From Gödel's First Incompleteness Theorem to the Uncountability of a Set of Natural Numbers

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Abstract

In the proof of Gödel's first incompleteness theorem, given a formal system F, Gödel defined a set B_F of natural numbers which are the Gödel numbers of formulas that can be proved in F. We show that there exists a formal system F' such that the characteristic function of $B_{F'}$ cannot be constructed in any consistent formal system. We show further that an infinite set S of natural numbers is countable if and only if S has a characteristic function. Excluding inconsistent formal systems, we conclude that $B_{F'}$ is uncountable, because if $B_{F'}$ were countable, we would have obtained a characteristic function of $B_{F'}$. $B_{F'}$ serves as a counterexample to the claim that "every set of natural numbers is countable," which appears in many textbooks on set theory, logic, or discrete mathematics.

1 Introduction

Gödel's two incompleteness theorems, published by Kurt Gödel in 1931, have great influence in logic and mathematics [4]. The first incompleteness theorem states that every consistent formal system expressive enough to formalize ordinary mathematics is incomplete in the sense that there exists a formula that neither be proved to be true nor disproved (its negation is proved to be true) in the system [4, 8]. To prove this theorem, Gödel used an ingenious encoding, called *Gödel numbering*, to represent any formula ϕ of the formal system by a natural number, denoted by $\ulcorner\phi\urcorner$, the *Gödel number* of ϕ. The set of all formals that can be proved by the formal system can be represented by the set B of all Gödel numbers of these formulas. Obviously, $B \subset \mathcal{N}$ is infinite, where $\mathcal{N}$ denotes the set of all natural numbers.

We say a total function $f : \mathcal{N} \mapsto \{0, 1\}$ is the *characteristic function* of a set $S \subseteq \mathcal{N}$ if for every $n \in \mathcal{N}$, $f(n) = 1$ if and only if $n \in S$. Gödel's first incompleteness theorem implies that if B has any characteristic function, say f_B, then f_B cannot be constructed in the same formal system, because we would have used f_B to decide if any formula can be proved or not (by checking whether its Gödel number is in B or not) in the same formal system, a contradiction to the first incompleteness theorem. From this result, we show further that there exists a set G of natural numbers whose characteristic function cannot be constructed in any consistent formal system.

By Cantor's definition [1], a set S is *countable* if either S is finite or there exists a bijection $f : \mathcal{N} \mapsto S$. In the infinite case, we say f is a *counting bijection* of S. S is *countably infinite* if S has a counting bijection. We show that when S is infinite, a counting bijection of S exists if and only if S has a characteristic function. Take $G \subset \mathcal{N}$ from the previous paragraph, it implies that no counting bijection of G can be constructed in any consistent formal system. We conclude that G is uncountable, unless we accept inconsistent formal systems. This conclusion is contradictory to the claim that any set of natural numbers is countable. In the following, we will present in detail how this conclusion is reached.

2 Does Every Subset of $\mathcal{N}$ have a Characteristic Function?

To answer the question in the section title, we first present Gödel's first incompleteness theorem, which is one of the most important results in mathematical logic.

In Gödel's milestone paper [4], Gödel presented a formal system P (stands for *Principia Mathematica*), a higher-order logic system. If we ignore variables of higher orders, P can be regarded as a small first-order language known today as *Peano arithmetic* [12].

The syntax of P uses the following symbols: '0' (constant), 's' (successor function symbol), '+' (addition function symbol), '=' (equality predicate), '$\neg$' (negation), '$\vee$' (logical disjunction), '$\forall$' (universal quantifier), 'x', 'y', 'z', ... (variables), '(' and ')' (grouping symbols). Popular logical operators can be defined from $\neg$, $\vee$, and $\forall$. Well-formed *terms* and *formulas* are created from these symbols as in any first-order language. *Axioms* are those formulas that are assumed to be true (e.g., $\forall x\ (x + 0 = x)$ and $\forall x \forall y\ (x + sy = s(x + y))$). The rules of *primitive recursive functions* are included as axioms. Popular inference rules are included and supported by the axioms. For a discussion on the minimal set of axioms that P accepts, please see [11]. As in Peano arithmetic, the set of all natural numbers, $\mathcal{N}$, is the default domain of all interpretations of the formulas in P. All primitive recursive functions, including all popular arithmetic functions, can be constructed in P. Now, the phrase "expressive enough to formalize ordinary mathematics" can be replaced by P or Peano arithmetic for simplicity.

