Three Untrue Statements in Computability Theory

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Abstract

The following general statements often appear in the textbooks on computability theory: (1) Every finite set of natural numbers is decidable; (2) every set of natural numbers has a characteristic function; and (3) every set of natural numbers is countable. In the textbooks, their proofs use implicit assumptions not mentioned in these statements. As a result, the validities of these general statements are questionable, as they fall to the fallacy of hasty generalization. Our starting point is that every function is defined in a formal system. By Gödel's incompleteness theorems, every formal system has its limitation, and there always exist other functions undefinable in the given formal system. We extend Tarski's undefinability theorem and show that certain functions are not definable in consistent formal systems. Using functions undefinable in consistent formal systems is error-prone and leads to unsound statements.

1 Introduction

The computability theory originated in the 1930s from mathematical logic, pioneered by Kurt Gödel, Alonzo Church, and Alan Turing, laying the groundwork for computer science. These pioneers introduced different formal systems (i.e., axiomatic systems, often represented by languages of logic) to define what is computable [9, 4, 26]. The equivalence of these formal systems is captured in the so-called Church-Turing thesis. Gödel's two incompleteness theorems, published in 1931, have profound influence on what is uncomputable and demonstrate clearly the limitation of formal systems. To prove the incompleteness theorems, Gödel used ingeniously the diagonal method, invented by Georg Cantor for the uncountability of real numbers [3], over the set of *Gödel numbers* of formulas in a formal system [9, 13, 17, 8].

This article is a continuation of Gödel's approach to the limitations of formal systems, including several closely related theorems, e.g., Tarski's undefinability theorem on the formal undefinability of truth [25] and Turing's theorem that there is no algorithm to solve the halting problem [26].

The following general statements often appear in the textbooks on the theory of computation [21]:

1. **The finiteness claim**: Every finite set of natural numbers is decidable.
2. **The oracle claim**: Every set of natural numbers has a characteristic function.
3. **The subset claim**: Every set of natural numbers is countable.

To refute the finiteness claim, it suffices to consider the universal halting problem of Turing machines. Let $f : \mathcal{N} \mapsto \{0, 1\}$ be a Boolean function, where $\mathcal{N}$ denotes the set of all natural numbers, such that $f(x) = 1$ iff (read "if and only if") $x = \langle M \rangle$, the encoding of Turing machine M that halts on every input. For any Turing machine M, let $R = \{f(\langle M \rangle)\}$. Apparently, $R \subset \{0, 1\}$ is a finite set. If R were decidable, by checking whether $1 \in R$ or not, the universal halting problem would be decidable. The textbook proof of the finiteness claim relies on two statements: (a) every finite formal language is regular [16] (page 271); and (b) every regular language is decidable. While (b) is true, R is a counterexample to (a). We will discuss this topic in Section 4.

We say a total function $f : \mathcal{N} \mapsto \{0, 1\}$ is a *characteristic function* of a set $S \subseteq \mathcal{N}$ if for every $n \in \mathcal{N}$, $f(n) = 1$ iff $n \in S$. If f exists, f is unique for S as f is required to be total. *Turing reduction* [21] is an important concept in the theory of computation that uses *oracle*, a synonym of characteristic function. A set A is *Turing reducible* to a set B if A has an *oracle machine*, which is a modified Turing machine that has the additional capability of querying the oracle of B, to decide any x is a member of A [21]. In other terms, A is *Turing reducible* to B if the characteristic function of A is computable, assuming we can query freely the characteristic function of B.

Turing reduction takes for granted the oracle claim, i.e., every set of natural numbers has a characteristic function. We assume that every function is defined in a formal system. By Gödel's incompleteness theorems, every formal system has its own limitation. In Section 3, we will investigate formal systems in which characteristic functions of some integer sets are not definable.

Our investigation leads to an extension of Tarski's undefinability theorem, which states that the arithmetic true cannot be defined in first-order arithmetic (i.e., the language of arithmetic). In Tarski's undefinability theorem, the *true arithmetic* is often denoted by T^*, the set of Gödel numbers of all true sentences in the structure of natural numbers. Tarski's undefinability theorem implies that the characteristic function of T^* cannot be defined in first-order arithmetic. Tarski's notion of *undefinability*, formally defined in the next section, involves two concepts: (i) a well-defined set of natural numbers, and (ii) a formal system (equivalently, a language of logic). In Tarski's undefinability theorem, the set is $T^* \subset \mathcal{N}$ and the formal system is a member of first-order arithmetic. Here, by *first-order arithmetic*, we mean a collection of axiomatic systems that formalize the natural numbers and their properties within the framework of first-order logic. In this article, we will extend Tarski's notion of undefinability from first-order arithmetic to any formal system, as carried out by Smullyan in [22], and extend Tarski's undefinability theorem to more sets of natural numbers.

Like Tarski's undefinability theorem, our result is based on Gödel's first incompleteness theorem, which states that every consistent formal system expressive enough to formalize ordinary mathematics is incomplete, in the sense that there exists a formula which can be neither proved to be true nor disproved (its negation is proved to be true) in the system

[9, 13]. Let B_L be the set of Gödel numbers of all formulas that can be proved by a formal system L (that is expressive enough to formalize ordinary mathematics). Gödel's first incompleteness theorem implies that if B_L has any characteristic function, say g, then g cannot be defined in L, because we would have used g to decide if any formula can be proved or disproved (by checking whether its Gödel number is in B_L or not) in the same L, a contradiction to the first incompleteness theorem. Hence, if L is first-order arithmetic, then the characteristic function of B_L cannot be defined in first-order arithmetic. From this result, we show further that there exists a consistent effective formal systema L such that the characteristic function of $G = B_L$ cannot be defined in any consistent effective formal system.

By Cantor's definition [3], a set S is *countable* if either S is finite or there exists a bijection $f : \mathcal{N} \mapsto S$. In the infinite case, we say S is *countably infinite* and f is a *counting bijection* of S. Hence, S is countably infinite iff S has a counting bijection. If f is increasing, i.e., $f(x+1) > f(x)$ for any $x \in \mathcal{N}$, we say S is *increasingly countable*. We show that S is increasingly countable iff S has a characteristic function. Take $G \subset \mathcal{N}$ from the previous paragraph, it implies that G cannot be shown to be increasingly countable in any consistent effective formal system. Hence, for any $S \subset \mathcal{N}$, if our task is to define increasing counting bijections of S only in formal systems where Gödel's incompleteness theorems hold, G will be a counterexample for this task.

In the known proof of the subset claim that every set of natural numbers is countable, an increasing counting bijection is defined in first-order arithmetic for any $S \subset \mathcal{N}$. That is, the proof actually shows a stronger statement than the subset claim: Every set of natural numbers is increasingly countable. Unfortunately, either the true arithmetic T^* from Tarski's undefinability theorem or G mentioned above makes the proof of the subset claim invalid, because no increasing counting bijections are definable for T^* or G in first-order arithmetic. In Section 5, we will present in detail how this conclusion is reached.

2 Background

We first present Gödel's first incompleteness theorem, which is one of the most important results in mathematical logic [8]. In his milestone paper [9], Gödel chose Hilbert's Principia Mathematica (P) [11], a higher-order logic system, as his formal system. If we ignore variables of higher orders, P can be regarded as a small first-order language known today as *Peano arithmetic* (PA), a well-known formal system of first-order arithmetic [18].

Nowadays, a formal system is regarded as a synonym of a language of logic, which uses an alphabet of *symbols* (constants, variables, function and predicate symbols, Boolean operators, etc.) to construct *syntactic items* such as *terms, formulas, axioms, inference rules*, etc. Each formal system or language L considered by Gödel contains the sub-language PA. PA uses symbols 0 (zero), s (successor), $+$ (addition), $*$ (multiplication), '=' (equality predicate), '$\neg$' (negation), '$\vee$' (logical disjunction), '$\forall$' (universal quantifier), 'x', 'y', 'z', ... (variables), '(' and ')' (grouping symbols). Popular logical operators can be defined from $\neg$, $\vee$, and $\forall$. Well-formed *terms* and *formulas* are created from these symbols as in any first-order language. *Axioms* are those formulas that are assumed to be true (e.g., $\forall x\ (x+0 = x)$ and $\forall x \forall y\ (x + s(y) = s(x + y))$). The rules of *primitive recursive functions* are included as axioms. Popular inference rules are included and supported by the axioms. For a discussion on the minimal set of axioms that P accepts, please see [2, 17]. All primitive recursive functions, including all popular arithmetic functions, can be defined in P. Now, the phrase "a formal system expressive enough to formalize ordinary mathematics" can be replaced by P or Peano arithmetic (PA) for simplicity.

