may derive from any shares. Yet the generality of the model of Arrow and Debreu is not lost by restricting ourselves to constant-returns technologies, see [26], chapters 5 and 17.

[3]Note that this requirement means that there is no feasible plan that makes positive profit. This rules out the trivial approach of ignoring the production units and computing an equilibrium for the resulting exchange model.

Production Planning. Our model has m agents, n goods and l factories. Each factory can produce one of the possible bundles of the goods. The set of possible bundles which a factory can produce is assumed to be a polytope contained in the positive orthant. This polytope is called the *production polytope* of the factory. Each factory is collectively owned by the m agents. Let γ_{ik} denotes the fraction of factory k owned by agent i. We assume that γ_{ik}'s are nonnegative and add up to one for any factory k. Each agent has a linear utility function, $u_i : \mathbf{R}_+^n \to \mathbf{R}_+$ of the form $u_i(x) = \sum_j u_{ij}x_j$, given by the coefficients $u_{ij} \geq 0$. A market equilibrium is an assignment of nonnegative prices π_j to every good j such that:

- Each factory produces a bundle $y \in \mathbf{R}_+^n$ from its production polytope that maximizes the revenue $\pi \cdot y$ at the given price vector. This revenue is shared by the agents in the ratio of γ_{ik}'s.
- Each agent buys a bundle of goods that maximizes her utility among bundles that cost no more than the money she received from the factories. (π should have the property that utility maximization is well-defined.)
- The market clears.

We assume that each agent owns at least one factory to a non-zero extent. If an agent does not own any factory then we can simply remove the agent from consideration. We also assume that no factory is capable of producing any good to an unbounded extent. (This implies that revenue maximization is well-defined at any price.) If some factory can produce any good to an unbounded extent then that good will be priced zero in the equilibrium and we can remove that good from consideration. We also assume that $u_{ij} > 0$. This is a simplifying assumption to make certain that at equilibrium every good has a non-zero price. We may remove this assumption along the lines of [19]. Note that utility maximization is well-defined for each agent at positive prices.

We assume that goods can be freely disposed. This is again without loss of generality. This assumption means that if a factory can produce a bundle of goods $\mathbf{b}$ then the factory can also produce a bundle of goods $\mathbf{b}'$, where $\mathbf{b}'$ is not bigger than $\mathbf{b}$ on any coordinate. If a factory does not satisfy this assumption then we can take the *disposal closure* of its production polytope. Disposal closure of a polytope is the smallest polytope satisfying the free disposal assumption and contains the original polytope. Since the price of each good is positive, an optimum production point over the disposal closure will be in the original polytope.

With the assumption of free disposal, production polytope of a factory k can be written as:

$$\forall t: \sum_j a_{kjt} y_{kj} \leq b_{kt}$$
$$\forall j: y_{kj} \geq 0$$

where y_{kj} is the amount of good j produced by factory k. Here t varies over the non-trivial constraints of the production polytope. Here by non-trivial, we mean the constraints which are other than non-negativity constraints. Since we assumed free disposal we get, a_{kjt}'s and b_{kt}'s are non-negative. For notational purposes we will use, i to vary over the agents, k over the factories, j over the goods and t over the non-trivial facets of a production polytope. The range of t is governed by the context.

3 Constant Returns Technologies

In this section, we derive our convex programs for the constant returns model.

3.1 Simplifying Consumers via Production

Let M denote an economy such as the one described above with m consumers, n goods, and l producers. We describe a transformation into an economy M' with m consumers, $n+m$ goods, and $l+m$ producers. For each consumer i, an additional good, which will be the $(n+i)$'th good, is added. The new utility function of the i'th consumer is $u'_i(x_1, \dots, x_{n+m}) = x_{n+i}$; that is, the i'th consumer only wants good $n+i$. The new initial endowment w'_i is the same as the old one; that is $w'_{ij} = w_{ij}$ if $j \leq n$, and $w'_{ij} = 0$ if $j > n$. The first l producers stay the same. That is, for $k \leq l$, the k'th producer outputs good o_k using the technology described by the function $f'_k(z_1, \dots, z_{n+m}) = f_k(z_1, \dots, z_n)$. For $1 \leq i \leq m$, the $(l+i)$'th producer outputs good $n+i$ using the technology described by the function $f'_{l+i}(z_1, \dots, z_{n+m}) = u_i(z_1, \dots, z_n)$.

