Day 9

1. Functions and Environments

To develop our approximation of local variables, we needed to move from a substution-based view of evaluation to an environment-based view. We'll have to do something similar for functions. So, let's get started!

$$\frac{}{H, x \Downarrow_{\mathsf{cbv}} H(x)} \qquad \frac{}{H, \lambda x.t \Downarrow_{\mathsf{cbv}} \lambda x.t} \qquad \frac{H, t_1 \Downarrow_{\mathsf{cbv}} \lambda x.t \quad H, t_2 \Downarrow_{\mathsf{cbv}} w \quad H[x \mapsto w], t \Downarrow_{\mathsf{cbv}} v}{H, t_1\, t_2 \Downarrow_{\mathsf{cbv}} v}$$

- Omitted rules for numeric constants, because they don't behave any different than they did in the last version
- Again, reusing syntax for 2-place and 3-place evaluation relations

We should confirm that it works. Let's try some simple reductions:

$$\dfrac{\dfrac{\dfrac{}{\emptyset, \lambda a.\lambda b.b \Downarrow \lambda a.\lambda b.b} \quad \dfrac{}{\emptyset, 3 \Downarrow 3} \quad \dfrac{}{\{a \mapsto 3\}, \lambda b.b \Downarrow \lambda b.b}}{\emptyset, (\lambda a.\lambda b.b) 3 \Downarrow \lambda b.b} \quad \dfrac{}{\emptyset, 2 \Downarrow 2} \quad \{b \mapsto 2\}, b \Downarrow 2}{\emptyset, (\lambda a.\lambda b.b)\, 3\, 2 \Downarrow_{\mathsf{cbv}} 2}$$

Looks good so far!

$$\dfrac{\dfrac{\dfrac{}{\emptyset, \lambda a.\lambda b.a \Downarrow \lambda a.\lambda b.a} \quad \dfrac{}{\emptyset, 3 \Downarrow 3} \quad \dfrac{}{\{a \mapsto 3\}, \lambda b.a \Downarrow \lambda b.a}}{\emptyset, (\lambda a.\lambda b.a)\, 3 \Downarrow \lambda b.a} \quad \dfrac{}{\emptyset, 2 \Downarrow 2} \quad \{b \mapsto \overset{\lightning}{2}\}, a \Downarrow 3}{\emptyset, (\lambda a.\lambda b.a)\, 3\, 2 \Downarrow_{\mathsf{cbv}} 3}$$

What's gone wrong?

- We're trying to use variable a when it's not apparently in scope. Fair enough—this *shouldn't* be derivable.
- Variable a should have gotten its meaning in reducing the left-hand argument, but it didn't. This is the real problem.
- Missing one aspect of substitution—although evaluation doesn't touch λs, substitution does!

Solution: λ terms need to carry their defining environments with them!

- Means we don't have to reintroduce substitution
- Combination of a function and its environment called a *closure.*

2. Closures

Let's recap our language:

$$\begin{aligned}
\mathcal{X} &\ni x \\
\mathcal{V} &\ni v ::= z \mid \langle H, \lambda x.t\rangle \\
\mathcal{T} &\ni t ::= z \mid t_1 \odot t_2 \mid x \mid \lambda x.t \mid t_1\, t_2
\end{aligned}$$

- New value form: closures. Package environment with function
- Values no longer subset of terms... but can think of $\langle H, \lambda x.t\rangle$ as being syntax for $(\lambda x.t)[v_i/y_i]$ where $H = \{y_i \mapsto v_i\}$.

Now we can adjust evaluation rules to construct and use closures.

$$\frac{}{H, \lambda x.t \Downarrow \langle H, \lambda x.t\rangle} \qquad \frac{H, t_1 \Downarrow \langle H', \lambda x.t\rangle \quad H, t_2 \Downarrow w \quad H'[x \mapsto w], t \Downarrow v}{H, t_1\, t_2 \Downarrow_{\mathsf{cbv}} v}$$

Does this work?

$$\dfrac{\dfrac{\dfrac{}{\emptyset, \lambda a.\lambda b.b \Downarrow \lambda a.\lambda b.b} \quad \dfrac{}{\emptyset, 3 \Downarrow 3} \quad \dfrac{}{\{a \mapsto 3\}, \lambda b.b \Downarrow \langle\{a \mapsto 3\}, \lambda b.b\rangle}}{\emptyset, (\lambda a.\lambda b.b)\, 3 \Downarrow \langle\{a \mapsto 3\}, \lambda b.b\rangle} \quad \dfrac{}{\emptyset, 2 \Downarrow 2} \quad \dfrac{}{\{a \mapsto 3, b \mapsto 2\}, b \Downarrow 2}}{\emptyset, (\lambda a.\lambda b.b)\, 3\, 2 \Downarrow_{\mathsf{cbv}} 2}$$

Looks promising.

$$\dfrac{\dfrac{\dfrac{}{\emptyset, (\lambda a.\lambda b.a) \Downarrow \langle\emptyset, \lambda a.\lambda b.a\rangle} \quad \dfrac{}{\emptyset, 3 \Downarrow 3} \quad \dfrac{}{\{a \mapsto 3\}, \lambda b.a \Downarrow \langle\{a \mapsto 3\}, \lambda b.a\rangle}}{\emptyset, (\lambda a.\lambda b.a)\, 3 \Downarrow \langle\{a \mapsto 3\}, \lambda b.a\rangle} \quad \dfrac{}{\emptyset, 2 \Downarrow 2} \quad \dfrac{}{\{a \mapsto 3, b \mapsto 2\}, a \Downarrow 3}}{\emptyset, (\lambda a.\lambda b.a)\, 3\, 2 \Downarrow_{\mathsf{cbv}} 3}$$

Seems to work!