A formal system L is *effective* if there exists an algorithm that tells whether any formula is an instance of an axiom of L or not and implements each inference rule of L. For practical purposes, we consider only effective formal systems because Gödel's completeness theorems apply to them [8]. L possesses some inference rules, which are used to prove *theorems* from axioms. Formally, a *proof* of formula ϕ_n in L is a list of formulas $(\phi_1, \phi_2, ..., \phi_n)$ such that for $1 \leq i \leq n$, ϕ_i is either an instance of an axiom of L or the result of an inference rule of L using ϕ_j, $j < i$, as the premises needed by the inference rule. In this case, we say ϕ_n is a *theorem* of L or ϕ_n is *proved* (to be true) in L, write $\vdash_L \phi_n$.

A *sentence* is a closed formula (i.e., no free variable in the formula). L is *inconsistent* if $\vdash_L \phi$ and $\vdash_L \neg\phi$ for some sentence ϕ; L is *consistent* if L is not inconsistent. L is *complete* if for any sentence ϕ, either $\vdash_L \phi$ or $\vdash_L \neg\phi$; L is *incomplete* if L is not complete.

A formal system L' is an *extension* of formal system L if L' is obtained by adding (function and predicate) symbols and axioms to L. The concept of "extension" is best understood when L and L' are regarded as languages, such that the set of symbols (terms, formulas, axioms, or inference rules) in L is a subset of the symbols (terms, formulas, axioms, or inference rules) in L'. L' is an effective extension, or *e-extension* of L, if L' is effective. As an extension of L, L' inherits all axioms and inference rules of L. Now, Gödel's first incompleteness theorem can be stated as follows:

Theorem 2.1 (Gödel's first incompleteness theorem) *Let L be an e-extension of PA. If L is consistent, then L is incomplete.*

The above theorem applies to PA, as PA is a trivial extension of PA. The theorem states that if L is a consistent e-extension of PA, then there exists a sentence ϕ such that neither $\vdash_L \phi$ nor $\vdash_L \neg\phi$. This ϕ is often called a *Gödel sentence* of L.

Given a formal system L, the semantics of syntactic items in L is specified in a metalanguage M which defines truths of the formulas in L. After Tarski's approach on undefinability, people often use an algebraic structure in M to describe the semantics of L. For instance, the structure of natural numbers is used to define *true arithmetic* T, the set of all sentences (i.e., closed formulas) of PA which are true in the structure. This structure consists of $\mathcal{N}$ (the set of all natural numbers, called the underlying set), a collection of arithmetic operations on $\mathcal{N}$, and a finite set of *axioms* that these operations must satisfy. Each natural number $n \in \mathcal{N}$ is represented by the *numeral* $\overline{n}$ (a term of PA), where $\overline{n} = s^n(0)$.

Definition 2.2 *A set $X \subseteq \mathcal{N}$ is said to be* definable *in a formal system L if there exists a formula $\phi(x)$ of L, where x is the only free variable, such that for every $n \in \mathcal{N}$, $n \in X$ iff $\phi(\overline{n})$ is true. X is said to be* arithmetic *if X is definable in L_E.*

To prove the incompleteness theorems, Gödel developed a set L_G of arithmetic relations and functions over $\mathcal{N}$ through *Gödel numbering*. L_G can be regarded as a metalanguage to specify the semantics of formulas in P (Principia Mathematica). Since L_G shares the symbols of PA (Peano arithmetic) and uses the same syntactic rules of P, L_G can be added into P as an extension of P, so that a sentence expressing "I'm not provable" (analogous to the liar paradox) can be specified in one language [9, 13, 17, 8].

Gödel established a one-to-one correspondence between the formulas of P and a set of natural numbers through *Gödel numbering*, which assigns a distinct natural number to each symbol and then constructs a unique natural number to each term, each formula, each list of formulas, etc. By convention, for any syntactic entity t of L, be it terms, formulas, or lists of formulas, we will use $\ulcorner t \urcorner \in \mathcal{N}$ to denote the Gödel number of t [13]. Using L_G, Gödel defined several dozens of primitive recursive relations and functions over $\mathcal{N}$ as *arithmetic interpretation* of predicates and functions in P. One notable primitive recursive relation in Gödel's proof is $pr \subset \mathcal{N}^2$: For $x, y \in \mathcal{N}$, $pr(x, y)$ is true (i.e., $\langle x, y\rangle \in pr$) iff $x = \ulcorner \phi_1, \phi_2, ..., \phi_n \urcorner$, $y = \ulcorner \phi_n \urcorner$, and the list $(\phi_1, \phi_2, ..., \phi_n)$ is a proof of ϕ_n in P. Hence, for any formula ϕ, $\vdash_P \phi$ iff $\exists x\ pr(x, \ulcorner \phi \urcorner)$ is true [13].

Let $\mathcal{A}$ be the structure of natural numbers and $\mathcal{A} \models \phi$ denote that formal ϕ is true in $\mathcal{A}$. *True arithmetic* is defined to be the set of all sentences (i.e., closed formulas) in PA that are true in $\mathcal{A}$, written $T = \{\phi \mid \mathcal{A} \models \phi\}$. That is, true arithmetic is the set of all true first-order statements about the arithmetic of natural numbers. Let $T^* = \{\ulcorner \theta \urcorner \mid \theta \in T\}$ be the set of Gödel numbers of all true sentences in $\mathcal{A}$. Tarski's theorem can be stated as follows [22], where "arithmetic sentences" means "sentences of PA".

Theorem 2.3 (Tarski's Theorem) *The set T^* of Gödel numbers of the true arithmetic sentences is not arithmetic.*

By Definition 2.2, "T^* is not arithmetic" means T^* is not definable in PA. Recall that for any set $X \subseteq \mathcal{N}$, the *characteristic function* of X is a total function $f : \mathcal{N} \mapsto \{0, 1\}$ such that for any $n \in \mathcal{N}$, $n \in X$ iff $f(n) = 1$. We say f is *definable* in L if X is definable in L. Hence, f is definable in L iff there exists a formula $\phi(x)$ of L such that for all $n \in \mathcal{N}$, $f(n) = 1$ iff $\phi(\overline{n})$ is true. In fact, $f(n)$ is the interpretation of $\phi(x)$ in the metalanguage M and $\phi(x)$ is the syntactical counterpart of $f(n)$ in L. This concept of definability can be easily extended to any relation of L if L is an extension of PA.

Definition 2.4 *Let L be an extension of PA and M be a structure whose underlying set is $\mathcal{N}$. A relation $r(n_1, n_2, ..., n_k)$ of M is said to be* definable *in L if there exists a formlua $\phi(x_1, x_2, ..., x_k)$ of L such that $r(n_1, n_2, ..., n_k)$ is true iff $\phi(\overline{n_1}, \overline{n_2}, ..., \overline{n_k})$ is true in M.*

In the above definition, the structure of natural numbers is a substructure of M. This definition applies to functions of M as function $f(x)$ is a relation $r(x, y)$ such that $f(x) = y$ iff $r(x, y)$ is true.

Theorem 2.3 states that T^* is undefinable in PA, hence, we obtain a version of Tarski's theorem in terms of characteristic function:

Theorem 2.5 *The characteristic function of T^* is undefinable in PA.*

We like to point out that the above version is better than Theorem 2.3 in that we require explicitly a *total* Boolean function for T^*, because Tarski's theorem requires a total truth definition [25, 22].