Note that good $n+i$ is only consumed by the i'th consumer and produced by the $(l+i)$'th producer. It is not demanded as the input of any producer either. Also note that the unit expenditure $e_i(\pi_1, \dots \pi_n)$ of trader i, for any $i \leq m$, in the market M, equals the unit cost $c'_{l+i}(\pi_1, \dots, \pi_n, \pi_{n+1}, \dots, \pi_{n+m})$ of producer $l+i$ in M', for any $\pi_{n+1}, \dots, \pi_{n+m} \in \mathbf{R}_+$.

Lemma 3.1. *If $\pi = (\pi_1, \dots, \pi_n) \in \mathbf{R}^n_+$ is an equilibrium for M, then $\bar\pi = (\pi_1, \dots, \pi_n, e_1(\pi), \dots, e_m(\pi))$ is an equilibrium for M'. Conversely, if $\bar\pi = (\pi_1, \dots, \pi_n, \pi_{n+1}, \dots, \pi_{n+m})$ is an equilibrium for M', then $\pi = (\pi_1, \dots, \pi_n)$ is an equilibrium for M, and $\pi_{n+i} = e_i(\pi)$ for $1 \leq i \leq m$.*

3.2 Inequalities Characterizing Equilibrium

We now characterize the equilibria for the market M' in terms of a system G of inequalities, in the following sets of non-negative variables: (1) $\pi_1, \dots, \pi_{n+m}$, for the prices; (2) $x_{i,n+i}$, for the demand of consumer i for the $(n+i)$'th good; (3) $z_k = (z_{k1}, \dots, z_{kn}) \in \mathbf{R}^n_+$, standing for the inputs used by the k'th production sector; and (4) q_{ko_k}, for the output of the good o_k by the k'th producer.

$$(3.1) \quad \pi_{n+i} x_{i,n+i} \geq \sum_{j=1}^{n} \pi_j w_{ij}, \text{ for } 1 \leq i \leq m$$

$$(3.2) \quad q_{ko_k} \leq f_k(z_k), \text{ for } 1 \leq k \leq l+m$$

$$(3.3) \quad \pi_{o_k} \leq c_k(\pi_1, \dots, \pi_n), \text{ for } 1 \leq k \leq l+m$$

$$(3.4) \quad \sum_k z_{kj} \leq \sum_i w_{ij} + \sum_{k:o_k=j} q_{kj}, \text{ for } 1 \leq j \leq n$$

$$(3.5) \quad x_{i,n+i} \leq q_{l+i,n+i} \text{ for } 1 \leq i \leq m$$

Here, $c_k()$ denotes the k-th producer's unit cost function, which only depends on the prices of the first n goods. Evidently, any equilibrium is a feasible solution to this system of inequalities G. What is not so evident is that any feasible solution of G is an equilibrium. To see this, we first note that the sets of inequalities (3.2) and (3.3) imply that no producer can make positive profit: we have $\sum_{j \leq n} \pi_j z_{kj} \geq \pi_{o_k} q_{ko_k}$ for each producer k. Adding up these inequalities, as well as the inequailities (3.1), we get a certain inequality that says that the cost of the consumer and producer demands is greater than or equal to the cost of the initial endowments and producer outputs. Whereas by multiplying each inequality in (3.4) and (3.5) by the corresponding price and adding up these inequalities, we get that the cost of the consumer and producer demands is less than or equal to the cost of the initial endowments and producer outputs.

This implies that the two costs must be equal. From this it follows that $\sum_{j \leq n} \pi_j z_{kj} = \pi_{o_k} q_{ko_k}$ for each producer k. Each production plan makes zero profit. Since (3.3) ensures that profit maximization is well defined, these are optimal production plans. Furthermore, we must have equality in (3.1): $x_{i,n+i}$ is the demand of good $n+i$ at price π. Since conservation of goods is assured by (3.4) and (3.5), we conclude that any solution of G is an equilibrium.

3.3 Implications for Nested CES functions: Non-negative ρ. Following Nenakov-Primak [27] and Jain [19], we make the substitution $\pi_j = e^{\psi_j}$ in the above system of inequalities. This makes all the constraints convex, except possibly for the ones in (3.3). Whenever each inequality in the set (3.3) also becomes a convex constraint, we get a convex feasibility characterization of positive equilibrium prices. We show that this is the case when all the production functions are nested CES with the ρ of all the nests being non-negative.

