Three Untrue Statements in Computability Theory*

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Abstract

The following general statements often appear in the textbooks on computability theory: (1) Every finite set of natural numbers is decidable; (2) every set of natural numbers has a characteristic function; and (3) every set of natural numbers is countable. The known proofs of these statements made implicit assumptions not mentioned in them. As a result, the validities of these general statements are questionable, as they fall victim to the fallacy of hasty generalization. We will examine each of them in detail and propose remedies.

1 Introduction

The computability theory originated in the 1930s from mathematical logic, pioneered by Kurt Gödel, Alonzo Church, and Alan Turing, who introduced different formal systems (i.e., axiomatic systems) to define what is computable, laying the groundwork for computer science [9, 4, 22]. The equivalence of these formal systems is captured in the so-called Church-Turing thesis. Gödel's two incompleteness theorems, published in 1931, have profound influence on what is uncomputable and demonstrate clearly the limitation of formal systems. To prove the incompleteness theorems, Gödel used ingeniously the diagonal method, originally invented by Georg Cantor for the uncountability of real numbers [3], over the set of *Gödel numbers* of formulas in a formal system.

This article is a continuation of Gödel's approach to the limitations of formal systems, including several closely related theorems, e.g., Tarski's undefinability theorem on the formal undefinability of truth [21] and Turing's theorem that there is no algorithm to solve the halting problem [22].

The following general statements often appear in the textbooks on the theory of computation [19]:

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1. **The finiteness claim**: Every finite set of natural numbers is decidable;
2. **The oracle claim**: Every set of natural numbers has a characteristic function; and
3. **The subset claim**: Every set of natural numbers is countable.

To refute the finiteness claim, it suffices to consider the universal halting problem of Turing machines. Let $f : \mathcal{N} \mapsto \{0, 1\}$ be a function. where $\mathcal{N}$ denotes the set of all natural numbers, such that $f(x) = 1$ if $x = \langle M \rangle$, the encoding of Turing machine M that halts on every input. For any Turing machine M, $R = \{f(\langle M \rangle)\}$ is either $\{0\}$ or $\{1\}$, a singleton set in either case. If R were decidable, so would be the universal halting problem by checking $1 \in R$ or not. The textbook proof of the finiteness claim relies on two statements: (a) every finite formal language is regular [16] (page 271); and (b) every regular language is decidable. While (b) is true, R is a counterexample to (a). We will discuss this topic in Section 4.

We say a total function $f : \mathcal{N} \mapsto \{0, 1\}$ is a *characteristic function* of a set $S \subseteq \mathcal{N}$ if for every $n \in \mathcal{N}$, $f(n) = 1$ if and only if $n \in S$. Function f is unique for S as f is required to be total. *Turing reduction* is an important concept in the theory of computation that uses *oracle*, a synonym of characteristic function. A set A is *Turing reducible* to a set B if A has an *oracle machine*, which is a modified Turing machine that has the additional capability of querying the oracle of B, to decide any x is a member of A [19]. In other terms, A is *Turing reducible* to B if the characteristic function of A is computable, assuming we can query freely the characteristic function of B.

Turing reduction takes for granted the oracle claim, i.e., every set of natural numbers has a characteristic function. We assume that every function is defined in a formal system which has its own limitation. In Section 3, we will investigate formal systems in which characteristic functions of some integer sets are not definable.

Our investigation leads to an extension of Tarski's undefinability theorem on the *true arithmetic*, which is often denoted by T^*, the set of Gödel numbers of all sentences true in the standard model of first-order arithmetic. Tarski's undefinability theorem claims that the characteristic function of T^* cannot be defined in first-order arithmetic. Tarski's notion of *undefinability* involves two concepts: (i) a well-defined set of natural numbers, and (ii) a formal system. In Tarski's undefinability theorem, the set is (the Gödel numbers of true sentences in) true arithmetic and the formal system is a typical one of first-order arithmetic, since by *first-order arithmetic*, we mean a collection of axiomatic systems that formalize the natural numbers and their properties within the framework of first-order logic. In this article, we will use Tarski's notion of undefinability and extend Tarski's undefinability theorem to more sets of natural numbers.

Like Tarski's undefinability theorem, our result is based on Gödel's first incompleteness theorem, which states that every consistent formal system expressive enough to formalize ordinary mathematics is incomplete in the sense that there exists a formula which can be neither proved to be true nor disproved (its negation is proved to be true) in the system [9, 13]. Let B_F be the set of Gödel numbers of all formulas that can be proved by a formal system F (that is expressive enough to formalize ordinary mathematics). Gödel's first incompleteness theorem implies that if B_F has any characteristic function, say g, then g cannot be defined in F, because we would have used g to decide if any formula can be proved or disproved (by checking whether its Gödel number is in B_F or not) in the same

F, a contradiction to the first incompleteness theorem. Hence, if F belongs to first-order arithmetic, then the characteristic function of B_F cannot be defined in first-order arithmetic. From this result, we show further that there exists a consistent effective formal systema F such that the characteristic function of $G = B_F$ cannot be defined in any consistent effective formal system.

By Cantor's definition [3], a set S is *countable* if either S is finite or there exists a bijection $f : \mathcal{N} \mapsto S$. In the infinite case, we say S is *countably infinite* and f is a *counting bijection* of S. Hence, S is countably infinite if and only if S has a counting bijection. We show that when S is infinite, an increasing counting bijection of S exists if and only if S has a characteristic function. Take $G \subset \mathcal{N}$ from the previous paragraph, it implies that no increasing counting bijections of G can be defined in any consistent effective formal system. Hence, for any $S \subset \mathcal{N}$, if our task is to define increasing counting bijections of S only in formal systems where Gödel's incompleteness theorems hold, G will be a counterexample for this task.