A function $f : \mathcal{N} \mapsto \{0, 1\}$ is called a *decision function* and defines uniquely $S = \{x \in \mathcal{N} \mid f(x) = 1\}$. If f is total, then f is the characteristic function of S. Otherwise, the characteristic function g of S can be obtained from f by $g(x) = 1$ if $f(x) = 1$ and $g(x) = 0$ if $f(x) \neq 1$. A function is *computable* if it can be computed by a Turing machine [27]. A Turing machine is called *algorithm* if it computes a total function. A decision function is *decidable* if it is total and computable (i.e., total computable). A set is *computable* if it is defined by a computable decision function. A set S is *decidable* if it is defined by a decidable decision function (which is the same as the characteristic function of S).

By the Church-Turing thesis, a function is computable iff it is general (or partial) recursive. All primitive recursive functions and relations are total computable, including the relation *pr* in Gödel's proof, where $pr(\ulcorner s \urcorner, \ulcorner t \urcorner)$ is true if s is a proof of t. However, $\exists x\ pr(x, y)$ is computable, not total computable. That is, the set $\{y \mid \exists x\ pr(x, y)\}$ is undecidable [13]. Other notable computable but undecidable sets include the set of natural numbers encoding the halting problem of Turing machines [26].

Let L be an e-extension of PA and $B_L = \{\ulcorner \phi \urcorner \mid \vdash_L \phi\}$. Since $\vdash_L \phi$ iff $\exists x\ pr(x, \ulcorner \phi \urcorner)$ is true, B_L is computable but undecidable [13]. In the next section, we will present Tarski's general form of the undefinability theorem that the characteristic function of B_L is not definable in L [22]. Based on this general form, we prove that there exists an infinite set G of natural numbers whose characteristic function cannot be defined in any consistent effective formal system.

3 Does Every Set Have a Characteristic Function?

To answer the question in the section title, we will inherit the notations from the previous section. Let L be any consistent e-extension of PA (Peano arithmetic) and

$$B_L = \{\ulcorner \phi \urcorner \mid \vdash_L \phi\}.$$

We say formal system L' is *more powerful* than L if $B_L \subset B_{L'}$. L' and L are *equivalent* if $B_L = B_{L'}$. Any non-trivial extension of L is more powerful than L.

Proposition 3.1 *Let L be any consistent e-extension of PA. If B_L has a characteristic function g, then (a) g is not definable in L; (b) g must be defined in a formal system that is equivalent to an extension of L.*

Proof. (a) Let us assume that $g : \mathcal{N} \mapsto \{0, 1\}$ is a characteristic function of B_L such that $g(n) = 1$ iff $n \in B_L$. Let ϕ be any Gödel sentence of L. If g can be defined in L, using g, we can decide in L if $\vdash_L \phi$ or $\vdash_L \neg\phi$:

- If $g(\ulcorner \phi \urcorner) = 1$, then $\ulcorner \phi \urcorner \in B_L$ and $\vdash_L \phi$.
- If $g(\ulcorner \phi \urcorner) = 0$, then $g(\ulcorner \neg\phi \urcorner) = 1$ because g is total and L is consistent. Hence $\ulcorner \neg\phi \urcorner \in B_L$ and $\vdash_L \neg\phi$.

In either case, we have a contradiction to Gödel's first incompleteness theorem.

(b) If g can be defined in another formal system, say L', then L' must be equivalent to a nontrivial extension of L, because (i) for any formula ϕ, $g(\ulcorner\phi\urcorner) = 1$ iff $\vdash_L \phi$, that is, L' can decide every theorem of L through g; (ii) g can only be defined in L', not in L, thus L' must be more powerful than L. □

Proposition 3.1(a) is a direct consequence of Gödel's first incompleteness theorem and often regarded as a general form of Tarski's undefinability theorem [22].

It is known that a Gödel sentence can become proved or disproved in an extension L' of L by adding axioms in L'. However, Gödel's first incompleteness theorem also applies to L' if L' is effective and consistent. That is, L' has its own Gödel sentences. Let $B_{L'} = \{\ulcorner\phi\urcorner \mid \vdash_{L'} \phi\}$. In general, $B_L \subseteq B_{L'}$ as more axioms lead to more theorems. We observe that the existence of characteristic functions for B_L and $B_{L'}$ has a similar property to the existence of Gödel sentences for L and L'. That is, assume L' is a consistent e-extension of L and a characteristic function can be defined for B_L in L', then there is no way to construct in L' any characteristic function of $B_{L'}$ as Proposition 3.1 also applies to L'.

Example 3.2 The *arithmetical hierarchy* proposed by Stephen Kleene classifies formulas of the first-order arithmetic by the use of quantifiers [19, 23]. The classifications of formulas are denoted Σ^0_n (and Π^0_n) for $n \in \mathcal{N}$ (0 in the superscript denotes first-order). For $n > 0$, a Σ^0_n formula is equivalent to a prenex formula that begins with some existential quantifiers and alternates $n-1$ times between series of existential and universal quantifiers. Analogously, a Π^0_n formula, $n > 0$, is equivalent to a prenex formula that begins with some universal quantifiers. It is known that each formula in $\Sigma^0_0 = \Pi^0_0$ defines a decidable set (we say Σ^0_0 *defines* decidable sets). Σ^0_1 defines computable sets, Π^0_1 defines co-computable sets, and Σ^0_n may define uncomputable sets for $n > 1$ [23].

We may define an infinite sequence of formal systems L_n, $n \in \mathcal{N}$, such that L_n deals exactly those formulas in Σ^0_n. Each L_{n+1} is an e-extension of L_n as $\Sigma^0_n \subset \Sigma^0_{n+1}$. We also define S_n, $n > 0$, an infinite sequence of formal systems that deal formulas in Σ^1_n (second-order formulas). Since $\Sigma^0_n \subset \Sigma^1_1$, S_1 is an e-extension of L_n for any $n \in \mathcal{N}$. P (Principia Mathematica) used by Gödel, a higher-order formal system, is an e-extension of S_n for any $n \in \mathcal{N}$. □

The following result is important to the rest of the article.

Theorem 3.3 *There exists a set G of natural numbers whose characteristic function cannot be defined in any consistent effective formal system.*

Proof. Let us consider $B_L = \{\ulcorner\phi\urcorner \mid \vdash_L \phi\}$ from Proposition 3.1. If the characteristic function of B_L cannot be defined in any consistent effective formal system, let G be B_L and the theorem is proved. If B_L has a characteristic function g, then Proposition 3.1 claims that g must be defined in a nontrivial extension of L, but not in L.

Among all consistent e-extensions of L in which g can be defined, we choose one of the most powerful e-extensions as L'. Here, "most powerful" means the set of theorems provable by a formal system is maximal among all considered formal systems. That is, we assume that L' does not have any consistent nontrivial e-extension in which g can be defined. Let $B_{L'} = \{\ulcorner\phi\urcorner \mid \vdash_{L'} \phi\}$. If the characteristic function of $B_{L'}$ cannot be defined in any consistent effective formal system, then let G be $B_{L'}$ and the theorem is proved. If $B_{L'}$ has a characteristic function g', then Proposition 3.1 claims that g' cannot be defined in L'. If g' is defined in a consistent e-extension L'' of L', since g can be also defined in L'', we have a contradiction to the assumption that L' does not have any consistent nontrivial e-extension in which g can be defined. □

Like Gödel's first incompleteness theorem, the above theorem applies to any known or unknown consistent e-extension of PA. The above result provides a negative answer to the question in the section title: If we consider only formal systems where Gödel's first incompleteness theorem holds, there exists a set $G \subset \mathcal{N}$ such that no characteristic functions of G can be defined in any consistent effective formal system.

The limitation of Gödel's first incompleteness theorem also applies to Theorem 3.3: We consider only consistent effective formal systems. Inconsistent formal systems are ruled out as theorems are meaningless in these systems. For non-effective systems, we do not know anyone in which the characteristic function of G can be defined. Moreover, we do not know how to check in general the consistency of non-effective formal systems.

