CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

Combining Theory Solvers with SAT solvers

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Checking the satisfiability of quantifier-free formulas

Theory solvers check the satisfiability of conjunctions of literals

What if we have formulas with disjunctions, like this $\mathcal{T}_{EUF}$-formula?

$$g(a) \doteq c \;\land\; (f(g(a)) \not\doteq f(c) \;\lor\; g(a) \doteq d) \;\land\; c \not\doteq d$$

What about arbitrary Boolean combinations of literals?

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Theorem 1

For all theories $\mathcal{T}$, the $\mathcal{T}$-satisfiability of quantifier-free formulas is decidable iff the $\mathcal{T}$-satisfiability of conjunctions/sets of literals is decidable.

Proof.

Convert the formula to DNF and check if any of its disjuncts is $\mathcal{T}$-satisfiable. □

Problem: the DNF conversion is very inefficient! (formula size can explode exponentially)

A better solution: exploit propositional satisfiability technology to deal with the Boolean structure

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Lifting SAT Technology to SMT

Two main approaches:

1. *Eager*

   - translate into an equisatisfiable propositional formula
   - feed it to any SAT solver

Notable systems: UCLID

2. *Lazy*

   - abstract the input formula to a propositional one
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   - use a theory decision procedure to refine the formula and guide the SAT solver

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Lazy Approach for SMT

Given a quantifier-free Σ-formula φ, for each atomic formula α in φ, we associate a unique propositional variable $e(\alpha)$

The *Boolean skeleton* of a formula φ is a propositional logic formula, where each atomic formula α in φ is replaced with $e(\alpha)$

Example:

$$\varphi := \underbrace{x < 0}_{p_1} \vee (\underbrace{x + y < 1}_{p_2} \wedge \neg(\underbrace{x < 0}_{p_1})) \Rightarrow \underbrace{y < 0}_{p_3}$$

Boolean skeleton of φ : $p_1 \vee (p_2 \wedge \neg p_1) \Rightarrow p_3$

with $e := \{ (x < 0) \mapsto p_1,\ (x + y < 1) \mapsto p_2,\ (y < 0) \mapsto p_3 \}$

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(Very) Lazy Approach for SMT – Example

$$g(a) \doteq c \;\wedge\; (f(g(a)) \mathrel{\dot{\neq}} f(c) \;\vee\; g(a) \doteq d) \;\wedge\; c \mathrel{\dot{\neq}} d$$

Simplest setting:

- Off-line SAT solver
- Non-incremental *theory solver* for conjunctions of equalities and disequalities
- Theory atoms (e.g., $g(a) \doteq c$) *abstracted* to propositional atoms (e.g., 1)

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$$g(a) \doteq c \;\wedge\; (f(g(a)) \not\doteq f(c) \vee g(a) \doteq d) \;\wedge\; c \not\doteq d$$

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$$\underbrace{g(a) \doteq c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\bar{2}} \vee \underbrace{g(a) \doteq d}_{3} \quad \wedge \quad \underbrace{c \not\doteq d}_{\bar{4}}$$

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Done! The original formula is unsatisfiable in $\mathcal{T}_{EUF}$

Eager Approach for SMT – Example

$$f(b) \doteq a \quad \vee \quad f(a) \not\doteq a$$

Step 1: Eliminate all function applications (Ackermann's encoding)

- introduce a constant symbol f_x to replace function application $f(x)$
- for each pair of introduced variables f_x, f_y, add the formula $x \doteq y \Rightarrow f_x \doteq f_y$

$$f(b) \Rightarrow f_b \qquad f(a) \Rightarrow f_a$$

$$(f_b \doteq a \;\vee\; f_a \not\doteq a) \;\wedge\; (a \doteq b \Rightarrow f_a \doteq f_b)$$

Now, atomic formulas are equalities between constants/variables

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Eager Approach for SMT – Example

Rename f_b as c and f_a as d:

$$(f_b \doteq a \ \vee \ f_a \not\doteq a) \ \wedge \ (a \doteq b \Rightarrow f_a \doteq f_b)$$

becomes

$$(c \doteq a \ \vee \ d \not\doteq a) \ \wedge \ (a \doteq b \Rightarrow d \doteq c)$$

Step 2: Eliminate all equalities

- replace each pair of constants x, y with a unique propositional variable $p_{x,y}$
- add facts about reflexivity, symmetry, transitivity

$$\begin{gathered}(p_{c,a} \ \vee \ \neg p_{d,a}) \ \wedge \ (p_{a,b} \Rightarrow p_{d,c}) \\ \wedge \ p_{a,a} \wedge p_{b,b} \wedge p_{c,c} \wedge p_{d,d} \wedge \ (p_{a,b} \Leftrightarrow p_{b,a}) \wedge (p_{a,c} \Leftrightarrow p_{c,a}) \wedge (p_{a,d} \Leftrightarrow p_{d,a}) \wedge \cdots \\ \wedge \ ((p_{a,b} \wedge p_{b,c}) \Rightarrow p_{a,c}) \wedge ((p_{a,c} \wedge p_{c,d}) \Rightarrow p_{a,d}) \wedge \cdots\end{gathered}$$

The resulting propositional formula is equisatisfiable with the original T_{EUF}-formula

Note: Not all the transitivity cases are needed

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Discussion: eager vs. lazy approach

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What are the pros and cons of the two approaches?

