Day 6.

1. Environments

Approximating the semantics of `let` has proved tricky:

- We can approximate the cbn reduction relation, but only at the cost of performing much of the substitution that we might hope to avoid
- We have even less success with the cbv reduction, as we can't substitute approximations into terms

Our approach to this is to reconsider the role that substitutions play in evaluation. Rather than applying substitutions immediately to terms, we'll preserve substitutions as *environments* in the reduction relation.

2. CBV with Environments

We define *environments* to be mappings from terms to values: $\mathrm{H} \in \mathcal{X} \rightharpoonup \mathcal{V}$.

(Aside: H is supposed to be a Greek capital eta, not a Latin H. What difference does it make? None.)

Now we can define a 3-place evaluation relation $\Downarrow \in (\mathcal{X} \rightharpoonup \mathcal{V}) \times \mathcal{E} \times V$

$$\frac{\mathrm{H} \vdash t_1 \Downarrow v_1 \quad \mathrm{H} \vdash t_2 \Downarrow v_2}{\mathrm{H} \vdash t_1 + t_2 \Downarrow v_1 + v_2} \qquad \frac{\mathrm{H} \vdash t_1 \Downarrow v_1 \quad \mathrm{H} \vdash t_2 \Downarrow v_2}{\mathrm{H} \vdash t_1 \div t_2 \Downarrow \lfloor {}^{t_1}/{}_{t_2} \rfloor} \; (v_2 \neq 0)$$

$$\frac{}{\mathrm{H} \vdash_{\mathsf{cbv}} x \Downarrow \mathrm{H}(x)} \qquad \frac{\mathrm{H} \vdash_{\mathsf{cbv}} t_1 \Downarrow v_1 \quad \mathrm{H}[x \mapsto v_1] \vdash_{\mathsf{cbv}} t_2 \Downarrow v_2}{\mathrm{H} \vdash_{\mathsf{cbv}} \texttt{let}\ x = t_1\ \texttt{in}\ t_2 \Downarrow v_2}$$

- We could introduce a new evaluation symbol (or new subscript) for the three-place version of the evaluation relation... but the context will always make it clear which version we mean.
- The constant rules behave the same in call-by-name and call-by-value
- We write $\mathrm{H}(x)$ to denote the value that x is mapped to in H, and $\mathrm{H}[x \mapsto v]$ to denote extending a partial function... formally:

$$\mathrm{H}[x \mapsto v](y) = \begin{cases} v & \text{if } x = y \\ H(y) & \text{otherwise} \end{cases}$$

We have some simple derivations:

$$\frac{\dfrac{}{\emptyset \vdash_{\mathsf{cbv}} 4 \Downarrow 4} \quad \dfrac{\dfrac{}{\{x \mapsto 4\} \vdash_{\mathsf{cbv}} x \Downarrow 4} \quad \dfrac{}{\{x \mapsto 4\} \vdash_{\mathsf{cbv}} x \Downarrow 4}}{\{x \mapsto 4\} \vdash_{\mathsf{cbv}} x \div x \Downarrow 1}}{\emptyset \vdash_{\mathsf{cbv}} \texttt{let}\ x = 4\ \texttt{in}\ x \div x \Downarrow 1}$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash_{\mathsf{cbv}} 4 \Downarrow 4} \quad \dfrac{}{\emptyset \vdash_{\mathsf{cbv}} 4 \Downarrow 4}}{\emptyset \vdash_{\mathsf{cbv}} 4 \div 4 \Downarrow 1} \quad \dfrac{\dfrac{}{\{x \mapsto 1\} \vdash_{\mathsf{cbv}} x \Downarrow 1} \quad \dfrac{}{\{x \mapsto 1\} \vdash_{\mathsf{cbv}} x \Downarrow 1}}{\{x \mapsto 1\} \vdash_{\mathsf{cbv}} x + x \Downarrow 2}}{\emptyset \vdash_{\mathsf{cbv}} \mathtt{let}\ x = 4 \div 4\ \mathtt{in}\ x + x \Downarrow 2}$$

- Starting with the empty environment ($\emptyset$), and $\emptyset[x \mapsto 1] = \{x \mapsto 1\}$.
- Results of evaluation equivalent to substitution version: same final value, same number of operations. However, we replace substitution with variable lookup. Complexity implications?