Theorem 3.3 is an extension of Tarski's undefinability theorem with evident difference. Both Tarski's undefinability theorem and Theorem 3.3 state that the characteristic function of a set of natural numbers cannot be defined in a formal system. The difference is that Tarski's true arithmetic, T^*, comes from the standard interpretation of arithmetic (a semantic concept), and his formal system is first-order arithmetic. On the other hand, Theorem 3.3 considers any consistent effective formal system, and the set G of natural numbers comes from the proving power of this formal system (a syntactic concept).

In the application of characteristic functions, people often take for granted that there exists a characteristic function for every set of natural numbers. As mentioned in the introduction, an *oracle* for a set $A \subseteq \mathcal{N}$ is an external device (other than a Turing machine) that can report whether any number $x \in \mathcal{N}$ is a member of A [21]. The oracle in this sense is a synonym of the characteristic function, and they always coexist. That is, if g is the characteristic function of A, we may use g as its oracle; if A does not have any characteristic function, then A does not have any oracle, because, otherwise, we might use the oracle as its characteristic function. If A does not have an oracle, then the Turing reduction from B to A does not make any sense, as any statement can be drawn from a false premise.

Two sets of natural numbers are *Turing equivalent* if they are Turing reducible to each other. A *Turing degree* is a set of Turing equivalent sets [5]. Turing reduction induces a partial order over the set of all Turing degrees to assess the level of unsolvability. A great deal of research has been conducted into the structure of the Turing degrees with this order [15]. However, none of these studies considered the non-existence of oracles in the application of Turing reduction. If the oracles used in these studies cannot be defined in consistent formal systems, then these oracles are either undefinable in any formal system or only definable in inconsistent formal systems. In either case, the degree structure obtained is meaningless, because it is based on a false premise and hence every statement can be proved to be true. It would be interesting to study degree structures with Theorem 3.3 in mind.

The *Turing jump*, also known as the *jump operator*, is an operation that maps each set of natural numbers $X \subseteq \mathcal{N}$ to a strictly more complex set X', where $'$ in X' is the so-called jump operator. It is known that the set of oracle machines is countable and each oracle machine computes a (partial) computable function. Let φ_e^X denote the decision function computed by the e^{th} oracle machine using the oracle of X, then the set Y of natural numbers defined by φ_e^X is Turing reducible to X. The *jump operator* is defined to be $X' = \{e \in \mathcal{N} \mid \varphi_e^X(e) \downarrow\}$, where $\downarrow$ denotes halting on input e [20]. An important property of X' is that X is Turing reducible to X' but X' is not Turing reducible to X. Let $\emptyset$ denote the degree of decidable sets. Then $\emptyset'$ is the set of computable sets (including undecidable sets). For $n \in \mathcal{N}$, let $X^{(n)}$ denote the set obtained from X by applying the jump operator n times. Post's theorem states that, for $n \in \mathcal{N}$, there exists a one-to-one mapping between $\emptyset^{(n)}$ and Σ_n^0 (the arithmetical hierarchy, see Example 3.2) [20].

Since the Turing jump uses oracle Turing machines and an oracle of a set is the characteristic function of the set, the existence of the oracles used in the Turing jump remains our focus. If the oracles used in Post's theorem cannot be defined in a consistent formal system, then Post's theorem is questionable because it is based on a false premise. The jump operator is foundational for studying relative computability, with applications in recursion theory, including proofs of density results and the existence of degrees without minimal covers. We should be vigilant on the existence of oracles in these applications.

To end this section, we like to point out that the existence of characteristic functions is a problem not only for infinite sets, but also for finite sets, as illustrated by the following example.

Example 3.4 For any $S \subset \mathcal{N}$, let $c(S) = 1$ if the characteristic function of S is definable in an effective consistent formal system and 0 otherwise. Take G from Theorem 3.3 and let $Z = \{0, c(G)\}$. Then $Z = \{0, 1\}$ if G's characteristic function is available; otherwise, $Z = \{0\}$. If Z has a characteristic function, say g, we may check whether $g(1) = 1$ or not, to decide if a characteristic function is available for G or not. By Theorem 3.3, g, the characteristic function of Z, cannot be defined in any consistent effective formal system. □

Given the fact that both G from Theorem 3.3 and Z in the above example do not have characteristic functions definable in any consistent effective formal system, we conclude that the following is true.

Proposition 3.5 *The oracle claim is not true if characteristic functions are required to be definable in consistent effective formal systems.*

4 Is Every Finite Set Decidable?

The *cardinality* of a set A, denoted by $|A|$, measures how many elements that A contains. In set theory, the natural numbers in $\mathcal{N}$ are defined as *ordinals*: $0 := \emptyset$ is the first ordinal number; given an ordinal n, the (immediate) *successor* of n is the set $n \cup \{n\}$. That is, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, ..., $n = \{0, 1, ..., n-1\}$, and so on.

According to [14] and [1], a set A is *finite* if there is a bijection $f : n \mapsto A$ for some $n \in \mathcal{N}$. This definition of "finite set" is problematic because we do not always know in what formal system f is defined. Take Z from Example 3.4, $|Z| = 1$ or 2. The exact value of $|Z|$ is open, but we know $|Z| \in \mathcal{N}$. If we know the exact value of $|Z|$, we can tell if G's characteristic function is available or not, where G is taken from Theorem 3.3. By Theorem 3.3, G's characteristic function cannot be defined in any consistent effective formal system. Hence, $|Z|$, as well as bijection $f : |Z| \mapsto Z$ (viewing $|Z|$ as an ordinal), cannot be defined in any consistent effective formal system.

By the definition of "finite set" in set theory [14], Z is not finite. A set is *infinite* if it is not finite [14, 1]. To avoid the awkward situation where Z is considered infinite, we will use the following definition:

Definition 4.1 *A set A is* finite *if $|A| \in \mathcal{N}$, i.e., the cardinality of A is a natural number.*

Even though this definition is against the *uniqueness of cardinality* (see Proposition 3.6.8 [24]), it allows Z of Example 3.4 to be a finite set.

In the introduction of this article, we showed that the finiteness claim, i.e., "every finite set of natural numbers is decidable," is false by reducing the universal halting problem to the decidability of a singleton set.

Proposition 4.2 *If every finite set is decidable, then every decision function is decidable.*

Proof. For every decision function, say $f : \mathcal{N} \mapsto \{0, 1\}$, and for every $x \in \mathcal{N}$, let $S = \{f(x)\}$. S is a finite set because $S \subset \{0, 1\}$. If S is always decidable, then there exists an algorithm (i.e., a Turing machine that halts on every input) $A(S, y)$ that takes S and y as input and returns 1 iff $y \in S$. To compute $f(x)$, we use algorithm $B(x)$ that creates first $S = \{f(x)\}$, then calls $A(S, 1)$ and returns what returned by $A(S, 1)$. That is, $B(x)$ returns 1 iff $A(S, 1)$ returns 1, iff $1 \in S$, and iff $f(x) = 1$. Hence, $B(x)$ is the algorithm that computes $f(x)$, and hence f is decidable (i.e., total computable). □

Corollary 4.3 *The finiteness claim is invalid.*

Since some decision functions are not decidable, it cannot be true that every finite set is decidable, and the finiteness claim must be invalid. Depending on the complexity of f and $x \in \mathcal{N}$, $S = \{f(x)\}$ could be regular, decidable, computable, or uncomputable.

The finiteness claim appears in a textbook on the computability theory [7] (page 8): "any finite set of natural numbers must be decidable." In Sipser's popular textbook on theory of computation, the sample solution of Exercise 3.22 states that "The language A is one of the two languages $\{0\}$ or $\{1\}$. In either case, the language is finite and hence decidable." [21] (page 191)

By convention, given a finite set Σ of symbols called *alphabet*, a *language* in the theory of computation is a subset of Σ^*, the set of all finite-length strings composed of symbols from Σ [21]. The empty string ϵ is always in Σ^*. Since there exists a computable bijection between $\mathcal{N}$ and Σ^*, all the computability results on sets of natural numbers can apply to languages and vice versa.