A formal system F is an *extension* of P if F is obtained by adding axioms to P and there exists an algorithm that tells whether any formula is an axiom or not. As a formal system, F inherits P's inference rules, which are used to prove *theorems* from axioms. A *proof* of formula ϕ_n in F is a list of formulas $(\phi_1, \phi_2, ..., \phi_n)$ such that for $1 \leq i \leq n$, ϕ_i is either an instance of an axiom of F or the result of an inference rule of F using ϕ_j, $j < i$, as the premises needed by the inference rule. In this case, we say ϕ_n is a *theorem* of F or ϕ_n is *proved* (to be true) in F, and write $\vdash_F \phi_n$. F is *inconsistent* if $\vdash_F \phi$ and $\vdash_F \neg\phi$ for some formula ϕ; F is *consistent* if F is not inconsistent. F is *complete* if for any formula ϕ, either $\vdash_F \phi$ or $\vdash_F \neg\phi$; F is *incomplete* if F is not complete. Now, Gödel's first incompleteness theorem can be stated as follows:

Theorem 2.1 (Gödel's First Incompleteness Theorem) *Let F be an extension of P. If F is consistent, then F is incomplete.*

This theorem applies to P, as P is a trivial extension of P. The theorem states that if F is a consistent extension of P, then there exists a formula ϕ such that neither $\vdash_F \phi$ nor $\vdash_F \neg\phi$. This ϕ is often referred to as a "Gödel sentence" of F.

An essential step of Gödel's proof is to establish a one-to-one correspondence between the formulas of F and a set of natural numbers through *Gödel numbering*, which assigns a distinct natural number to each symbol and then constructs a unique natural number to each term, each formula, each list of formulas, etc. [8]. By convention, for any syntactic entity t of F, be it terms, formulas, or lists of formulas, we will use $\ulcorner t \urcorner \in \mathcal{N}$ to denote the Gödel number of t [8].

Using Gödel numbers, Gödel developed several dozens of primitive recursive relations and functions over $\mathcal{N}$ as *arithmetic interpretation* of predicates and functions in F. One notable primitive recursive relation in Gödel's proof is $pr \subset \mathcal{N}^2$: For $a, b \in \mathcal{N}$, $pr(a, b)$ is true (i.e., $\langle a, b\rangle \in pr$) if and only if $a = \ulcorner \phi_1, \phi_2, ..., \phi_k \urcorner$, $b = \ulcorner \phi_k \urcorner$, and the list $(\phi_1, \phi_2, ..., \phi_k)$ is a proof of ϕ_k in F. Hence, for any formula ϕ, $\vdash_F \phi$ if and only if $\exists x\ pr(x, \ulcorner \phi \urcorner)$ is true [8].

Lemma 2.2 *Let F be a consistent extension of P and*

$$B_F = \{\ulcorner \phi \urcorner \mid \vdash_F \phi\}.$$

If B_F has a characteristic function g, then (a) g cannot be constructed in F; (b) g must be constructed in a formal system that is equivalent to an extension of F.

Proof. (a) Let us assume that $g : \mathcal{N} \mapsto \{0, 1\}$ is a characteristic function of B_F such that $g(n) = 1$ if and only if $n \in B_F$. Let ϕ be any Gödel sentence of F. If g can be constructed in F, using g, we can decide in F if $\vdash_F \phi$ or $\vdash_F \neg\phi$:

- If $g(\ulcorner \phi \urcorner) = 1$, then $\ulcorner \phi \urcorner \in B_F$ and $\vdash_F \phi$.
- If $g(\ulcorner \phi \urcorner) = 0$, then $g(\ulcorner \neg\phi \urcorner) = 1$ because g is total and F is consistent. Hence $\ulcorner \neg\phi \urcorner \in B_F$ and $\vdash_F \neg\phi$.

In either case, we have a contradiction to Gödel's first incompleteness theorem.

