# Day 9.

## 1. Basic Proof Theory

Historical notes:

- Hilbert's *axiomatic proof theory*: 1. Choose axioms and basic objects 2. Prove consistency 3. Explore independence and completeness 4. Decision procedure
- Aims: geometry, arithmetic, analysis
- Gentzen's development of formal proof theory.
  - Based on work by Frege
  - Starting the above program with logic

Gentzen's observation: rather than starting from *axioms*, most proofs start from a set of *assumptions*. There are then two categories of operations:

- Assumptions are analyzed into parts—*eliminating* them
- Conclusions are analyzed into parts—*introducing* them
- Ideally, you meet in the middle

This means that, to formalize proofs, we want to provide each *logical connective* with a set of introduction rules and a set of elimination rules.

Conjunction:

$$(\wedge \mathrm{I})\ \frac{A \quad B}{A \wedge B} \qquad (\wedge \mathrm{E}_1)\ \frac{A \wedge B}{A} \qquad (\wedge \mathrm{E}_2)\ \frac{A \wedge B}{B}$$

Disjunction:

$$(\vee \mathrm{I}_1)\ \frac{A}{A \vee B} \qquad (\vee \mathrm{I}_2)\ \frac{B}{A \vee B} \qquad (\vee \mathrm{E})\ \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

- Bracketed propositions *may* be used in the derivation, as often as needed, but are not required to be.

Implication:

$$(\Rightarrow \mathrm{I})\ \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Rightarrow B} \qquad (\Rightarrow \mathrm{E})\ \frac{A \Rightarrow B \quad A}{B}$$

Now, we can put together some simple derivations:

$$(\Rightarrow \mathrm{I})^p\,\dfrac{(\vee \mathrm{E})^{q,r}\,\dfrac{(\wedge \mathrm{E}_2)\,\dfrac{[A \wedge (B \vee C)]^p}{B \vee C} \qquad (\vee \mathrm{I}_1)\,\dfrac{(\wedge \mathrm{I})\,\dfrac{(\wedge \mathrm{E}_1)\,\dfrac{[A \wedge (B \vee C)]^p}{A} \qquad [B]^q}{A \wedge B}}{(A \wedge B) \vee (A \wedge C)} \qquad (\vee \mathrm{I}_2)\,\dfrac{(\wedge \mathrm{I})\,\dfrac{(\wedge \mathrm{E}_1)\,\dfrac{[A \wedge (B \vee C)]^p}{A} \qquad [C]^r}{A \wedge C}}{(A \wedge B) \vee (A \wedge C)}}{(A \wedge B) \vee (A \wedge C)}}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}$$

- We label rules that introduce assumptions and the corresponding uses of those assumptions... for example, the hypothesis introduced at the base of the derivation is used at the points labeled $p$.

We can extend this approach to the logical constants as well:

$$(\top\,\mathrm{I})\,\dfrac{}{\top} \qquad \text{(No elimination rule for truth)} \qquad (\bot\,\mathrm{E})\,\dfrac{\bot}{A} \qquad \text{(No introduction rule for falsity)}$$

- We define negation in terms of implication and falsity: $\neg A = A \Rightarrow \bot$. This gives, as we expect, $A \wedge \neg A \implies \bot$.
- Don't actually need $(\bot\,\mathrm{E})$ (also called ECQ). Result is called *minimal* logic.

Key idea: *normalization*

- Eliminate detours (i.e. lemmas) in proofs
- Consistency as a consequence (i.e., because there are no proofs of $\bot$, and normalized proof can only prove $\bot$ if it's assumed it.

Conjunction:

$$(\wedge \mathrm{E}_1)\,\dfrac{(\wedge \mathrm{I})\,\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}}{A} \;\rightsquigarrow\; \begin{matrix}\vdots \\ A\end{matrix} \qquad\qquad (\wedge \mathrm{E}_2)\,\dfrac{(\wedge \mathrm{I})\,\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}}{B} \;\rightsquigarrow\; \begin{matrix}\vdots \\ B\end{matrix}$$

Disjunction:

$$(\vee \mathrm{E})\,\dfrac{(\vee \mathrm{I}_1)\,\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B} \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \;\rightsquigarrow\; \begin{matrix}\vdots \\ A \\ \vdots \\ C\end{matrix} \qquad\qquad (\vee \mathrm{E})\,\dfrac{(\vee \mathrm{I}_2)\,\dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B} \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \;\rightsquigarrow\; \begin{matrix}\vdots \\ B \\ \vdots \\ C\end{matrix}$$

Implication:

$$(\Rightarrow \mathrm{E})\,\dfrac{(\Rightarrow \mathrm{I})\,\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B} \;\rightsquigarrow\; \begin{matrix}\vdots \\ A \\ \vdots \\ B\end{matrix}$$

Key observation: these transformations correspond to evaluation rules for functional languages!

- $\lambda$-calculus as notation for proofs and proof transformations
- Soundness of proof systems corresponds to *termination* of proof normalization
- So, simply typed $\lambda$-calculus, as we've been developing it, is terminating.

*Aside.* We're actually talking here about *intuitionistic* logic, not the classical logic you've likely been taught to this point. The key distinction is that intuitionistic logic is a logic of *proof*. That is to say, that statement $A$ should be interpreted as "$A$ is provable", not as "$A$ is, in some absolute sense, true". One consequence of this view is that axioms of classical logic, like the law of the excluded middle, do not hold in intuitionistic logic. While it might be intuitive to say every proposition is either true or false, it is not intuitive to say that every proposition is either proved or contradicted.