A language $L \subseteq \Sigma^*$ is *regular* if L is one of the three basic cases: $\emptyset$, $\{\epsilon\}$, or $\{a\}$ if $a \in \Sigma$, or L is recursively constructed from the basic cases by applying a finite number of *regular operations*: union, concatenation, and Kleene star [21]. It is known that every regular language is decidable. However, some finite sets are not regular. In particular, some singleton sets are not regular because their definitions involve complex functions which may not be computable, e.g., $S = \{f(x)\}$. If $f : \mathcal{N} \mapsto \{0,1\}$ is uncomputable, S cannot be regular for every $x \in \mathcal{N}$.

To see what went wrong in the finiteness claim, we propose the following concept.

Definition 4.4 *A set A is called* unambiguous *if A is the union of zero or more singleton sets each of which has a unique candidate for its element.*

By the above definition, the empty set is unambiguous as it is the union of zero sets. A singleton set is unambiguous if its member is fixed. The union of two unambiguous sets is unambiguous. If $f(x)$ is an uncomputable decision function, then the set $S = \{f(x)\}$ is not unambiguous for every $x \in \mathcal{N}$, because we do not know the value of $f(x)$ for every $x \in \mathcal{N}$. The set Z in Example 3.4 is ambiguous, i.e., not unambiguous.

Proposition 4.5 *Every regular set is unambiguous.*

Proof. The three basic cases of regular sets are unambiguous. If A and B are unambiguous, so are $A \cup B$ and wB, where $w \in A$. Hence AB is unambiguous because $AB = \bigcup_{w \in A} wB$. Since $A^* = \{\epsilon\} \cup A^1 \cup A^2 \cup \cdots \cup A^i \cup \cdots$, by induction on i, A^i is unambiguous for any $i \in \mathcal{N}$, hence A^* is unambiguous. □

Proposition 4.6 *Every finite unambiguous set is regular and hence decidable.*

Proof. Every unambiguous singleton set is regular. If A is finite and unambiguous, then A is the union of a finite number of regular singleton sets. □

It is clear now that the finiteness claim is false because it made the implicit assumption that every finite set is unambiguous.

The case analysis is a popular proof technique based on the law of excluded middle. Suppose D is a decision problem, e.g., Goldbach's conjecture, such that $D = 1$ if the decision problem is true and $D = 0$ otherwise. The case analysis usually considers $D = 1$ and $D = 0$ as the two cases, and $\{D\} = \{0\}$ or $\{1\}$. From the view point of finite sets, the case analysis makes the assumption that the set $\{D\}$ is unambiguous in each case. Under this assumption, $\{D\}$ is regular in each case. Without this assumption, $\{D\}$ is not known to be regular, as $\{D\}$ may be ambiguous. If $\{D\}$ were regular in general, we would have designed a finite-state machine to solve D. Similarly, the case analysis is not a method for showing that D is computable in constant time. In either case of $D = 0$ or $D = 1$, the value of D is returned in constant time. However, it does not lead to the conclusion that D is computable in constant time. Such a conclusion falls victim to the fallacy of circular reasoning, because in each case of D, the value of D is assumed to be available. In short, the case analysis is not the method for showing D is decidable or computable, as D may be uncomputable.

If D is known to be a decidable problem, then the value of D will be known if we spend enough time to compute D. In intuitionistic logic, the law of excluded middle does not hold, and D's value is considered unknown if we have not found it yet, even if D is decidable. That is, unambiguous sets in classic logic are not necessarily unambiguous in intuitionistic logic.

Example 4.7 Let $p : \mathcal{N} \mapsto \{0, 1\}$ be the characteristic function for the set P of all prime numbers. It is known that P is decidable, hence the value of $p(x)$ can be obtained in finite time, and $\{p(x)\}$ is unambiguous in classic logic. However, it does not mean that $\{p(x)\}$ is regular now for any $x \in \mathcal{N}$; it means $\{p(x)\}$ is regular after we have computed the value of $p(x)$. As of 2025, the largest known prime number is $M_{136,279,841} = 2^{136,279,841} - 1$, a Mersenne prime of $41,024,320$ digits, reportedly found on October 12, 2024. Hence, $\{p(M_{136,279,841})\} = \{1\}$, an unambiguous set in intuitionistic logic. However, for an infinite number of $n \in \mathcal{N}$, we do not know the value of $p(n)$. Hence, for these n, $\{p(n)\}$ is ambiguous in intuitionistic logic. If n is found to be a new prime, the set $\{p(n)\}$ becomes unambiguous in intuitionistic logic, as the number of candidates for $\{p(n)\}$ becomes 1. □

In the computability theory, an *enumerator* is a variant of Turing machine that writes strings on an output tape [21]. A known result of enumerators is that an enumerator enumerates a language L in canonical order iff L is decidable (Theorem 8.9 of [16]; Theorem 11.4.5 of [27]). This result is true if L is infinite. The result is false when L is finite because an enumerator may not terminate when enumerating a finite language, which is not necessarily decidable.

5 Is Every Set of Natural Numbers Countable?

The subset claim that "every subset of $\mathcal{N}$ is countable" is widely accepted and appears in many textbooks on set theory, logic, discrete mathematics, or theory of computation [10, 12]. In a textbook on set theory [6], it states that "Obviously every subset of a countable set is countable." In a textbook on theory of computation [16], the claim appears as Theorem 8.25. The claim also appears in an influential textbook by Terence Tao [24] (Proposition 8.1.5).

Let us recall some concepts from the literature. A *bijection* $f : A \mapsto B$ is a total, injective and surjective function from A to B. The *inverse* of f, f^{-1}, is a bijection from B to A. A bijection $f : \mathcal{N} \mapsto S$ is called a *counting bijection* of S. We call f^{-1} a *ranking bijection* of S and $f^{-1}(x)$ is the *rank* of $x \in S$. When $S \subseteq \mathcal{N}$, we say f (or f^{-1}) is *increasing* if $f(x) < f(y)$ (or $f^{-1}(x) < f^{-1}(y)$) whenever $x < y$ for every $x, y \in \mathcal{N}$ (or S). If S has an increasing counting bijection, we say S is *increasingly countable*.

A typical proof of the subset claim uses the fact that if we can construct a total function $f : \mathcal{N} \mapsto S$ such that $f(n)$ returns the $(n+1)^{th}$ minimal number of S, then f is an increasing counting bijection of S [24]:

- Let $f(0)$ be the smallest natural number in S.
- For each $n \in \mathcal{N}$, the set $S - \{f(0), f(1), ..., f(n)\}$ is not empty since S is infinite. Define $f(n + 1)$ to be the smallest natural number in $S - \{f(0), f(1), ..., f(n)\}$.

The existence of "the smallest natural number" is backed by the well-ordering principle. By induction on n for any $n \in \mathcal{N}$, we can show that $f(n)$ is the $(n+1)^{th}$ minimal number of S. It is an easy exercise to check that f is an increasing counting bijection of S. In other words, S is proved to be increasingly countable.

To construct f in the proof of the subset claim, strictly speaking, we need the characteristic function of S, i.e., a total function $g : \mathcal{N} \mapsto \{0, 1\}$, such that $x \in S$ iff $g(x) = 1$, to tell us which number is or is not a member of S. Without g, we cannot exclude non-members of S as candidates for the smallest number of S. In other words, the proof of the subset claim assumes implicitly the existence of g. From the viewpoint of logic, this assumption is natural because "$\forall x \in S\ \phi(x)$" is logically equivalent to "$\forall x\ g(x) \to \phi(x)$" for any predicate $\phi(x)$.

The constructed f in the proof of the subset claim is an increasing counting bijection of S [24]. Its inverse, f^{-1}, is an increasing ranking bijection of S. We often assume that f^{-1} can be obtained from f and vice versa. This assumption also assumes implicitly the existence of g. That is, we need g in any application of f^{-1}: $f^{-1}(x)$ is meaningful only when $g(x) = 1$.

It turns out that "having an increasing counting bijection" is equivalent to "having a characteristic function" for any infinite set of natural numbers in PA (Peano Arithmetic) or its extension.