In the known proof of the subset claim that every set of natural numbers is countable, an increasing counting bijection is defined in first-order arithmetic for any $S \subset \mathcal{N}$. Either the true arithmetic T^* from Tarski's undefinability theorem or G mentioned above makes the proof of the claim invalid, because no increasing counting bijections are definable for T^* or G in first-order arithmetic. In Section 5, we will present in detail how this conclusion is reached.

2 Background

We first present Gödel's first incompleteness theorem, which is one of the most important results in mathematical logic [8]. In his milestone paper [9], Gödel presented a formal system P (stands for Principia Mathematica), a higher-order logic system. If we ignore variables of higher orders, P can be regarded as a small first-order language known today as *Peano arithmetic* (PA), a formal system of *first-order arithmetic* [18].

The syntax of P uses the following symbols: '0' (constant), 's' (successor function symbol), '+' (addition function symbol), '=' (equality predicate), '$\neg$' (negation), '$\vee$' (logical disjunction), '$\forall$' (universal quantifier), 'x', 'y', 'z', ... (variables), '(' and ')' (grouping symbols). Popular logical operators can be defined from $\neg$, $\vee$, and $\forall$. Well-formed *terms* and *formulas* are created from these symbols as in any first-order language. *Axioms* are those formulas that are assumed to be true (e.g., $\forall x\ (x + 0 = x)$ and $\forall x \forall y\ (x + s(y) = s(x + y))$). The rules of *primitive recursive functions* are included as axioms. Popular inference rules are included and supported by the axioms. For a discussion on the minimal set of axioms that P accepts, please see [2, 17]. As in Peano arithmetic, the set of all natural numbers, $\mathcal{N}$, is the default domain of all interpretations of the formulas in P. All primitive recursive functions, including all popular arithmetic functions, can be defined in P. Now, the phrase "expressive enough to formalize ordinary mathematics" can be replaced by P or Peano arithmetic (PA) for simplicity.

A formal system F is *effective* if there exists an algorithm that tells whether any formula is an axiom of F or not and implements each inference rule of F. For practical purposes, we consider only effective formal systems because Gödel's first completeness theorem applies to them [8]. F possesses some inference rules, which are used to prove *theorems* from axioms. Formally, a *proof* of formula ϕ_n in F is a list of formulas $(\phi_1, \phi_2, ..., \phi_n)$ such that for $1 \leq i \leq n$, ϕ_i is either an instance of an axiom of F or the result of an inference rule of F using ϕ_j, $j < i$, as the premises needed by the inference rule. In this case, we say ϕ_n is a *theorem* of F or ϕ_n is *proved* (to be true) in F, write $\vdash_F \phi_n$.

A *sentence* is a closed formula (i.e., no free variable in the formula). F is *inconsistent* if $\vdash_F \phi$ and $\vdash_F \neg\phi$ for some sentence ϕ; F is *consistent* if F is not inconsistent. F is *complete* if for any sentence ϕ, either $\vdash_F \phi$ or $\vdash_F \neg\phi$; F is *incomplete* if F is not complete.

A formal system F' is an *extension* of formal system F if F' is obtained by adding (function and predicate) symbols and axioms to F; F' is an *e-extension* of F if F' is also effective. As an extension of F, F' inherits all axioms and inference rules of F. Now, Gödel's first incompleteness theorem can be stated as follows:

Theorem 2.1 (Gödel's first incompleteness theorem) *Let F be an e-extension of PA. If F is consistent, then F is incomplete.*

The above theorem applies to PA, as PA is a trivial extension of PA. The theorem states that if F is a consistent e-extension of PA, then there exists a formula ϕ such that neither $\vdash_F \phi$ nor $\vdash_F \neg\phi$. This ϕ is often referred to as a "Gödel sentence" of F.

An essential step of Gödel's proof is to establish a one-to-one correspondence between the formulas of F and a set of natural numbers through *Gödel numbering*, which assigns a distinct natural number to each symbol and then constructs a unique natural number to each term, each formula, each list of formulas, etc. By convention, for any syntactic entity t of F, be it terms, formulas, or lists of formulas, we will use $\ulcorner t \urcorner \in \mathcal{N}$ to denote the Gödel number of t [13].

Using Gödel numbers, Gödel developed several dozens of primitive recursive relations and functions over $\mathcal{N}$ as *arithmetic interpretation* of predicates and functions in P (Principia Mathematica). One notable primitive recursive relation in Gödel's proof is $pr \subset \mathcal{N}^2$: For $x, y \in \mathcal{N}$, $pr(x, y)$ is true (i.e., $\langle x, y \rangle \in pr$) if and only if $x = \ulcorner \phi_1, \phi_2, ..., \phi_n \urcorner$, $y = \ulcorner \phi_n \urcorner$, and the list $(\phi_1, \phi_2, ..., \phi_n)$ is a proof of ϕ_n in P. Hence, for any formula ϕ, $\vdash_P \phi$ if and only if $\exists x\ pr(x, \ulcorner \phi \urcorner)$ is true [13].

Let $\mathcal{A}$ be the standard model of first-order arithmetic and $\mathcal{A} \models \phi$ denote that formal ϕ is true in $\mathcal{A}$. *True arithmetic* is defined to be the set of all sentences (i.e., closed formulas) in the language of first-order arithmetic that are true in $\mathcal{A}$, written $Th(\mathcal{A}) = \{\phi \mid \mathcal{A} \models \phi\}$. That is, true arithmetic is the set of all true first-order statements about the arithmetic of natural numbers. Let F be any consistent formal system of first-order arithmetic, by Gödel's first incompleteness theorem, F cannot prove every member of $Th(\mathcal{A})$. Tarski's undefinability theorem [21] states that there is no total function $g : \mathcal{N} \mapsto \{0, 1\}$ definable in first-order arithmetic such that, for every sentence θ of first-order arithmetic, $\mathcal{A} \models \theta$ if and only if $g(\ulcorner \theta \urcorner) = 1$. Let $T^* = \{\ulcorner \theta \urcorner \mid \theta \in Th(\mathcal{A})\}$, the set of Gödel numbers of all true sentences in the standard model. Then g is the characteristic function of T^* and we obtain a succinct version of Tarski's undefinability theorem.