(b) If g can be constructed by another formal system, say F', then F' must be equivalent to a nontrivial extension of F, because (i) for any formula ϕ, $g(\ulcorner \phi \urcorner) = 1$ if and only if $\vdash_F \phi$, that is, F' can decide every theorem of F through g; (ii) g can only be constructed in F', not in F, thus F' must be more powerful than F. □

It is known that a Gödel sentence can become proved or disproved in an extension F' of F by adding axioms in F'. However, Gödel's first incompleteness theorem also applies to F' if F' is consistent. That is, F' has its own Gödel sentences. Let $B_{F'} = \{\ulcorner \phi \urcorner \mid \vdash_{F'} \phi\}$. In general, $B_F \subseteq B_{F'}$ as more axioms lead to more theorems. We observe that the existence of characteristic functions for B_F and $B_{F'}$ has a similar property to the existence of Gödel sentences for F and F'. That is, assume F' is a consistent extension of F and a characteristic function can be constructed for B_F in F', then there is no way to construct in F' any characteristic function of $B_{F'}$ as Lemma 2.2 also applies to F'. We will use this property to prove the following theorem.

Theorem 2.3 *There exists a set G of natural numbers whose characteristic function cannot be constructed in any consistent formal system.*

Proof. Let us consider $B_F = \{\ulcorner\phi\urcorner \mid \vdash_F \phi\}$ from Lemma 2.2. If the characteristic function of B_F cannot be constructed in any consistent formal system, let G be B_F and the theorem is proved. If B_F has a characteristic function g, then Lemma 2.2 claims that g must be constructed in a nontrivial extension of F, but not in F.

Among all consistent extensions of F in which g can be constructed, we choose one of the maximal extensions as F'. Here, "maximal" means the set of theorems provable by a formal system is maximal among all considered formal systems. That is, we assume that F' does not have any nontrivial consistent extension in which g can be constructed. Let $B_{F'} = \{\ulcorner\phi\urcorner \mid \vdash_{F'} \phi\}$. If the characteristic function of $B_{F'}$ cannot be constructed in any consistent formal system, then let G be $B_{F'}$ and the theorem is proved. If $B_{F'}$ has a characteristic function g', then Lemma 2.2 claims that g' cannot be constructed in F'. If g' is constructed in a consistent extension F'' of F', since g can be also constructed in F'', we have a contradiction to the assumption that F' does not have any nontrivial consistent extension in which g can be constructed. □

Like Gödel's first incompleteness theorem, the above theorem applies to any known or unknown consistent formal system. The above result provides a negative answer to the question in the section title: There exists a set $G \subset \mathcal{N}$ such that no characteristic functions of G can be constructed in any consistent formal system.

In computation theory, an *oracle* for a set $A \subseteq \mathcal{N}$ is an external device (other than a Turing machine) that is capable of reporting whether any number $x \in \mathcal{N}$ is a member of A [13]. The oracle in this sense is a synonym of characteristic function, and they always coexist. That is, if a set A has a characteristic function g, we may use g as its oracle; if A does not have any characteristic function, then A does not have any oracle, because, otherwise, we might use the oracle as its characteristic function.

Turing reduction is an important concept in theory of computation that uses oracles: A set A is *Turing reducible* to a set B if A has an *oracle machine*, which is a modified Turing machine that has the additional capability of querying the oracle of B, to decide any x is a member of A. In other terms, A is Turing reducible to B if the characteristic function of A is computable, assuming we can query freely the characteristic function of B. Two sets of natural numbers are *Turing equivalent* if they are Turing reducible to each other. A *Turing degree* is a set of Turing equivalent sets [2]. Turing reduction induces a partial order over the set of all Turing degrees to assess the level of unsolvability. A great deal of research has been conducted into the structure of the Turing degrees with this order [9]. However, none of these studies considered the non-existence of oracles in the application of Turing reduction. It would be interesting to study the structure of the Turing degrees with Theorem 2.3 in mind.

3 Is Every Set of Natural Numbers Countable?

Throughout this article, by *the subset claim* we mean the claim that "every subset of $\mathcal{N}$ is countable." The subset claim is widely accepted and appears in many textbooks on set theory, logic, discrete mathematics, or theory of computation [3, 7, 10, 14].