Proposition 5.1 *Let $S \subseteq \mathcal{N}$ be infinite and L an extension of PA. The following statements are logically equivalent:*

1. *S has an increasing counting bijection definable in L.*
2. *S has a characteristic function definable in L.*

Proof. (1) $\to$ (2): Let $f : \mathcal{N} \mapsto S$ be an increasing bijection definable in L. For any $n \in \mathcal{N}$, we define $g(n) = h(n, 0)$, where $h : \mathcal{N} \times \mathcal{N} \mapsto \{0, 1\}$ is defined as follows:

$$\begin{array}{rcl} h(n,m) & = & \textbf{if } (f(m) = n) \textbf{ then } 1 \\ & & \textbf{else if } (f(m) > n) \textbf{ then } 0 \\ & & \qquad \textbf{else } h(n, m+1) \end{array}$$

When $h(n, m)$ returns 1, there exists m such that $f(m) = n$. When $h(n, m)$ returns 0, there exists no $x \geq m$ such that $f(x) = n$, because f is increasing, i.e., $f(x) \geq f(m) > n$. h is a total function and g is the characteristic function of S, since $g(x) = 1$ iff $x \in S$.

(2) $\to$ (1): The above proof of the subset claim could be used here, but using explicitly the characteristic function of S gives us a neater proof. Let $g : \mathcal{N} \mapsto \{0, 1\}$ be the characteristic function of S definable in L. For any $n \in \mathcal{N}$, define $f(n) = h(n, 0)$, where $h : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$ is defined as follows:

$$\begin{array}{rcl} h(n,m) & = & \textbf{if } (g(m) = 1) \\ & & \textbf{then if } (n = 0) \textbf{ then } m \textbf{ else } h(n-1, m+1) \\ & & \textbf{else } h(n, m+1) \end{array}$$

Let $S = \{a_0, a_1, a_2, ...\}$ such that $a_i < a_{i+1}$ for $i \in \mathcal{N}$. For any $n \in \mathcal{N}$, $h(n, 0)$ visits $0, 1, ..., a_0$ before that n is decreased by 1 in the recursive calls of h. Similarly, $h(n-1, a_0+1)$ will visit

$a_0+1, ..., a_1$; $h(n-2, a_1+1)$ will visit $a_1+1, ..., a_2$, and so on, during the recursive calls. Finally, $h(0, a_{n-1}+1)$ will return a_n, the minimal number of $S - \{f(0), f(1), ..., f(n-1)\}$. It is ready to check that

$$f(n) = h(n,0) = h(n-1, a_0+1) = h(n-2, a_1+1) = \cdots = h(0, a_{n-1}+1) = a_n$$

$h(n,m)$ is well-defined because S is infinite and $g(m) = 1$ for an infinite number of m. It is easy to check that f is an increasing counting bijection of S. □

The above proposition shows the coexistence of the increasing counting bijection and the characteristic function for any infinite set of natural numbers. That is, if the increasing counting bijection f of S is definable in PA, so is the characteristic function g of S, and vice versa, because the function h in both cases is definable in PA. If f is total computable, g is also total computable (i.e., decidable), and vice versa, because the function h in both cases is total computable. We summarize these properties in the following corollary of Proposition 5.1.

Corollary 5.2 *Let $S \subseteq \mathcal{N}$ be infinite. The characteristic function of S is definable in PA iff the increasing counting bijection of S is definable in PA.*

Combining Tarski's undefinability theorem (Theorem 2.5) and Proposition 5.1, we have the following result.

Corollary 5.3 *The increasing counting bijection of true arithmetic T^* is not definable in PA.*

Proof. If T^* has an increasing counting bijection $f : \mathcal{N} \mapsto T^*$ definable in Peano arithmetic, by Proposition 5.1, the characteristic function g of T^* is definable in first-order arithmetic, a contradiction to Theorem 2.5. □

Proposition 5.4 *The subset claim lacks valid proof.*

Proof. A typical proof of the subset claim is given by Tao [24] (*Proposition* 8.1.5). The proof defines in PA an increasing counting bijection for any subset of $\mathcal{N}$, including T^*. This is a contradiction to Corollary 5.3. □

Based on the above proposition, we can say that the subset claim is not true, unless we find a valid proof.

The following result tells the relationship between characteristic functions and ranking bijections.

Proposition 5.5 *Let $S \subseteq \mathcal{N}$ be infinite. If the characteristic function of S is definable in PA, then the increasing ranking bijection of S is definable in PA.*

Proof. Let $g : \mathcal{N} \mapsto \{0, 1\}$ be the characteristic function of S. For any $n \in \mathcal{N}$, define $r, t : \mathcal{N} \mapsto \mathcal{N}$ as follows:

$$\begin{array}{rcl} r(x) & = & \textbf{if } (g(x) \neq 1) \textbf{ then } \textit{undefined} \textbf{ else } t(x) - 1 \\ t(x) & = & \textbf{if } (x = 0) \textbf{ then if } (g(0) = 1) \textbf{ then } 1 \textbf{ else } 0 \\ & & \textbf{else if } (g(x) = 1) \textbf{ then } t(x-1) + 1 \textbf{ else } t(x-1) \end{array}$$

For $x \in \mathcal{N}$, $t(x)$ returns the number of elements in S less than or equal to x. If $n \in S$, then $t(n) - 1$ is exactly the rank of n in S. It is an easy exercise to check that r is an increasing ranking bijection of S. Note that (a) $r(n)$ is undefined if $n \notin S$; (b) $t(n) > 0$ if $n \in S$. □

Comparing the above proposition to Proposition 5.1, from the existence of an increasing ranking bijection $r(n)$ of S, we cannot define the characteristic function $g(n)$ of S, because $r(n)$ is a total function from S to $\mathcal{N}$ and undefined for $n \notin S$. Any usage of $r(n)$ requires the value of $g(n)$. By Propositions 5.1 and 5.5, the existence of increasing counting bijection implies the existence of increasing ranking bijection for the same set. However, the inverse is not true if the set's characteristic function is not available. Hence, the coexistence of increasing counting and ranking bijections relies on the oracle claim which is not always true in every consistent formal system (Proposition 3.5).

In [27], it is shown that the properties of counting bijections are related to the computability of a set.

Proposition 5.6 (Proposition 11.4.9 [27]) *Let $S \subseteq \mathcal{N}$ be infinite.*

1. *S is computable iff S has a computable counting bijection.*
2. *S is decidable iff S has a computable increasing counting bijection.*

Proposition 5.6(2) provides a positive example showing that the characteristic function of a set can be defined in the same formal system where the set itself is defined. The above proposition shows clearly the difference between being increasing or not for a counting bijection. By Proposition 5.1, if a set does not have an increasing counting bijection, its characteristic function cannot be defined in the same formal system.

Example 5.7 Recall that, in the proof of the first incompleteness theorem, Gödel defined a primitive recursive relation $pr(x, y)$, such that $pr(x, y)$ is true iff x is the Gödel number of a proof of formula ϕ and $y = \ulcorner\phi\urcorner$. Let $S = \{\ulcorner\phi\urcorner \mid \exists x\ pr(x, \ulcorner\phi\urcorner)\}$, then S is computable but not decidable. By Proposition 5.6(1), S has a computable counting bijection. By Proposition 5.6(2), S does not have a computable increasing counting bijection. We may expect to define an increasing counting bijection of S in a formal system that is more powerful than Turing machines. However, Gödel's first incompleteness theorem implies that we cannot do so in P (Principia Mathematica), a higher order logic system. □

The above example illustrates that it is more difficult to find an increasing counting bijection than a counting bijection for a set, as the former needs more powerful formal systems for its definition.

The failed proof of the subset claim attempted to prove that every set of natural numbers is increasingly countable. An increasingly countable set S has an advantage: Both S's characteristic function and its ranking bijection can be defined. Hence, we may use either bijection as in standard practice.

To see the difficulty of proving or disproving the subset claim, or answering the question in the section title, let us look at the following result.

Theorem 5.8 *There exists an infinite set G of natural numbers whose increasing counting bijection cannot be defined in any consistent effective formal system.*

Proof. Consider G in Theorem 3.3. G is an infinite set because the set of theorems provable by any extension of formal system P is infinite. Assume that G has an increasing counting bijection, say f, which is defined in a consistent effective formal system, say L. By Proposition 5.1, we might define the characteristic function of G in L, a contradiction to Theorem 3.3. □

In Cantor's definition of countable sets, no restriction is given on the formal system in which we define counting bijections. Any attempt to prove the subset claim must consider G in Theorem 5.8, and there are at least three approaches for choosing formal systems in which counting bijections of G are defined.