Let us first consider what happens to the constraint in (3.3) corresponding to a CES production function $f_k(z_1, \dots, z_n) = (\sum_j a_{kj} x_j^\rho)^{1/\rho}$, where $0 < \rho < 1$.The corresponding constraint is $\pi_{o_k} \le c_k(\pi) = (\sum_j a_{kj}^\sigma \pi_j^{1-\sigma})^{1/1-\sigma}$, where $\sigma = 1/(1-\rho)$ (We use a standard expression for the cost function corresponding to the CES production function f_k). Raising both sides to the power $(1-\sigma)$, and noting that $1-\sigma < 0$, this constraint becomes

$$\pi_{o_k}^{1-\sigma} \ge \left(\sum_j a_{kj}^\sigma \pi_j^{1-\sigma} \right).$$

It is now easy to see that the substitution $\pi_j = e^{\psi_j}$ turns this into a convex constraint.

It is also easy to verify, using standard formulas for the cost functions, that the constraint in (3.3) corresponding to a linear production function ($\rho = 1$) or a Cobb-Douglas production function ($\rho = 0$) also becomes convex after the substitution $\pi_j = e^{\psi_j}$.

We now turn to nested CES functions. Suppose that $h_1, \dots, h_t$ are n-variate CES functions, and g is an m-variate CES function. Let the corresponding unit cost functions be $c_{h_1}, \dots, c_{h_t}$ and c_g. Then the unit cost function for the n-variate nested CES function $f(x) = g(h_1(x), \dots, h_t(x))$ is given by $c_f(\pi) = c_g(c_{h_1}(\pi), \dots, c_{h_t}(\pi))$.

To write the constraint $\pi_o \le c_f(\pi)$ corresponding to f, we may introduce t new variables $\pi_{h_1}, \dots, \pi_{h_t}$, and write the set of constraints

$$\pi_o \le c_g(\pi_{h_1}, \dots, \pi_{h_t}),$$

and

$$\pi_{h_i} \le c_{h_i}(\pi), \text{ for } 1 \le i \le t.$$

If g and $h_1, \dots, h_t$ are CES functions with $\rho \ge 0$, our discussion above implies that this set of inequalities becomes convex after the substitution $\pi_j = e^{\psi_j}$ and $\pi_{h_i} = e^{\psi_{h_i}}$. It is clear that our discussion, which focussed on 2-level nested CES functions, generalizes to arbitrary nesting structures.

Let P denote the above system of inequalities thus derived from G.

THEOREM 3.1. *If each production function in the market M' is nested CES with each nest having a non-negative ρ, then the above system of inequalities P characterize the positive price equilibria of the market M'.*

We note that this yields the first convex programs even when the original economy M is an exchange model with such nested CES functions. To see that the programs of Nenakov-Primak [27] and Jain [19] do not work in this case, consider the nested CES function $u(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/2} + x_3$. Then the log of the function

$$\frac{\sum_{j=1}^3 x_j \frac{\partial u(x)}{\partial x_j}}{\frac{\partial u(x)}{\partial x_j}} = 2(x_1 + \frac{x_1^{1/2} x_3}{x_2^{1/2}}),$$

is easily seen to be not a concave one even in the variables x_3 and x_2.

4 Production Planning with Linear Utilities

In this section, we derive a polynomial sized convex program whose solutions correspond to the equilibria of the production planning model with linear utilities (recall Section 2).

Let π_j's denote the prices at an equilibrium. Each factory, k, solves the following maximization problem.

$$\text{(4.6)} \qquad \begin{aligned} &\text{maximize } \sum_j \pi_j y_{kj} \\ &\forall t: \ \sum_j a_{kjt} y_{kj} \le b_{kt} \\ &\forall j: \ y_{kj} \ge 0 \end{aligned}$$

Its dual is:

$$\text{(4.7)} \qquad \begin{aligned} &\text{minimize } \sum_t b_{kt} \alpha_{kt} \\ &\forall j: \ \sum_t a_{kjt} \alpha_{kt} \ge \pi_j \\ &\forall t: \ \alpha_{kt} \ge 0 \end{aligned}$$

From the weak duality theorem of linear programming we know that the dual solution is at least as big as the primal solution. If we enforce the primal to be as big as dual then we get the optimality condition for producer k.

$$\text{(4.8)} \quad \sum_j \pi_j y_{kj} \ge \sum_t b_{kt} \alpha_{kt}$$

We will use this as a certification of optimality. We can also use the dual to derive an upper bound on the money available to an agent. An upper bound on the money available to an agent i is:

$$\sum_k \gamma_{ik} \sum_t b_{kt}\alpha_{kt}.$$

We also need to argue that the agents are consuming optimally. Let us write agents optimality conditions as a linear program too. For an agent i we have:

$$\text{(4.9)} \qquad \text{maximize} \sum_j u_{ij}x_{ij}$$
$$\sum_j \pi_j x_{ij} \le \sum_k \gamma_{ik} \sum_t b_{kt}\alpha_{kt}$$
$$\forall j: \ x_{ij} \ge 0$$