Theorem 2.2 (Tarski's undefinability theorem) *The characteristic function of T^* is not definable in first-order arithmetic.*

Tarski proved a stronger theorem than Theorem 2.2, referred as the general form of Tarski's undefinability theorem [2]. This theorem states that the characteristic function of T^* is not definable in any extension of PA and with sufficient capability for *self-reference* that the diagonal lemma holds. First-order arithmetic satisfies these preconditions, but the theorem applies to much more general formal systems.

A function $f : \mathcal{N} \mapsto \{0, 1\}$ is called a *decision function* and defines uniquely $S = \{x \in \mathcal{N} \mid f(x) = 1\}$. If f is total, then f is the characteristic function of S. Otherwise, the characteristic function g of S can be obtained from f by $g(x) = 1$ if $f(x) = 1$ and $g(x) = 0$ if $f(x) \neq 1$. A function is *computable* if it can be computed by a Turing machine [23]. A Turing machine is called *algorithm* if it computes a total function. A set S is *decidable* if it is defined by a total computable decision function (which is the same as the characteristic function of S). A set is *computable* if it is defined by a computable decision function f.

By the Church-Turing thesis, a function is computable if and only if it is general (or partial) recursive. All primitive recursive functions and relations are total computable, including the relation pr in Gödel's proof, where $pr(\ulcorner s \urcorner, \ulcorner t \urcorner)$ is true if s is a proof of t. However, $\exists x\ pr(x, y)$ is computable, not total computable. Let F be an e-extension of PA and $B_F = \{\ulcorner \phi \urcorner \mid \vdash_F \phi\}$. Since $\vdash_F \phi$ if and only if $\exists x\ pr(x, \ulcorner \phi \urcorner)$ is true, B_F is computable but undecidable [13, 19]. Other notable computable but undecidable sets include the set of natural numbers encoding the halting problem of Turing machines [22].

In the next section, we will show that the characteristic function of B_F is not definable in F. In particular, there exists an infinite set G of natural numbers whose characteristic function cannot be defined in any consistent effective formal system.

3 Does Every Set Have a Characteristic Function?

To answer the question in the section title, we will inherit the notations from the previous section. Let F be any consistent e-extension of PA (Peano arithmetic) and

$$B_F = \{\ulcorner \phi \urcorner \mid \vdash_F \phi\}.$$

We say formal system F' is *more powerful* than F if $B_F \subset B_{F'}$. F' and F are *equivalent* if $B_F = B_{F'}$. Any non-trivial extension of F is more powerful than F.

Proposition 3.1 *Let F be any consistent e-extension of PA. If B_F has a characteristic function g, then (a) g cannot be defined in F; (b) g must be defined in a formal system that is equivalent to an extension of F.*

Proof. (a) Let us assume that $g : \mathcal{N} \mapsto \{0, 1\}$ is a characteristic function of B_F such that $g(n) = 1$ if and only if $n \in B_F$. Let ϕ be any Gödel sentence of F. If g can be defined in F, using g, we can decide in F if $\vdash_F \phi$ or $\vdash_F \neg\phi$:

- If $g(\ulcorner \phi \urcorner) = 1$, then $\ulcorner \phi \urcorner \in B_F$ and $\vdash_F \phi$.
- If $g(\ulcorner \phi \urcorner) = 0$, then $g(\ulcorner \neg\phi \urcorner) = 1$ because g is total and F is consistent. Hence $\ulcorner \neg\phi \urcorner \in B_F$ and $\vdash_F \neg\phi$.

In either case, we have a contradiction to Gödel's first incompleteness theorem.

(b) If g can be defined in another formal system, say F', then F' must be equivalent to a nontrivial extension of F, because (i) for any formula ϕ, $g(\ulcorner\phi\urcorner) = 1$ if and only if $\vdash_F \phi$, that is, F' can decide every theorem of F through g; (ii) g can only be defined in F', not in F, thus F' must be more powerful than F. □

It is known that a Gödel sentence can become proved or disproved in an extension F' of F by adding axioms in F'. However, Gödel's first incompleteness theorem also applies to F' if F' is effective and consistent. That is, F' has its own Gödel sentences. Let $B_{F'} = \{\ulcorner\phi\urcorner \mid \vdash_{F'} \phi\}$. In general, $B_F \subseteq B_{F'}$ as more axioms lead to more theorems. We observe that the existence of characteristic functions for B_F and $B_{F'}$ has a similar property to the existence of Gödel sentences for F and F'. That is, assume F' is a consistent e-extension of F and a characteristic function can be defined for B_F in F', then there is no way to construct in F' any characteristic function of $B_{F'}$ as Proposition 3.1 also applies to F'.

Example 3.2 Let L be an effective first-order axiomatic system other than first-order arithmetic. We assume that L has a finite alphabet for function and predicate symbols which are different from those of PA. To combine L and PA into one formal system, we introduce a new unary predicate, $num(x)$, with the following axioms:

$$num(0),\ \forall x\ num(s(x)) \leftrightarrow num(x),\ \neg num(f(...)) \text{ for every function symbol } f \text{ of } L.$$

Intuitively, $num(x)$ defines the *type* of natural numbers represented by 0, $s(0)$, $s(s(0))$, etc. Before combining L and PA, we perform the following transformation: For every occurrence of $\forall x\ \phi(x)$ in any formula of PA, either axiom or theorem, we replace $\forall x\ \phi(x)$ by $\forall x\ (\neg num(x) \vee \phi(x))$. For every occurrence of $\forall x\ \phi(x)$ in any formula of L, we replace $\forall x\ \phi(x)$ by $\forall x\ (num(x) \vee \phi(x))$. Let F be the union of L and PA after applying the transformation just described. It is easy to see that the theorems of PA and L remain to be the theorems of F after the transformation. Hence, F is an extension of PA and L.