1. Inconsistent formal systems are considered. When Cantor proposed the concept of countable sets, naïve set theory, an inconsistent theory, was mainstream. Such formal systems are ruled out because every formula is a theorem in these systems.

2. Non-effective formal systems are considered. In non-effective formal systems, we do not have any algorithm to decide if a formula is an instance of an axiom or implement its inference rules. No such formal systems are known for defining a counting bijection of G. The subset claim has its best chance to be proved in such formal systems. However, it is widely believed that Gödel's incompleteness theorems may be extended to non-effective formal systems. If this is the case, it will invalidate the proof of the subset claim. For instance, the general form of Tarski's undefinability theorem applies to formal systems with sufficient capability for *self-reference* that the diagonal lemma holds. The subset claim will fail in such formal systems. Moreover, it is exceedingly difficult to ensure the consistency of non-effective formal systems.

3. Only consistent and effective formal systems are considered. Theorem 5.8 rules out any such formal system in which an increasing counting bijection of G is definable. As the application of any ranking bijection of G requires the characteristic function of G, the only hope lies on finding a non-increasing counting bijection.

If we take the third approach, as millions of mathematicians and logicians do, the set G from Theorem 5.8 cannot be increasingly countable. This theorem lays the ground for thinking that some sets of natural numbers are not increasingly countable, because the increasing counting bijections of these sets cannot be defined using today's "ordinary" mathematical methods and axioms, such as people find in mathematical textbooks.

The subset claim has several equivalent statements (Exercise 1.25 [27]).

Proposition 5.9 *The following statements are logically equivalent:*

1. *Any subset of a countable set is countable.*
2. *Any subset of $\mathcal{N}$ is countable.*
3. *If there is an injective function from set S to $\mathcal{N}$, then S is countable.*
4. *If there is a surjective function from $\mathcal{N}$ to S, then S is countable.*

Since Proposition 5.4 shows that the second statement of the above proposition is not true, the other three statements cannot be true.

By Cantor's definition, every finite set is countable. From the definition of "finite set" in set theory [14], a set A is *finite* if there exists a bijection $f : n \mapsto A$, where n is a finite ordinal. This f is also called *counting bijection* of A. We have argued in the previous section that this definition of "finite set" is problematic, because the counting bijection of Z in Example 3.4 cannot be defined. Hence, the word "countable" does not mean that we can count one by one the elements of every finite set, because its characteristic function may be unknown due to the high complexity of the set's definition and we do not know exactly what elements this set has.

If Cantor's original intention is that "countable" means the counting function of a set, finite or not, is definable in an effective formal system, then we have shown that this counting function is not always definable in a consistent effective formal system. It is counter-intuitive to believe that a countable set contains an uncountable subset, just like believing that a set has the same size as its proper subset, as shown by Hilbert's hotel puzzle [11]. To overcome this counter-intuitivity, we may adopt a new definition of being *uncountable*:

> A set X is *uncountable* if either its cardinality is greater than that of $\mathcal{N}$, or the characteristic function of X cannot be defined in any consistent effective formal system.

Either of the above two conditions prevents X, finite or not, from having any increasing counting bijection.

When a subset of a countable set is uncountable, it is false to claim that $S \cup T$ and $S - T$ are uncountable when S is uncountable and T is countable. For instance, let $S \subset \mathcal{N}$ be uncountable, then both $S \cup \mathcal{N} = \mathcal{N}$ and $S - \mathcal{N} = \emptyset$ are countable. It is interesting to investigate the closure properties of set operations if the subset claim is false.

ZFC (Zermelo-Fraenkel set theory with the axiom of choice) is the standard form of first-order axiomatic set theory and serves as the most common foundation of mathematics. A first-order axiomatic system can be combined with PA (Peano arithmetic) into one formal system of first-order arithmetic. Let L be the combination of PA and ZFC, and $B_L = \{\ulcorner \phi \urcorner \mid \vdash_L \phi\}$. L is effective because both PA and ZFC are effective. Then the characteristic function of B_L is not definable in ZFC (Proposition 3.1). Hence, the increasing counting bijection of B_L cannot be defined in ZFC. The hope of proving the subset claim in ZFC is slim, because ZFC is a first-order formal system while the sets like G in Theorem 3.3 involves formal systems more complex than higher-order formal systems.

6 Conclusions

In summary, from the limitations of formal systems in which functions are defined, we have examined the following general statements in the computability theory:

1. The finiteness claim: Every finite set of natural numbers is decidable.

2. The oracle claim: Every set of natural numbers has a characteristic function.

3. The subset claim: Every set of natural numbers is countable.

We found that the known proofs of these statements made implicit assumptions which are missing in the general statements. As a result, the validities of these general statements are questionable, as they fall victim to the fallacy of hasty generalization.

For the finiteness claim, we found that the definition of "finite set" in set theory is problematic because it relies on the existence of counting functions. We have introduced the concept of "unambiguous set" and shown that every unambiguous finite set is regular and hence decidable. The finiteness claim is false because it assumes that every finite set is unambiguous.

For the oracle claim, we have used the set of Gödel numbers of all theorems in the proof of Gödel's first incompleteness theorem and extended Tarski's undefinability theorem to more sets of natural numbers. We have shown that there exists a set of natural numbers whose characteristic functions cannot be defined in any effective consistent formal system. Hence, the oracle claim is not true if we require that oracles be definable in effective consistent formal systems. If set B does not have an oracle, i.e., a characteristic function, then the Turing reduction from set A to B does not make any sense, as any statement can be drawn from a false premise.

For the subset claim, we have shown the coexistence of the characteristic function and the increasing counting bijection of an infinite set. Combining this result with Tarski's undefinability theorem, we conclude that the subset claim lacks valid proof. Hence, the subset claim remains an open problem. We have suggested a new concept of "uncountability" based on the existence of characteristic functions.

To check the difficulty of proving the subset claim, we introduced the concept of being "increasingly countable", i.e., a set that has an increasing counting bijection. We have shown the existence of a set G of natural numbers which is not increasingly countable if we consider only consistent effective formal systems. That is, G has neither characteristic functions nor increasing counting bijections that can be defined in any consistent effective formal system. This result shows the everlasting influence of Gödel's incompleteness theorems.

To prove or disprove the oracle claim or the subset claim, that is the question for further investigation. In the spirit of Gödel's incompleteness theorems, no matter how powerful a formal system is, we believe that there always exists a set of natural numbers whose characteristic function or counting bijections are not definable in the same formal system. Hence, it is unlikely that the oracle claim or the subset claim are proved.

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Appendix: Rejection Letters from Journal Editors

This article was submitted subsequently to three journals on mathematical logic and received a rejection in each case. In this appendix, we present the letters from the editors in its entirety, except that the names of journals and editors are omitted. We also provide a brief comment on each letter. We hope this representation will be helpful to create rational reviews for this article.