Here x_{ij} is the amount of good j consumed by i. Its dual program is:

$$\text{(4.10)} \qquad \text{minimize } \lambda_i \sum_k \gamma_{ik} \sum_t b_{kt}\alpha_{kt}$$
$$\forall j: \ \pi_j \lambda_i \ge u_{ij}$$
$$\lambda_i \ge 0$$

Again using the weak duality theorem of linear programming we know that the dual solution is at least as big as the primal solution. So if we enforce the primal to be as big as dual then we get optimality condition for agent i. This gives us for an agent i:

$$\text{(4.11)} \ \sum_j u_{ij}x_{ij} \ge \lambda_i \sum_k \gamma_{ik} \sum_t b_{kt}\alpha_{kt}$$

The market equilibrium problem becomes finding feasible solutions of linear programs 4.6, 4.7, 4.9, and 4.10 together with their optimality condition 4.8 and 4.11 and the following global condition of market clearing.

$$\forall j: \ \sum_i x_{ij} = \sum_k y_{kj}$$

Note that we are already given γ_{ik}'s such that

$$\text{(4.12)} \ \forall k: \ \sum_i \gamma_{ik} = 1$$

It turns out that the global market clearing condition is quite powerful and helps simplifying the other conditions. We do that in the next section.

4.1 A Simple Non-Convex Program

First let us simplify an agent i dual program. Agent i dual program gives $\lambda_i = \max_j\{u_{ij}/\pi_j\}$. Eliminating λ_i simplifies condition 4.11 to:

$$\text{(4.13)} \ \forall j': \ \sum_j u_{ij}x_{ij} \ge \frac{u_{ij'}}{\pi_{j'}} \sum_k \gamma_{ik} \sum_t b_{kt}\alpha_{kt}$$

We do not need the primal feasibility to be enforced explicitly in the primal programs of agents. Instead we will show that it follows from other constraints. We enforce the feasibility of primal and dual programs for factories but drop the optimality condition 4.8. We will show that even this condition follows. All together we get the following non-convex program:

$$\text{(4.14)} \quad \forall i, j': \ \sum_j u_{ij}x_{ij} \ge \frac{u_{ij'}}{\pi_{j'}} \sum_k \gamma_{ik} \sum_t b_{kt}\alpha_{kt}$$
$$\text{(4.15)} \quad \forall k, t: \ \sum_j a_{kjt}y_{kj} \le b_{kt}$$
$$\text{(4.16)} \quad \forall k, j: \ \sum_t a_{kjt}\alpha_{kt} \ge \pi_j$$
$$\text{(4.17)} \quad \forall j: \ \sum_i x_{ij} = \sum_k y_{kj}$$
$$\text{(4.18)} \quad \forall j: \ \pi_j > 0$$
$$\text{(4.19)} \quad \forall i, j, k, t: \ x_{ij}, \alpha_{kt}, y_{kj} \ge 0$$

THEOREM 4.1. *All the market equilibria satisfy the non-convex program 4.14–4.19 and conversely all the feasible solutions of the non-convex program 4.14–4.19 are the market equilibria.*

Proof. Forward direction is clear and already argued to be correct. For the reverse direction note that α's form a feasible dual to 4.7 hence using the weak duality theorem of linear programming and constraint 4.14 we get:

$$\forall i, j': \sum_j u_{ij}x_{ij} \ge \frac{u_{ij'}}{\pi_{j'}} \sum_k \gamma_{ik} \sum_j \pi_j y_{kj}$$

Multiply the above inequality by $x_{ij'}\pi_{j'}$ and add over j'. We get, for all i:

$$\sum_{j'} x_{ij'}\pi_{j'} \sum_j u_{ij}x_{ij} \ge \sum_{j'} u_{ij'}x_{ij'}(\sum_k \gamma_{ik} \sum_j \pi_j y_{kj})$$

Canceling $\sum_j u_{ij}x_{ij}$ on both sides we get:

$$\forall i: \sum_j x_{ij}\pi_j \ge \sum_k \gamma_{ik} \sum_j \pi_j y_{kj}$$

Summing this over i we get:

$$\sum_{i,j} x_{ij}\pi_j \geq \sum_{i,k,j} \gamma_{ik}\pi_j y_{kj}$$

Using 4.12, we get:

$$\sum_{i,j} x_{ij}\pi_j \geq \sum_{k,j} \pi_j y_{kj}$$

Using 4.17, the above inequality must have been equality. This means it must have been equality all the way. This means the instances of weak duality must have been strong duality. Which gives the optimality of the production for factories. We already assumed the strong duality for the agent's program (constraint 4.14). Though we did not assume the primal feasibility for an agent's program. Note that an agent has only one constraint in the primal program. If that constraint is violated then the primal would be strictly bigger than the dual - since all the complementary slackness conditions hold. Since this is not the case we have primal feasibility for an agent's program. Further details of this argument can be found in [19]. This proves the theorem.