Since any interpretation of L can be represented using the Herbrand universe of L, we may use $\mathcal{N}$ as the default universe for every interpretation of F with the assumption that each term $s^i(0)$ is represented by $2i \in \mathcal{N}$ and each ground term of L by an odd natural number. Hence, F can be regarded as an axiomatic system of first-order arithmetic. In other words, first-order arithmetic is as expressive as general first-order logic.

Applying Proposition 3.1 to F, we conclude that the characteristic function of $B_F = \{\ulcorner\phi\urcorner \mid \vdash_F \phi\}$ cannot be defined in F. Here, F is a combination of L and PA. If we use P (Principia Mathematica) instead of PA, we can also combine L and P into one formal system, say F', which is a high-order axiomatic system. Since P is an e-extension of PA, F' is an e-extension of F. By Proposition 3.1, $B_{F'}$ cannot be defined in F'. □

The following result is important to the rest of the article.

Theorem 3.3 *There exists a set G of natural numbers whose characteristic function cannot be defined in any consistent effective formal system.*

Proof. Let us consider $B_F = \{\ulcorner\phi\urcorner \mid \vdash_F \phi\}$ from Proposition 3.1. If the characteristic function of B_F cannot be defined in any consistent effective formal system, let G be B_F and the theorem is proved. If B_F has a characteristic function g, then Proposition 3.1 claims that g must be defined in a nontrivial extension of F, but not in F.

Among all consistent e-extensions of F in which g can be defined, we choose one of the most powerful e-extensions as F'. Here, "most powerful" means the set of theorems provable by a formal system is maximal among all considered formal systems. That is, we assume that F' does not have any consistent nontrivial e-extension in which g can be defined. Let $B_{F'} = \{\ulcorner \phi \urcorner \mid \vdash_{F'} \phi\}$. If the characteristic function of $B_{F'}$ cannot be defined in any consistent effective formal system, then let G be $B_{F'}$ and the theorem is proved. If $B_{F'}$ has a characteristic function g', then Proposition 3.1 claims that g' cannot be defined in F'. If g' is defined in a consistent e-extension F'' of F', since g can be also defined in F'', we have a contradiction to the assumption that F' does not have any consistent nontrivial e-extension in which g can be defined. □

Like Gödel's first incompleteness theorem, the above theorem applies to any known or unknown consistent e-extension of PA. The above result provides a negative answer to the question in the section title: If we consider only formal systems where Gödel's first incompleteness theorem holds, there exists a set $G \subset \mathcal{N}$ such that no characteristic functions of G can be defined in any consistent effective formal system.

The limitation of Gödel's first incompleteness theorem also applies to Theorem 3.3: We consider only consistent effective formal systems. Inconsistent formal systems are ruled out as theorems are meaningless in these systems. For non-effective systems, we do not know anyone in which the characteristic function of G can be defined. Moreover, we do not know how to check in general the consistency of non-effective formal systems.

Theorem 3.3 is an extension of Tarski's undefinability theorem with evident difference. Both Tarski's undefinability theorem and Theorem 3.3 state that the characteristic function of a set of natural numbers cannot be defined in a formal system. The difference is that Tarski's true arithmetic, T^*, comes from the standard interpretation of arithmetic (a semantic concept), and his formal system is first-order arithmetic. On the other hand, Theorem 3.3 considers any consistent effective formal system, and the set G of natural numbers comes from the proving power of this formal system (a syntactic concept). Tarski's undefinability theorem has been extended to formal systems with sufficient capability for *self-reference* that the diagonal lemma holds. We can also extend Theorem 3.3 to these formal systems if they are consistent.

In the application of characteristic functions, people often take for granted that there exists a characteristic function for every set of natural numbers. As mentioned in the introduction, an *oracle* for a set $A \subseteq \mathcal{N}$ is an external device (other than a Turing machine) that can report whether any number $x \in \mathcal{N}$ is a member of A [19]. The oracle in this sense is a synonym of the characteristic function, and they always coexist. That is, if g is the characteristic function of A, we may use g as its oracle; if A does not have any characteristic function, then A does not have any oracle, because, otherwise, we might use the oracle as its characteristic function. If A does not have an oracle, then the Turing reduction from B to A does not make any sense, as any statement can be drawn from a false premise.

Two sets of natural numbers are *Turing equivalent* if they are Turing reducible to each other. A *Turing degree* is a set of Turing equivalent sets [5]. Turing reduction induces a partial order over the set of all Turing degrees to assess the level of unsolvability. A great deal of research has been conducted into the structure of the Turing degrees with this order [15]. However, none of these studies considered the non-existence of oracles in the application of Turing reduction. It would be interesting to study the structure of the Turing degrees with Theorem 3.3 in mind.

To end this section, we like to point out that the existence of characteristic functions is a problem not only for infinite sets, but also for finite sets, as illustrated by the following example.

Example 3.4 For any $S \subset \mathcal{N}$, let $c(S) = 1$ if the characteristic function of S is available (i.e., definable in a formal system) and 0 otherwise. Take G from Theorem 3.3 and let $Z = \{0, c(G)\}$. Then $Z = \{0, 1\}$ if G's characteristic function is available; otherwise, $Z = \{0\}$. If Z has a characteristic function, say g, we may check whether $g(1) = 1$ or not, to decide if a characteristic function is available for G or not. By Theorem 3.3, g cannot be defined in any consistent effective formal system, hence, the characteristic function of Z is unknown. □

Given the fact that both G from Theorem 3.3 and Z in the above example do not have characteristic functions definable in any known formal system, we conclude that the following corollary is true.