Call by name variation: just replace $H \in \mathcal{X} \rightharpoonup \mathcal{V}$ with $H \in \mathcal{X} \rightharpoonup \mathcal{T}$ and:

$$\frac{H, H(x) \Downarrow_{\mathsf{cbn}} v}{H, x \Downarrow_{\mathsf{cbn}} v} \qquad \frac{H, t \Downarrow_{\mathsf{cbn}1} \langle H', \lambda x.t\rangle \quad H'[x \mapsto t_2], t \Downarrow_{\mathsf{cbn}} v}{H, t_1\, t_2 \Downarrow_{\mathsf{cbn}} v}$$

Historical note. Early implementations of LISP, including some still in use (ELISP), got closures wrong. Some people like to present this as a design choice; they call it "dynamic scope" or similar euphemisms. This is not a design choice, any more than $2 + 2 = 5$ would be a design choice for addition. It is a system that fails to match the semantics of the λ-calculus.

3. Typing Functions

What can go wrong? $1\, 2$, $(\lambda c.c) + 1$.

We need to extend our grammar of types:

$$\mathcal{Y} \ni T ::= \texttt{Int} \mid T_1 \to T_2$$

- Why don't closures need to be reflected in the types of functions?

As before, we define a variation of the evaluation relation that characterizes the types of values: $\Gamma \vdash t : T$.

- Syntax: $\vdash$ denotes *consequence*—under the assumptions in Γ, the typing on the right holds. $:$ was originally $\in$.
- $\Gamma : \mathcal{X} \rightharpoonup \mathcal{Y}$ map from variables to their types.
- More about the typing relation... and the significance of our notational choices... to come.

Typing rules:

$$\frac{}{\Gamma \vdash z : \texttt{Int}} \qquad \frac{\Gamma \vdash t_1 : \texttt{Int} \quad \Gamma \vdash t_2 : \texttt{Int}}{\Gamma \vdash t_1 + t_2 : \texttt{Int}} \qquad \cdots$$

$$\frac{}{\Gamma \vdash x : \Gamma(x)} \qquad \frac{\Gamma[x \mapsto T_1] \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \qquad \frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\, t_2 : T_2}$$

- Common notation for $\Gamma[x \mapsto T_1]$ is $\Gamma, x{:}T_1$. May fall into this later, but not yet.
- Why don't we have to represent the closure in the application rule?

Let's look at some simple derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{a \mapsto \texttt{Int}, b \mapsto \texttt{Int} \to \texttt{Int}\} \vdash a : \texttt{Int}}}{\{a \mapsto \texttt{Int}\} \vdash \lambda b.a : (\texttt{Int} \to \texttt{Int}) \to \texttt{Int}}}{\emptyset \vdash (\lambda a.\lambda b.a) : \texttt{Int} \to (\texttt{Int} \to \texttt{Int}) \to \texttt{Int}} \quad \dfrac{}{\emptyset \vdash 3 : \texttt{Int}}}{\emptyset \vdash (\lambda a.\lambda b.a)\, 3 : (\texttt{Int} \to \texttt{Int}) \to \texttt{Int}} \quad \dfrac{\dfrac{}{\{c \mapsto \texttt{Int}\} \vdash c : \texttt{Int}}}{\emptyset \vdash \lambda c.c : \texttt{Int} \to \texttt{Int}}}{\emptyset \vdash (\lambda a.\lambda b.a)\, 3\, (\lambda c.c) : \texttt{Int}}$$

$$\dfrac{\dfrac{\dfrac{}{\{a \mapsto \texttt{Int} \to \texttt{Int}\} \vdash a : \texttt{Int} \to \texttt{Int}}}{\emptyset \vdash (\lambda a.a) : (\texttt{Int} \to \texttt{Int}) \to (\texttt{Int} \to \texttt{Int})} \quad \dfrac{\dfrac{}{\{b \mapsto \texttt{Int}\} \vdash b : \texttt{Int}}}{\emptyset \vdash (\lambda b.b) : \texttt{Int} \to \texttt{Int}}}{\emptyset \vdash (\lambda a.a)\, (\lambda b.b) : \texttt{Int} \to \texttt{Int}}$$

- Check typing of functions at *construction*, not at *use*. So: more structure under the typing of a λ, but less at their uses.
- Same term may have more than one typing derivation: $\lambda a.a$ (up to α-equivalence) given both $\texttt{Int} \to \texttt{Int}$ and $(\texttt{Int} \to \texttt{Int}) \to (\texttt{Int} \to \texttt{Int})$.