Exam II — Sample Solutions

Problem 1.
Fill in the blanks and explain: if G is a connected simple (undirected) graph with no self-loops having 70 nodes, then the number of edges in G is between <u>69</u> and <u>2415</u>.

The fewest edges for a connected graph occurs in a tree, and this e = v-1, 69 in this case. The most edges occurs in the complete graph, K_{70}. We can count the edges node by node as follows:

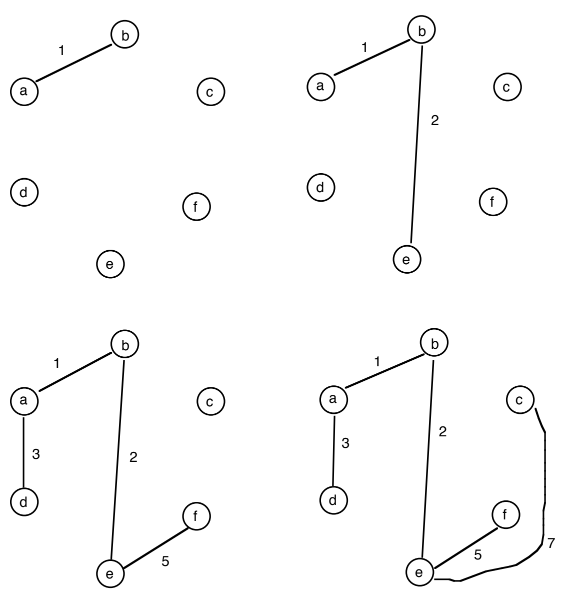
- node 1: an edge to each of the other nodes — 69
- node 2: besides the edge already counted in the first step, an edge to each of the remaining — 68;
- node 3: besides the edges already counted in the first two steps, and edge to each of the remaining — 67;
- etc.

Hence the total number of edges is 69+68+ … + 1 = $\sum_{k=1}^{69} k$ = 69•70/2 = 2415.

Problem 2.
Digraphs G_2 and G_3 are isomorphic — the renaming correspondence is

	node correspondence			
G_2	e	f	g	h
G_3	k	j	i	l

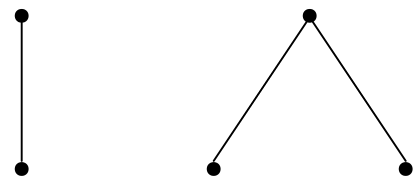
Problem 3.
This solution uses Prim's algorithm. We start with no edges and at each step add the least cost edge that connects nodes in the partial tree to those not yet included. Since there are 6 nodes, the tree must have five edges and we have the following five steps.



Problem 4

Each connected component is by definition connected, and since there are no simple cycles, each component must be a tree.

(a) Since there are no self-loops and no isolated vertices, a connected component cannot have a single vertex. Hence one component must have two vertices and the other three. Since both must be trees, there is only one possibility, namely



Notice there are 5–2 = 3 edges.

(b) In general, each connected component must be a tree — one with e_1 edges and v_1 vertices, the other with e_2 edges and v_2 vertices. Hence, it must be that $e_1 = v_1 - 1$ and $e_2 = v_2 - 1$. Since every edge and every vertex is in one connected component or the other, the number of edges e in the entire graph is $e = e_1 + e_2 = v_1 - 1 + v_2 - 1 = v_1 + v_2 - 2 = v - 2$.

Note that this generalizes to the case when there are multiple (i.e., one or more) connected components, say c of them — such a graph is called a *forest* (i.e., a collection of trees). Then $e = v - c$.