4.2 Compact Convex Program. In this section we give a compact convex program which can be efficiently solved using interior point methods [35]. The main idea in this section is to introduce a new set of variables ξ_{ktj} for α_{kt}/π_j. Note that ξ variables are neither primal nor dual variables. We will rewrite the linear programs 4.6 and 4.7 for factories to give ξ a dual interpretation and that's the reason to use Greek letter instead of Latin. With new variables ξ_{ktj}, constraints 4.14 becomes:

$$\forall i, j' : \sum_j u_{ij}x_{ij} \geq u_{ij'} \sum_k \gamma_{ik} \sum_t b_{kt}\xi_{ktj'} \tag{4.20}$$

It is very tempting to replace constraints 4.16 by:

$$\forall k, j : \sum_t a_{kjt}\xi_{ktj} \geq 1$$

It is not clear whether this transformation would work. If this transformation could have worked then we would get a linear program instead of a convex program. We actually do the following transformation to constraints 4.16:

$$\forall k, j, j' : \sum_t a_{kjt}\xi_{ktj'} \geq \pi_j/\pi_{j'} \tag{4.21}$$

Theorem 4.2. *. All the market equilibria satisfy the non-convex program 4.15, 4.17–4.19, 4.20, and 4.21. Conversely all the feasible solution of the non-convex program are the market equilibria.*

Proof. Let us rewrite a factory's primal program 4.6 several times. Each time we use the unit of value a good j' instead of money. This gives us the following set of primal programs:

$$\begin{aligned} \text{maximize} \quad & \sum_j \frac{\pi_j}{\pi_{j'}} y_{kj} \\ \forall t : \quad & \sum_j a_{kjt}y_{kj} \leq b_{kt} \\ \forall j : \quad & y_{kj} \geq 0 \end{aligned} \tag{4.22}$$

It is clear that the above linear program is optimized simultaneously for all j'. Its dual has the variable $\xi_{ktj'}$.

$$\begin{aligned} \text{minimize} \quad & \sum_t b_{kt}\xi_{ktj'} \\ \forall j : \quad & \sum_t a_{kjt}\xi_{ktj'} \geq \frac{\pi_j}{\pi_{j'}} \\ \forall t : \quad & \xi_{ktj'} \geq 0 \end{aligned} \tag{4.23}$$

Now let us repeat the proof of Theorem 4.1. As in Theorem 4.1 forward direction is clear. For the reverse direction note that $\xi_{ktj'}$'s form a feasible dual to 4.23 hence using the weak duality theorem of linear programming and constraint 4.20 we get:

$$\forall i, j' : \sum_j u_{ij}x_{ij} \geq \frac{u_{ij'}}{\pi_{j'}} \sum_k \gamma_{ik} \sum_j \pi_j y_{kj}$$

The rest of the proof of this theorem is the same as the proof of Theorem 4.1

The substitution $\pi_j = e^{\psi_j}$ turns the non-convex program in Theorem 4.2 into the following convex program:

$$\begin{aligned} \forall i, j' : \quad & \sum_j u_{ij}x_{ij} \geq u_{ij'} \sum_k \gamma_{ik} \sum_t b_{kt}\xi_{ktj'} \\ \forall k, t : \quad & \sum_j a_{kjt}y_{kj} \leq b_{kt} \\ \forall k, j, j' : \quad & \sum_t a_{kjt}\xi_{ktj'} \geq e^{\psi_j - \psi_{j'}} \\ \forall j : \quad & \sum_i x_{ij} = \sum_k y_{kj} \\ \forall i, k, j, j', t : \quad & x_{ij}, \xi_{ktj'}, y_{kj} \geq 0 \end{aligned} \tag{4.24}$$

Theorem 4.3. *Given any feasible solution ψ, x, y, ξ to the convex program (4.24), the price vector $(e^{\psi_1}, \ldots, e^{\psi_n})$ is an equilibrium for the production* *planning economy. Conversely, any equilibrium price vector $(e^{\psi_1}, \ldots, e^{\psi_n})$ of the production planning economy can be extended to a solution ψ, x, y, ξ to the convex program (4.24).*

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