Corollary 3.5 *The oracle claim lacks a valid proof.*

4 Is Every Finite Set Decidable?

The *cardinality* of a set A, denoted by $|A|$, measures how many elements that A contains. In set theory, the natural numbers in $\mathcal{N}$ are defined as *ordinals*: $0 = \emptyset$ is the first ordinal number; given an ordinal n, the (immediate) *successor* of n is the set $n \cup \{n\}$. That is, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, ..., $n = \{0, 1, ..., n - 1\}$, and so on.

According to [14] and [1], a set A is *finite* if there is a bijection $f : n \mapsto A$ for some $n \in \mathcal{N}$. This definition of "finite set" is problematic because we do not always know the cardinality of A and in what formal system f is defined. Take Z from Example 3.4, $|Z| = 1$ or 2. The exact value of $|Z|$ is open, but we know $|Z| \in \mathcal{N}$. If we know the exact value of $|Z|$, we can tell if G's characteristic function is available or not, where G is taken from Theorem 3.3. By Theorem 3.3, G's characteristic function cannot be defined in any consistent effective formal system. Hence, $|Z|$, as well as bijection $f : |Z| \mapsto Z$ (viewing $|Z|$ as an ordinal), cannot be decided in any consistent effective formal system. By the above definition of "finite set", Z is not finite. In set theory, a set is *infinite* if it is not finite. To avoid this awkward situation, we use this definition: a set A is *finite* if $|A| \in \mathcal{N}$, i.e., the cardinality of A is a natural number. Even though this new definition is against the uniqueness of cardinality (see Proposition 3.6.8 [20]), it allows Z of Example 3.4 to be a finite set.

In the introduction of this article, we showed that the finiteness claim, i.e., "every finite set of natural numbers is decidable," is false by reducing the universal halting problem to the decidability of a singleton set. In fact, for every decision problem, say $p : \mathcal{N} \mapsto \{0, 1\}$, and for every $x \in \mathcal{N}$, the value of $p(x)$ can be decided if $S = \{g(x)\}$ is decidable, where

$g(x) = 1$ if $p(x) = 1$ and $g(x) = 0$ if $p(x) \neq 1$ (i.e., g is a total extension of p). Depending on the complexity of p, S could be regular, decidable, computable, or uncomputable. If the finiteness claim is true, S must be decidable because a singleton set is finite.

The finiteness claim appears in a textbook on the computability theory [7] (page 8): "any finite set of natural numbers must be decidable." In Sipser's popular textbook on the theory of computation [19] (page 191), the sample solution of Exercise 3.22 states that "The language A is one of the two languages $\{0\}$ or $\{1\}$. In either case, the language is finite and hence decidable."

By convention, a *language* in the theory of computation is a subset of Σ^*, the set of all finite-length strings composed of symbols from Σ, a finite set of symbols called *alphabet* [19]. The empty string ϵ is always in Σ^*. Since there exists a computable bijection between $\mathcal{N}$ and Σ^*, all the computability results on sets of natural numbers can apply to languages and vice versa.

A language $L \subseteq \Sigma^*$ is *regular* if L is one of the three basic cases: $\emptyset$, $\{\epsilon\}$, or $\{a\}$ if $a \in \Sigma$, or L is recursively constructed from the basic cases by applying a finite number of *regular operations*: union, concatenation, and Kleene star [19]. It is known that every regular language is decidable. However, some finite sets are not regular. In particular, some singleton sets are not regular because their definitions involve complex functions which may not be computable, e.g., $S = \{g(x)\}$. If $g : \mathcal{N} \mapsto \{0, 1\}$ is not a computable, S cannot be regular.

Definition 4.1 *A set A is called* unambiguous *if A is the union of zero or more singleton sets each of which has a unique candidate for its element.*

By the above definition, the set $S = \{g(x)\}$ is not always unambiguous. The set Z in Example 3.4 is ambiguous.

Proposition 4.2 *Every regular set is unambiguous.*

Proof. The three basic cases of regular sets are unambiguous. If A and B are unambiguous, so are $A \cup B$ and wB, where $w \in A$. Hence AB is unambiguous because $AB = \bigcup_{w \in A} wB$. Since $A^* = \{\epsilon\} \cup A^1 \cup A^2 \cup \cdots \cup A^i \cup \cdots$, by induction on i, A^i is unambiguous for any $i \in \mathcal{N}$, hence A^* is unambiguous. □

Proposition 4.3 *Every finite unambiguous set is regular and hence decidable.*

Proof. Every unambiguous singleton set is regular. If A is finite and unambiguous, then A is the union of a finite number of regular singleton sets. □

It is clear now that the finiteness claim is false because it made the implicit assumption that every finite set is unambiguous.

In the computability theory, an *enumerator* is a variant of Turing machine that writes strings on an output tape [19]. A known result of enumerators is that an enumerator enumerates a language L in canonical order if and only if L is decidable (Theorem 8.9 of [16]; see also Exercise 3.18 of [19]). This result is true if L is infinite. The result is false when L is finite because an enumerator may not terminate when enumerating a finite language, which is not necessarily decidable.

5 Is Every Set of Natural Numbers Countable?

The subset claim that "every subset of $\mathcal{N}$ is countable" is widely accepted and appears in many textbooks on set theory, logic, discrete mathematics, or theory of computation [10, 12]. In a textbook on set theory [6], it states that "Obviously every subset of a countable set is countable." In a textbook on theory of computation [16], the claim appears as Theorem 8.25. The claim also appears in an influential textbook by Terence Tao [20] (Proposition 8.1.5).

A *bijection* $f : A \mapsto B$ is a total, injective and surjective function from A to B. The *inverse* of f, f^{-1}, is a bijection from B to A. Recall that a bijection $f : \mathcal{N} \mapsto S$ is called a *counting bijection* of S. We call f^{-1} a *ranking bijection* of S and $f^{-1}(x)$ is the *rank* of $x \in S$. We say f (or f^{-1}) is *increasing* if $f(x) < f(y)$ (or $f^{-1}(x) < f^{-1}(y)$) whenever $x < y$ for every $x, y \in \mathcal{N}$ (or S).