Recall that a bijection $f : \mathcal{N} \mapsto S$ is called a *counting bijection* of S. We say f is *increasing* if $f(n) < f(n+1)$ for every $n \in \mathcal{N}$. A typical proof of the subset claim uses the fact that if we can construct a total function $f : \mathcal{N} \mapsto S$ such that $f(n)$ returns the $(n+1)^{th}$ minimal number of S, then f is an increasing counting bijection of S [14]:

- Let $f(0)$ be the smallest natural number in S.
- For each $n \in \mathcal{N}$, the set $S - \{f(0), f(1), ..., f(n)\}$ is not empty since S is infinite. Define $f(n+1)$ to be the smallest natural number in $S - \{f(0), f(1), ..., f(n)\}$.

The existence of "the smallest natural number" is backed by the well-ordering principle. By induction on n for any $n \in \mathcal{N}$, we can show that $f(n)$ is the $(n+1)^{th}$ minimal number of S. It is an easy exercise to check that f is an increasing counting bijection of S.

To construct f in the proof of the subset claim, strictly speaking, we need the characteristic function of S, i.e., total function $g : \mathcal{N} \mapsto \{0, 1\}$, such that $x \in S$ if and only if $g(x) = 1$, to tell us which number is or is not a member of S. Without g, we cannot exclude non-members of S as candidates for the smallest number of S. In other words, the above proof of the subset claim assumes implicitly the existence of g. It turns out that "having a counting bijection" is equivalent to "having a characteristic function" for any infinite set of natural numbers.

Proposition 3.1 *Let $S \subseteq \mathcal{N}$ be infinite. The following statements are logically equivalent in first-order logic:*

1. *S has an increasing counting bijection.*
2. *S is countable.*
3. *S has a characteristic function.*

Proof. (1) → (2): "S has a counting bijection" means "S is countably infinite."

(2) → (3): Since S is countable, let $f : \mathcal{N} \mapsto S$ be a bijection. For any $x \in S$, define $g(x) = 1$ if $\exists n \in \mathcal{N}\ (f(n) = x)$ is true and 0, otherwise. Then g is a characteristic function of S, since $g(x) = 1$ if and only if $x \in S$.

(3) → (1): The above proof of the subset claim can be used here. Formally, let $g : \mathcal{N} \mapsto \{0, 1\}$ be the characteristic function of S. For any $n \in \mathcal{N}$, define $f(n) = h(n, 0)$, where $h : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$ is defined as follows:

$$
\begin{aligned}
h(n, m) \quad = \quad & \textbf{if } (g(m) = 1) \\
& \quad \textbf{then if } (n = 0) \textbf{ then } m \textbf{ else } h(n-1, m+1) \\
& \quad \textbf{else } h(n, m+1)
\end{aligned}
$$

For any $n, m \in \mathcal{N}$, $h(n, m)$ is well-defined because S is infinite and $g(m) = 1$ for an infinite number of m. It is easy to check that $f(n)$ is the $(n+1)^{th}$ minimal element of S, and f is an increasing counting bijection of S. □

The above proposition shows the coexistence of a counting bijection and a characteristic function for any infinite set of natural numbers. A function $f : \mathcal{N} \mapsto \{0, 1\}$ is called a *decision function* and defines uniquely $S = \{x \in \mathcal{N} \mid f(x) = 1\}$. The characteristic function of S can be obtained from f by $g(x) = 1$ if $f(x) = 1$ and $g(x) = 0$ if $f(x) \neq 1$. A function is *computable* if it can be computed by a Turing machine [13]. A set S is *decidable* if it is defined by a total computable decision function (which is the same as the characteristic function of S). A set is *computable* if it is defined by a computable decision function f. In [15], it is shown that the properties of counting bijections are related to the computability of a set. We combine these results in the following theorem.

Theorem 3.2 *Let $S \subseteq \mathcal{N}$ be infinite and $g : \mathcal{N} \mapsto \{0, 1\}$ be a characteristic function of S.*

1. *g exists if and only if S has a counting bijection.*
2. *S is computable if and only if S has a computable counting bijection.*
3. *S is decidable if and only if S has a computable increasing counting bijection.*

Proof. (1) comes from Proposition 3.1. (2) and (3) come from Proposition 11.4.9 of [15]. The proof of Proposition 3.1 can be modified to show (2) and (3). □

Now we are ready to answer the question in the section title.