3. CBN with Environments

Intuitively:

- CBV evaluates *before* substituting
- CBN evaluates *after* substituting

To map this intuition to environments, we have:

- CBV environments store *values*
- CBN envirnoments store *terms*

So for CBN, we define $\mathrm{H} \in \mathcal{X} \rightharpoonup \mathcal{E}$, and have evaluation rules

$$\dfrac{\mathrm{H} \vdash_{\mathsf{cbn}} \mathrm{H}(x) \Downarrow v}{\mathrm{H} \vdash_{\mathsf{cbn}} x \Downarrow v} \qquad \dfrac{\mathrm{H}[x \mapsto t_1] \vdash_{\mathsf{cbn}} t_2 \Downarrow v}{\mathrm{H} \vdash_{\mathsf{cbn}} \mathtt{let}\ x = t_1\ \mathtt{in}\ t_2 \Downarrow v}$$

Again, we can consider some simple derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} 4 \Downarrow 4} \quad \dfrac{}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} 4 \Downarrow 4}}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} 4 \div 4 \Downarrow 1}}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} x \Downarrow 1} \quad \dfrac{\dfrac{\dfrac{}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} 4 \Downarrow 4} \quad \dfrac{}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} 4 \Downarrow 4}}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} 4 \div 4 \Downarrow 1}}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} x \Downarrow 1}}{\{x \mapsto 4 \div 4\} \vdash_{\mathsf{cbn}} x + x \Downarrow 2}}{\emptyset \vdash_{\mathsf{cbn}} \mathtt{let}\ x = 4 \div 4\ \mathtt{in}\ x + x \Downarrow 2}$$

and:

$$\dfrac{\dfrac{}{\{x \mapsto 4 \div 0\} \vdash_{\mathsf{cbn}} 3 \Downarrow 3}}{\emptyset \vdash_{\mathsf{cbn}} \mathtt{let}\ x = 4 \div 0\ \mathtt{in}\ \Downarrow 3}$$

- Same properties of evaluation: evaluation repeated for each use of a variable, but unused variables don't stop evalution.

4. Approximating Evaluation

We build a type system from evaluation with environments following the same approach we've used for earlier evaluation relations.

Let environments $\Gamma \in \mathcal{X} \rightharpoonup \mathcal{T}$ map variables to approximations of values.

6.

Now, we define approximation by:

$$\frac{}{\Gamma \vdash z : \texttt{Int}} \qquad \frac{}{\Gamma \vdash b : \texttt{Bool}} \qquad \frac{\Gamma \vdash e_1 : \texttt{Int} \quad \Gamma \vdash e_2 : \texttt{Int}}{\Gamma \vdash e_1 \odot e_2 : \texttt{Int}} \ (\odot \in \{+, \times\})$$

$$\frac{\Gamma \vdash e_1 : \texttt{Bool} \quad \Gamma \vdash e_2 : t \quad \Gamma \vdash e_3 : t}{\Gamma \vdash \texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 : t}$$

$$\frac{}{\Gamma \vdash x : \mathrm{H}(x)} \qquad \frac{\Gamma \vdash t_1 : S_1 \quad \Gamma[x \mapsto S_1] \vdash t_2 : S_2}{\Gamma \vdash \texttt{let}\ x = t_1\ \texttt{in}\ t_2 : S_2}$$

As we would hope, the approximation relation now exactly follows the pattern of the evaluation relation!

This is still a conservative approximation of the meanings of terms:

$$\frac{\dfrac{\Gamma \vdash 1 : \texttt{Bool} \quad \dfrac{}{\Gamma \vdash 2 : \texttt{Int}} \quad \dfrac{}{\Gamma \vdash 3 : \texttt{Int}}}{\Gamma \vdash \texttt{if}\ 1\ \texttt{then}\ 2\ \texttt{else}\ 3 : \texttt{Int}} \qquad \dfrac{}{\Gamma \vdash 10 : \texttt{Int}}}{\Gamma \vdash \texttt{let}\ x = \texttt{if}\ 1\ \texttt{then}\ 2\ \texttt{else}\ 3\ \texttt{in}\ 10 : \texttt{Int}}$$

But this is no different than our previous approximations, so we should not be surprised.