A typical proof of the subset claim uses the fact that if we can construct a total function $f : \mathcal{N} \mapsto S$ such that $f(n)$ returns the $(n+1)^{th}$ minimal number of S, then f is an increasing counting bijection of S [20]:

- Let $f(0)$ be the smallest natural number in S.
- For each $n \in \mathcal{N}$, the set $S - \{f(0), f(1), ..., f(n)\}$ is not empty since S is infinite. Define $f(n+1)$ to be the smallest natural number in $S - \{f(0), f(1), ..., f(n)\}$.

The existence of "the smallest natural number" is backed by the well-ordering principle. By induction on n for any $n \in \mathcal{N}$, we can show that $f(n)$ is the $(n+1)^{th}$ minimal number of S. It is an easy exercise to check that f is an increasing counting bijection of S.

To construct f in the proof of the subset claim, strictly speaking, we need the characteristic function of S, i.e., a total function $g : \mathcal{N} \mapsto \{0, 1\}$, such that $x \in S$ if and only if $g(x) = 1$, to tell us which number is or is not a member of S. Without g, we cannot exclude non-members of S as candidates for the smallest number of S. In other words, the proof of the subset claim assumes implicitly the existence of g. From the viewpoint of logic, this assumption is natural because "$\forall x \in S\ \phi(x)$" is logically equivalent to "$\forall x\ g(x) \rightarrow \phi(x)$" for any predicate $\phi(x)$.

The defined f in the proof of the subset claim is a counting bijection of S. Its inverse, f^{-1}, is a ranking bijection of S. We often assume that f^{-1} can be obtained from f and vice versa. This assumption also assumes implicitly the existence of g. That is, we need g in the application of f^{-1}: $f^{-1}(x)$ is meaningful only when $g(x) = 1$.

It turns out that "having an increasing counting bijection" is equivalent to "having a characteristic function" for any infinite set of natural numbers.

Proposition 5.1 *Let $S \subseteq \mathcal{N}$ be infinite and F an extension of first-order arithmetic. The following statements are logically equivalent:*

1. *S has an increasing counting bijection definable in F.*
2. *S has a characteristic function definable in F.*

Proof. (1) → (2): Let $f : \mathcal{N} \mapsto S$ be an increasing bijection definable in F. For any $n \in \mathcal{N}$, we define $g(n) = h(n, 0)$, where $h : \mathcal{N} \times \mathcal{N} \mapsto \{0, 1\}$ is defined as follows:

$$\begin{array}{rcl} h(n,m) & = & \textbf{if } (f(m) = n) \textbf{ then } 1 \\ & & \textbf{else if } (f(m) > n) \textbf{ then } 0 \\ & & \quad \textbf{else } h(n, m+1) \end{array}$$

When $h(n, m)$ returns 1, there exists m such that $f(m) = n$. When $h(n, m)$ returns 0, there exists no $x \geq m$ such that $f(x) = n$, because f is increasing, i.e., $f(x) \geq f(m) > n$. h is a total function and g is the characteristic function of S, since $g(x) = 1$ if and only if $x \in S$.

(2) → (1): The above proof of the subset claim could be used here, but using explicitly the characteristic function of S gives us a neater proof. Let $g : \mathcal{N} \mapsto \{0, 1\}$ be the characteristic function of S definable in F. For any $n \in \mathcal{N}$, define $f(n) = h(n, 0)$, where $h : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$ is defined as follows:

$$\begin{array}{rcl} h(n,m) & = & \textbf{if } (g(m) = 1) \\ & & \textbf{then if } (n = 0) \textbf{ then } m \textbf{ else } h(n-1, m+1) \\ & & \textbf{else } h(n, m+1) \end{array}$$

Let $S = \{a_0, a_1, a_2, ...\}$ such that $a_i < a_{i+1}$ for $i \in \mathcal{N}$. For any $n \in \mathcal{N}$, $h(n, 0)$ visits $0, 1, ..., a_0$ before that n is decreased by 1 in the recursive calls of h. Similarly, $h(n-1, a_0+1)$ will visit $a_0 + 1, ..., a_1$; $h(n-2, a_1+1)$ will visit $a_1 + 1, ..., a_2$, and so on, during the recursive calls. Finally, $h(0, a_{n-1}+1)$ will return a_n, the minimal number of $S - \{f(0), f(1), ..., f(n-1)\}$. It is ready to check that

$$f(n) = h(n, 0) = h(n-1, a_0+1) = h(n-2, a_1+1) = \cdots = h(0, a_{n-1}+1) = a_n$$

$h(n, m)$ is well-defined because S is infinite and $g(m) = 1$ for an infinite number of m. It is easy to check that f is an increasing counting bijection of S. □

The above proposition shows the coexistence of the increasing counting bijection and the characteristic function for any infinite set of natural numbers in an extension of first-order arithmetic. That is, if the increasing bijection f of S is definable in first-order arithmetic, so is the characteristic function g of S, and vice versa, because the function h in both cases is definable in first-order arithmetic. If f is total computable, g is also total computable (i.e., decidable), and vice versa, because the function h in both cases is total computable. According to the Church-Turing thesis, the formal system of Turing machines is equivalent to the formal system of general (or partial) recursive functions. We summarize the above properties in the following corollary of Proposition 5.1.

Corollary 5.2 *Let $S \subseteq \mathcal{N}$ be infinite. The characteristic function of S is definable in an extension of general recursive functions if and only if the increasing counting bijection of S is definable in the same extension.*

Combining Tarski's undefinability theorem (Theorem 2.2) and Proposition 5.1, we have the following result.