Letter 1

```
Dear Professor Zhang,
   Thank you for allowing us to consider your manuscript

            Three Untrue Statements in Computability Theory

for publication in ***.
   We do not have a full referee report, but quick opinions gathered suggest
that, even assuming its correcness, this paper would not be accepted by our journal.
Our journal's very strict standards force the editors to be extremely selective
among competing articles.  Therefore, rather than begin a refereeing process
that could take months, I am returning your manuscript to you now, to give you
the chance to submit it elsewhere without delay.
   I hope that you will consider submitting other work of yours to *** in the
future.
   Sincerely,
   ***
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My Comments

This is a universal rejection letter that could reject anyone's submission, because it is based on unspoken "very strict standards", not on the content of the submission.

Letter 2

Thank you for submitting your manuscript to ***.

I had a quick look at your submission and came to the conclusion that it cannot be published in ***. Please find my comments below.

We appreciate you submitting your manuscript to *** and thank you for giving us the opportunity to consider your work.

Kind regards,

***

Editor comments:

The main claims in the paper are false and rely on basic misconceptions of the notions involved. I only show this here for the first claim that not every finite subset of $\mathcal{N}$ and even not every singleton subset of $\mathcal{N}$ would be dedicable (i.e. having a computable characteristic function): the author considers some in general noncomputable characteristic function f and forms for each $x \in \mathcal{N}$ the singleton subset $S := \{f(x)\}$ of $\mathcal{N}$. He then claims that this set could not have a computable characteristic function as such a function would allow one to determine the values of f. This, however, is not correct: consider the (primitive recursively) computable function $h : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$, defined by $h(y, z) := 1$, if $y = z$ and $h(y, z) = 0$, if $y \neq z$. Then for each **fixed** number $x \in \mathcal{N}$, the function $g_x : \mathcal{N} \mapsto \mathcal{N}$ defined by $g_x(y) := h(y, f(x))$ clearly is primitively recursively computable as well (as a function in y) and is the characteristic function of S. It is nowhere claimed that g_x is uniformly in x computable (i.e. that $g_x(y)$ would be computable as a function in y *and in x*) which it clearly is not if f was not computable. All what is relevant here is the true fact that for each $x \in \mathcal{N}$ there is exists (noneffectively in x) a computable function, namely g_x which is the characteristic function of $S = \{f(x)\}$.

My Comments

The statement that "$g_x(y) := h(y, f(x))$ *clearly is primitively recursively computable*" is clearly wrong, because if f is not computable, then $f(x)$ may be undefined (i.e., not a member of $\mathcal{N}$), and $h(y, f(x))$ is undefined, too.

Lettter 3

Dear Hantao Zhang

I am writing regarding your submission 'Three untrue statements in computability theory'.

We have received a report on your paper, which the editors have considered. Following the referee's recommendation, they regret that their decision is to reject the paper.

Sincerely

***

======== report =======

The manuscript aims to extend Tarski's undefinability theorem and to demonstrate the falsity of three claims that often appear in textbooks on computability theory. However, it is not clear what the manuscript actually succeeds in showing. A serious problem is the lack of clarity concerning the central notion of definability. The manuscript first (p. 2) discusses definability in the context of Tarski's undefinability theorem. Here, the relevant notion is arithmetical definability. (Contrary to what the manuscript suggests, the theorem does not essentially concern any particular formal system, but only a language.) The manuscript then moves on to Gödel's first incompleteness theorem and states that the characteristic function of the set of (Gödel numbers of) formulas provable in F is not definable in L.

However, this is false if \definable" is understood in the same sense as in Tarski's theorem. Indeed, Gödel's proof centrally relies on defining this set within F (and the definition of its characteristic function follows easily). Only if \definable" now means strongly representable does the claim make sense. But such an unnoticed shift in the meaning of a central concept is unacceptable. Moreover, if \definable" is taken to mean strongly representable, then it is a basic and well-known fact that no recursively axiomatized formal system can define (in this sense) any undecidable set. The standard Representability Theorem states that, for any recursively axiomatized formal system F that contains enough arithmetic, a set is recursive (decidable) iff it is strongly representable in L. A recursively axiomatized formal system cannot provide a decision method for an undecidable set. The corresponding facts for characteristic functions follow quite trivially.

Another issue concerns the imprecision surrounding the notion of an extension of a formal system L. The definition given on p. 4 allows the addition of function and predicate symbols to L. The manuscript then states a general form of Tarski's theorem claiming that T^* is not definable in any extension of PA. But this is false: if new function or predicate symbols may be introduced, then T^* can be defined. The theorem holds only for the unextended language.

These conceptual unclarities permeate the later parts of the manuscript.

Further problematic claims include:

p. 6: \Among all consistent e-extensions of F in which g can be defined, we choose one of the most powerful e-extensions as F'." | By Gödel's incompleteness theorems, no such extension exists.

p. 6: \For non-effective systems, we do not know of any in which the characteristic function of G can be defined." | In fact, we do: true arithmetic can define it, as can already the familiar \semiformal" system which adds to PA all true Pi-1-0 sentences.

Regarding the three claims the manuscript purports to refute: these are obviously intended to be interpreted extensionally. Consider, for example, the claim about the decidability of finite sets. The intended idea clearly is that any finite set can be represented in a decidable manner|for instance, by an explicit finite list. It is trivial that any decidable set, even the singleton $\{0\}$, can also be represented in a non-computable way. But this does not show that all decidable (in the standard sense) sets are really undecidable.

For these reasons, I unfortunately cannot recommend the publication of this manuscript in ***.

My Comments

This referee's report is full of logic errors. My comments are divided into six sections, corresponding to the six parts of the referee's report.

1 Notion of Definability

This section addresses the first paragraph of the report.

In my article, I use "formal system" in order to pay tribute to Gödel. It is common knowledge in logic that Gödel's *formal system* is a language L of logic. This is evident from Smullyan's book, where both Gödel's incompleteness theorems and Tarski's undefinability theorem are treated in a language.

In short, a formal system S is a language L of logic; an extension of S is an extension of L. A set X of natural numbers is not definable in L if the characteristic function of X is not definable as the interpretation of a formula of L. The set of Gödel numbers of proved formulas (by the inference rules specified in L) is just a subset of $\mathcal{N}$ and can be definable or undefinable in L. As carried out in Smullyan's book, we just apply Tarski's notion of definability from L_E to any L. Why an expert of logic failed to see Gödel's formal system as a language and considered using S instead of L as "a serious problem" is beyond my comprehension.

2 Total Functions vs Computable Functions

This section addresses the second paragraph of the report.

As stated in the previous section, we require explicitly that the characteristic function of a set is a *total* Boolean function, not a computable function. This is because Tarski's theorem requires a total truth definition. The above quoted critique says we use computable characteristic functions. That is a typical example of the straw man fallacy.

In my article, I showed that a set X of natural numbers has a characteristic function iff X has an increasing counting bijection. Terrence Tao showed that every set of natural numbers has an increasing counting bijection. If he is right, then every set of natural numbers has a characteristic function. If the referee is also right, then every set of natural number is computable. We questioned Tao's result by saying that some sets may not have a characteristic function, as there exist uncomputable characteristic functions.

3 Extension of Tarski's Theorem

This section addresses the third paragraph of the report.

In my article, nowhere says "T^* is not definable in any extension of PA." This is another example of the straw man fallacy. Our extension of Tarski's theorem is the following: There exists a set of natural numbers whose characteristic function cannot be defined in any effective formal system containing PA.

4 Existence of G

This section addresses the review on p. 6: "Among all consistent e-extensions of L in which g can be defined, we choose one of the most powerful e-extensions as L'." The statement "By Gödel's incompleteness theorems, no such extension exists." is totally wrong. Here, for a given formal system L, g is the characteristic function of the set $B_L = \{\ulcorner\phi\urcorner \mid \vdash_L \phi\}$, all Gödel numbers of the sentences that can be proved in L. By Gödel's incompleteness theorems, g is undefinable in L. Gödel's incompleteness theorems don't say that no extension L' of L exist such that g of L is definable in L'. If it happens that no effective consistent extension L' of L exist such that g of L is definable in L', we take B_L as G. If there exists such a L', we consider L' as L and repeat the process of looking for an extension of L. In either case, G exists.

5 Existence of G

This section addresses the review on p. 6: "For non-effective systems, we do not know of any in which the characteristic function of G can be defined." The statement after "In fact" is totally wrong. G is not true arithmetic T^*. $G = B_L = \{\ulcorner\phi\urcorner \mid \vdash_L \phi\}$, and L is not PA. It is known that there exists a non-effective consistent "semiformal" system for T^*, but it is not known that there exist no effective consistent formal systems for T^*.

6 The Finiteness Claim

This section addresses the review before the last sentence of the report. My article says the proofs of the three claims use implicit assumptions. The review just provided such an assumption: Not "interpreted extensionally". Regarding the finiteness claim, there is no support for the statement that "it is trivial that any decidable set, even the singleton $\{0\}$, can also be represented in a non-computable way." The statement "all decidable (in the standard sense) sets are really undecidable." is self-contradictory and not a statement of my article (a straw man error).