Solving Recurrence Relations

An alternative approach to solving recurrence relations employs the idea of the *difference operator*. This is a discrete analog to the differential operator of calculus. It can be applied when we have a recursively described infinite sequence, or equivalently a function f: **Nat** $\to$ **Nat**. We will use the function notation. When the difference operator Δ is applied to function f, we obtain a new function Δf defined by

$\Delta f(n) = f(n+1) - f(n)$.

This can be compared to the differential calculus operator

$$\frac{d}{dx} f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} .$$

In the case of the natural numbers, arguments cannot be chosen to be arbitrarily close together — ε can be no smaller than 1. In that sense, Δ is the best approximation to differentiation that we could hope for. The properties of the discrete difference operator strongly resemble those of the differential operator of calculus. For instance, if we define addition of functions f+g by [f+g] (n) = f(n) + g(n), then $\Delta(f+g) = \Delta f + \Delta g$. While not everything is the same (e.g., the product rule does not hold for the difference operator), we illustrate some of the numerous analogies in the table below.

$\frac{d}{dx}$	Δ
$\frac{d}{dx} c = 0$	$\Delta c = 0$
$\frac{d}{dx}(ax + b) = a$	$\Delta(an + b) = a(n+1) + b - (an + b) = a$
$\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$	$\Delta(an^2 + bn + c) = a(n+1)^2 + b(n+1) + c - (an^2 + bn + c) = 2an + a+b$
$\frac{d}{dx}(e^x + c) = e^x$	$\Delta(2^n + c) = 2^{n+1} + c - (2^n + c) = 2^n$

Comparison of differentiation and difference

So the difference of a constant is 0, the difference of a linear function is a constant, the difference of a quadratic is linear, etc. — all like differentiation. The "natural" base for the exponential is 2 rather than e, and such small distinctions are typical. In particular, when solving difference equations, we may use the knowledge of the solution of differential equations to strong advantage. For instance, to solve the differential equation $\frac{d}{dx} f(x) = 2x-1$, we know f must be a quadratic, and hence solutions are of the form $x^2 - x + c$ (for an arbitrary constant c). We may make similar deductions when we are presented with a difference equation we wish to solve.

Example

Consider the recurrence defining the sum of the first n natural numbers

$s(0) = 0$

$s(n+1) = s(n) + n+1, n \geq 0$.

The difference equation is $\Delta s(n) = s(n+1) - s(n) = n+1$. This linear difference indicates that s(n) is quadratic, So we assume $s(n) = an^2 + bn + c$, and determine the required values of the constants by comparing values with those from the recurrence computation. Since s(0) = 0, c = 0. Also s(1) = 1, so a + b = 1. Finally, s(2) = 3, so 4a + 2b = 3. Then we have 2 equations in 2 unknowns, and we can solve to obtain $a = \frac{1}{2}$ and $b = \frac{1}{2}$. Hence the solution is $s(n) = \frac{1}{2} n^2 + \frac{1}{2} n = \frac{n(n+1)}{2}$ which we see leaves us with integer results even though the coefficients are not integers.