Corollary 5.3 *The increasing counting bijection of true arithmetic T^* is not definable in first-order arithmetic.*

Proof. If T^* has an increasing counting bijection $f : \mathcal{N} \mapsto T^*$ definable in first-order arithmetic, by Proposition 5.1, the characteristic function g of T^* is definable in first-order arithmetic, a contradiction to Theorem 2.2. □

Proposition 5.4 *The subset claim lacks valid proof.*

Proof. A typical proof of the subset claim is given by Tao [20] (*Proposition* 8.1.5). The proof defines in first-order arithmetic an increasing counting bijection for any subset of $\mathcal{N}$, including T^*. This is a contradiction to Corollary 5.3. □

Based on the above proposition, we can say that the subset claim is not true, unless we find a valid proof.

The following result tells the relationship between characteristic functions and ranking bijections.

Proposition 5.5 *Let $S \subseteq \mathcal{N}$ be infinite. If the characteristic function of S is definable in a formal system, then the increasing ranking bijection of S is definable in the same formal system.*

Proof. Let $g : \mathcal{N} \mapsto \{0, 1\}$ be the characteristic function of S. For any $n \in \mathcal{N}$, define $r, t : \mathcal{N} \mapsto \mathcal{N}$ as follows:

$$\begin{array}{rcl} r(x) & = & \textbf{if } (g(x) \neq 1) \textbf{ then } \textit{undefined} \textbf{ else } t(x) - 1 \\ t(x) & = & \textbf{if } (x = 0) \textbf{ then if } (g(0) = 1) \textbf{ then } 1 \textbf{ else } 0 \\ & & \textbf{else if } (g(x) = 1) \textbf{ then } t(x-1) + 1 \textbf{ else } t(x-1) \end{array}$$

For $x \in \mathcal{N}$, $t(x)$ returns the number of elements in S less than or equal to x. If $n \in S$, then $t(n) - 1$ is exactly the rank of n in S. It is an easy exercise to check that r is an increasing ranking bijection of S. Note that (a) $r(n)$ is undefined if $n \notin S$; (b) $t(n) > 0$ if $n \in S$. □

Comparing the above proposition to Proposition 5.1, from the existence of an increasing ranking bijection of S, we cannot define the characteristic function of S, because $r(n)$ is a total function from S to $\mathcal{N}$ and undefined for $n \notin S$. Any usage of $r(n)$ requires the existence of g. By Propositions 5.1 and 5.5, the existence of increasing counting bijection implies the existence of increasing ranking bijection for the same set. However, the inverse is not true if the set's characteristic function is not available. It would be interesting to investigate general conditions for the coexistence of counting and ranking bijections when considering the limitation of formal systems.

In [23], it is shown that the properties of counting bijections are related to the computability of a set.

Proposition 5.6 (**Proposition 11.4.9** [23]) *Let $S \subseteq \mathcal{N}$ be infinite.*

1. *S is computable if and only if S has a computable counting bijection.*

2. S is decidable if and only if S has a computable increasing counting bijection.

Proposition 5.6(2) provides a positive example showing that the characteristic function of a set can be defined in the same formal system (i.e., of the total general recursive functions) where the set itself is defined. The above proposition shows clearly the difference between being increasing or not for a counting bijection. By Proposition 5.1, if a set does not have an increasing counting bijection, its characteristic function cannot be defined in the same formal system.

Example 5.7 Recall that, in the proof of the first incompleteness theorem, Gödel defined a primitive recursive relation $pr(x, y)$, such that $pr(x, y)$ is true if and only if x is the Gödel number of a proof of formula ϕ and $y = \ulcorner(\neg\phi)$. Let $S = \{\ulcorner\phi\urcorner \mid \exists x\ pr(x, \ulcorner\phi\urcorner)\}$, then S is computable but not decidable. By Proposition 5.6(1), S has a computable counting bijection. By Proposition 5.6(2), S does not have a computable increasing counting bijection. We may expect to define an increasing counting bijection of S in a formal system that is more powerful than Turing machines. However, Gödel's first incompleteness theorem implies that we cannot do so in P (Principia Mathematica), a higher order logic system. □

The above example illustrates that it is more difficult to find an increasing counting bijection than a counting bijection for a set, as the former needs more powerful formal systems for its definition.

Definition 5.8 *A set S of natural numbers is* strongly countable *if either S is finite or S has an increasing counting bijection.*

The failed proof of the subset claim attempted to prove that every set of natural numbers is strongly countable. A strongly countable set S has an advantage: Both S's characteristic function and its increasing ranking bijection can be defined. Hence, we may use either bijection or its inverse as in standard practice.

To see the difficulty of proving or disproving the subset claim, or answering the question in the section title, let us look at the following result.

Theorem 5.9 *There exists an infinite set G of natural numbers whose increasing counting bijection cannot be defined in any consistent effective formal system.*

Proof. Consider G in Theorem 3.3. G is an infinite set because the set of theorems provable by any extension of formal system P is infinite. Assume that G has an increasing counting bijection, say f, which is defined in a consistent effective formal system, say F. By Proposition 5.1, we might define the characteristic function of G in F, a contradiction to Theorem 3.3. □

In Cantor's definition of countable sets, no restriction is given on the formal system in which we define counting bijections. Any attempt to prove the subset claim must consider G in Theorem 5.9, and there are at least three approaches for choosing formal systems in which counting bijections of G are defined.

1. Inconsistent formal systems are considered. When Cantor proposed the concept of countable sets, naïve set theory, an inconsistent theory, was mainstream. Such formal systems are ruled out because every formula is a theorem in these systems.

2. Non-effective formal systems are considered. In non-effective formal systems, we do not have any algorithm to decide if a formula is an instance of an axiom or implement its inference rules. No such formal systems are known for defining a counting bijection of G. The subset claim has its best chance to be proved in such formal systems. However, it is widely believed that Gödel's incompleteness theorems may be extended to non-effective formal systems. If this is the case, it will invalidate the proof of the subset claim. For instance, the general form of Tarski's undefinability theorem applies to formal systems with sufficient capability for *self-reference* that the diagonal lemma holds. The subset claim will fail in such formal systems. Moreover, it is very difficult to ensure the consistency of non-effective formal systems.