Theorem 3.3 *There exists an infinite set G of natural numbers whose counting bijections cannot be constructed in any consistent formal system.*

Proof. Consider set G in Theorem 2.3. G is infinite because the set of theorems provable by any extension of formal system P is infinite. If G is countable, let f be a counting bijection of G that is constructed in a consistent formal system, say F. By Theorem 3.2(1), we might construct the characteristic function of G in F, a contradiction to Theorem 2.3. □

Ignoring inconsistent formal systems, which treat all formulas as theorems, Theorem 3.3 implies that G is not countable, because no counting bijections of G can be constructed in an acceptable formal system. G serves as a counterexample to the subset claim that "every subset of $\mathcal{N}$ is countable." The subset claim appears in many textbooks on set theory or theory of computation [5, 7]. In a textbook on set theory [3] published in 1977, the author simply wrote without proof that "Obviously every subset of a countable set is countable." In a textbook on theory of computation [10], the claim appears as Theorem 8.25. It also appears in an influential textbook by Terence Tao [14] (*Proposition* 8.1.5).

The subset claim has several equivalent statements.

Proposition 3.4 *The following statements are logically equivalent:*

1. *Any subset of a countable set is countable.*
2. *Any subset of $\mathcal{N}$ is countable.*
3. *If there is an injective function from set S to $\mathcal{N}$, then S is countable.*
4. *If there is a surjective function from $\mathcal{N}$ to S, then S is countable.*

Since Theorem 3.3 shows that the second statement of the above proposition is false, the other three statements cannot be true.

When a subset of a countable set is uncountable, it is false to claim that $S \cup T$ and $S - T$ are uncountable when S is uncountable and T is countable. For instance, let S be set G in Theorem 3.3 and $T = \mathcal{N}$, then S is uncountable and $S \subset T$. However, both $S \cup T = \mathcal{N}$ and $S - T = \emptyset$ are countable. It is interesting to investigate the closure properties of set operations in light of Theorem 3.3.

4 Conclusion

Using the set of Gödel numbers of all theorems in the proof of Gödel's first incompleteness theorem, we have shown that this set leads to an uncountable set G of natural numbers. That is, G has neither characteristic functions nor counting bijections that can be constructed in any consistent formal system. This result shows the everlasting influence of Gödel's incompleteness theorems and refutes the popular subset claim that "every set of natural numbers is countable."

Since every set specifiable in first-order logic has a characteristic function (i.e., the formula defining the set has a model as its characteristic function), the subset claim remains true by this modification: If a subset of $\mathcal{N}$ is specifiable in first-order logic, then it is countable. The modified subset claim works well in ZFC (Zermelo-Fraenkel set theory with the axiom of choice), which is the standard form of first-order axiomatic set theory and serves as the most common foundation of mathematics. In other words, correcting the subset claim has little impact on ZFC.

For infinite sets, it is counter-intuitive to believe that a countable set contains an uncountable subset, just like believing a set has the same size as its subset, as shown by Hilbert's hotel puzzle [6]. To overcome this counter-intuitivity, we propose a new view of countability: An uncountable set is either larger in size than $\mathcal{N}$, or too difficult to define its counting bijection (due to high complexity). In both cases, a counting bijection of the set cannot be constructed.

A majority of astronomers believe today that the universe is infinite, like an endless flat disk. Using a 3D coordinate system, we may map each star to a coordinate of three integers. Since the set of all coordinates is countable, if the subset claim is true, then the set of all stars is countable. A star, as well as each unit cube in the coordinate system, contains a finite number of atoms. We may draw the conclusion that the set of all atoms in the universe is countable. However, the existence of black holes prevents us from constructing a counting bijection for the set of all atoms in the universe, because of the high complexity of black holes. Black holes are very much like set G in Theorem 3.3.

By Cantor's definition, every finite set is countable. However, the word "countable" does not mean that we can count one by one the elements of every finite set, because the size of this set may be unknown due to the high complexity of the set. For example, how many atoms does a given black hole contain? For further research, we are interested in the following questions:

- Is it true or false that every finite set has a characteristic function?

- What is the relation between the size of a finite set and the existence of its characteristic function?
- If we do not know the size of a finite set, how can we show that this set is computable?

If Cantor's original intention is that "countable" means the counting process of a set is somehow constructible, we may adapt a new definition of being *uncountable*:

> A set X is *uncountable* if either its size is larger than that of $\mathcal{N}$, or the characteristic function of X cannot be constructed in any consistent formal system.

For infinite sets, either of the above two conditions prevents X from having any counting bijection. For finite sets, its countability depends on the existence of its characteristic function.

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