3. Only consistent and effective formal systems are considered. Theorem 5.9 rules out any such formal system in which an increasing counting bijection of G is definable. As the application of any ranking bijection of G requires the characteristic function of G, the only hope lies on finding a non-increasing counting bijection.

If we take the third approach, as millions of mathematicians and logicians do, the set G from Theorem 5.9 cannot be strongly countable. This theorem lays the ground for thinking that some sets of natural numbers are not strongly countable, because the increasing counting bijections of these sets cannot be defined using today's "ordinary" mathematical methods and axioms, such as people find in mathematical textbooks.

The subset claim has several equivalent statements (Exercise 1.25 [23]).

Proposition 5.10 *The following statements are logically equivalent:*

1. *Any subset of a countable set is countable.*

2. *Any subset of* $\mathcal{N}$ *is countable.*

3. *If there is an injective function from set* S *to* $\mathcal{N}$*, then* S *is countable.*

4. *If there is a surjective function from* $\mathcal{N}$ *to* S*, then* S *is countable.*

Since Proposition 5.4 shows that the second statement of the above proposition is not true, the other three statements cannot be true. When a subset of a countable set is uncountable, it is false to claim that $S \cup T$ and $S - T$ are uncountable when S is uncountable and T is countable. For instance, let $S \subset \mathcal{N}$ be uncountable, then both $S \cup \mathcal{N} = \mathcal{N}$ and $S - \mathcal{N} = \emptyset$ are countable. It is interesting to investigate the closure properties of set operations if the subset claim is false.

By Cantor's definition, every finite set is countable. From the definition of "finite set" in set theory [14], a set A is *finite* if there exists a bijection $f : n \mapsto A$, where n is a finite ordinal. This f is also called *counting bijection* of A. We have argued in the previous section that this definition of "finite set" is problematic, because the counting bijection of

Z in Example 3.4 cannot be defined. Hence, the word "countable" does not mean that we can count one by one the elements of every finite set, because the membership of this set, i.e., its characteristic function, may be unknown due to the high complexity of the set's definition.

If Cantor's original intention is that "countable" means the counting function of a set, finite or not, is definable in an effective formal system, then we have shown the limitation of effective formal systems for this counting function. It is counter-intuitive to believe that a countable set contains an uncountable subset, just like believing that a set has the same size as its proper subset, as shown by Hilbert's hotel puzzle [11]. To overcome this counter-intuitivity, we may adopt a new definition of being *uncountable*:

> A set X is *uncountable* if either its size is larger than that of $\mathcal{N}$, or the characteristic function of X cannot be defined in any consistent effective formal system.

Either of the above two conditions prevents X, finite or not, from having any counting bijection.

ZFC (Zermelo-Fraenkel set theory with the axiom of choice) is the standard form of first-order axiomatic set theory and serves as the most common foundation of mathematics. Example 3.2 demonstrates that a first-order axiomatic system can be combined with PA (Peano arithmetic) into one formal system of first-order arithmetic. Let F be the combination of PA and ZFC, and $B_F = \{\ulcorner\phi\urcorner \mid \vdash_F \phi\}$. F is effective because both PA and ZFC are effective. Then the characteristic function of B_F is not definable in ZFC (Proposition 3.1). Hence, the increasing counting bijection of B_F cannot be defined in ZFC. The hope of proving the subset claim in ZFC is slim, because ZFC is a first-order formal system while the sets like G in Theorem 3.3 involves formal systems more complex than higher-order formal systems.

6 Conclusions

In summary, from the limitations of formal systems in which functions are defined, we have examined the following general statements in the computability theory:

1. The finiteness claim: Every finite set of natural numbers is decidable.
2. The oracle claim: Every set of natural numbers has a characteristic function.
3. The subset claim: Every set of natural numbers is countable.

We found that the known proofs of these statements made implicit assumptions which are missing in the general statements. As a result, the validities of these general statements are questionable, as they fall victim to the fallacy of hasty generalization.

For the finiteness claim, we found that the definition of "finite set" in set theory is problematic because it relies on the existence of counting functions. We have introduced the concept of "unambiguous set" and shown that every unambiguous finite set is regular and hence decidable. The finiteness claim is false because it assumes that every finite set is unambiguous.

For the oracle claim, we have used the set of Gödel numbers of all theorems in the proof of Gödel's first incompleteness theorem and extended Tarski's undefinability theorem to more sets of natural numbers. We have shown that there exists a set of natural numbers whose characteristic functions cannot be defined in any effective consistent formal system. Hence, the oracle claim does not necessarily hold for arbitrary sets of natural number. If B does not have an oracle, i.e., a characteristic function, then the Turing reduction from A to B does not make any sense, as any statement can be drawn from a false premise.

For the subset claim, we have shown the coexistence of the characteristic function and increasing counting bijection of an infinite set. Combining the two results, we conclude that the subset claim that "every subset of $\mathcal{N}$ is countable" lacks valid proof. Hence, the subset claim remains an open problem. We have suggested a new concept of "uncountability" based on the existence of characteristic functions.

To study the difficulty of proving the subset claim, we introduced the concept of being "strongly countable", i.e., a set is either finite or has an increasing counting bijection. We have shown the existence of a set G of natural numbers which is not strongly countable if we consider only effective formal systems. That is, G has neither characteristic functions nor increasing counting bijections that can be defined in any consistent effective formal system. This result shows the everlasting influence of Gödel's incompleteness theorems.

To prove or disprove the oracle claim or the subset claim, that is the question for further investigation. In the spirit of Gödel's incompleteness theorems, no matter how powerful a formal system is, we believe that there always exists a set of natural numbers whose characteristic function or counting bijections are not definable in the same formal system. Hence, it is unlikely that these general statements are proved.

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