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Eager

- Can always use the best SAT solver off the shelf
- Requires care in encoding
- Tends not to scale well

Lazy

- Theory-specific reasoning
- Designing new theory solvers can be challenging
- Might require extension of a SAT solver for more efficiency interplay with theory solver
- Scales much better

Lazy Approach – Enhancements

Several enhancements are possible to increase efficiency:

- Check $\mathcal{T}$-satisfiability only of full propositional model

  Check $\mathcal{T}$-satisfiability of partial assignment M as it grows

- If M is $\mathcal{T}$-unsatisfiable, add $\neg M$ as a clause

  If M is $\mathcal{T}$-unsatisfiable, identify a $\mathcal{T}$-unsatisfiable subset M_0 of M and add $\neg M_0$ as a clause

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Lazy Approach – Main Benefits

Every tool does what it is good at:

- SAT solver takes care of Boolean information
- Theory solver takes care of theory information

The theory solver works only with conjunctions of literals

Modular approach:

- SAT and theory solvers communicate via a simple API
- SMT for a new theory only requires new theory solver
- An off-the-shelf SAT solver can be embedded in a lazy SMT system with low effort

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An Abstract Framework for Lazy SMT

Several variants and enhancements of lazy SMT solvers exist

They can be modeled a satisfiability proof system like those for Abstract DPLL and Abstract CDCL

Review: Abstract DPLL

States:

$$\textsc{unsat} \qquad \langle M, \Delta \rangle$$

where

- M is a *sequence of literals* and *decision points* ($\bullet$) denoting a partial variable assignment
- Δ is a *set of clauses* denoting a CNF formula

Review: Abstract DPLL

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Note: When convenient, we treat M as a set

Provided M contains no complementary literals it determines the assignment

$$v_M(p) = \begin{cases} \text{true} & \text{if } p \in M \\ \text{false} & \text{if } \neg p \in M \\ \text{undef} & \text{otherwise} \end{cases}$$

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Notation: If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level i* of M
- $M^{[i]}$ denotes the subsequence $M_0 \bullet \cdots \bullet M_i$, from decision level 0 through decision level i

Review: Abstract DPLL

States:

UNSAT $\quad \langle M, \Delta \rangle$

Initial state:

- $\langle \epsilon, \Delta_0 \rangle$ where ϵ is the empty assignment and Δ_0 is to be checked for satisfiability

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Expected final states:

- $\textsc{unsat}$ if Δ_0 is unsatisfiable
- $\langle M, \Delta_n \rangle$ otherwise, where Δ_n is equisatisfiable with Δ_0 and satisfied by M

Review: Abstract CDCL

States:

$$\text{UNSAT} \qquad \langle M, \Delta, C \rangle$$

where

- M and Δ are as in Abstract DPLL
- C is either $\texttt{no}$ or a *conflict clause*

Initial state:

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Review: CDCL proof rules

$$\textsc{Propagate}\ \frac{\{l_1,\ldots,l_n,l\}\in\Delta \qquad \bar{l}_1,\ldots,\bar{l}_n\in \mathsf{M} \qquad l,\bar{l}\notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$$

$$\textsc{Decide}\ \frac{l\in \mathtt{Lits}(\Delta) \qquad l,\bar{l}\notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\bullet l}$$

$$\textsc{Explain}\ \frac{\mathsf{C}=\{l\}\cup C' \quad \{l_1,\ldots,l_n,\bar{l}\}\in\Delta \quad \bar{l}_1,\ldots,\bar{l}_n,\bar{l}\in \mathsf{M} \quad \bar{l}_1,\ldots,\bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := \{l_1,\ldots,l_n\}\cup C'}$$

$$\textsc{Fail}\ \frac{\mathsf{C}\neq \mathtt{no} \qquad \bullet\notin \mathsf{M}}{\mathsf{UNSAT}}$$

$$\textsc{Backjump}\ \frac{\mathsf{C}=D \quad D=\{l_1,\ldots,l_n,l\} \quad \mathtt{lev}(\bar{l}_1),\ldots,\mathtt{lev}(\bar{l}_n)\leq i<\mathtt{lev}(\bar{l})}{\mathsf{M} := \mathsf{M}^{[i]}\, l \qquad \mathsf{C} := \mathtt{no} \qquad \Delta := \Delta\cup\{D\}}$$

$$\textsc{Restart}\ \frac{}{\mathsf{M} := \mathsf{M}^{[0]} \qquad \mathsf{C} := \mathtt{no}}$$

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$$\textsc{Learn}\ \frac{D \text{ is a clause} \qquad \Delta\models D \qquad D\notin\Delta}{\Delta := \Delta\cup\{D\}}$$

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We are going to extend this abstract framework to lazy SMT

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From SAT to SMT

Same state components and transitions as in Abstract CDCL except that

- Δ contains quantifier-free clauses in some theory $\mathcal{T}$
- M is a sequence of theory literals (i.e., atomic formulas or their negations) and decision points
- CDCL Rules operate on the Boolean skeleton of Δ,
  given by a mapping from theory literals to propositional literals
- The proofs system is augmented with SMT-specific rules based on $\models_{\mathcal{T}}$:
  $\mathcal{T}$-**Conflict**, $\mathcal{T}$-**Propagate** and $\mathcal{T}$-**Explain**
- We assume an oracle, the theory solver, for $\models_{\mathcal{T}}$ over theory literals
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SMT-level Rules

At SAT level:

$$\textsc{Conflict}\ \frac{\mathsf{C} = \texttt{no} \qquad \{l_1, \ldots, l_n\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathsf{M}}{\mathsf{C} := \{l_1, \ldots, l_n\}}$$

At SMT level:

$$\mathcal{T}\textsc{-Conflict}\ \frac{\mathsf{C} = \texttt{no} \qquad \overline{l}_1 \wedge \cdots \wedge \overline{l}_n \models_{\mathcal{T}} \bot \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathsf{M}}{\mathsf{C} := \{l_1, \ldots, l_n\}}$$

If a set of literals in M are unsatisfiable in $\mathcal{T}$, make their negation a conflict clause

SMT-level Rules

At SAT level:

$$\textsc{Propagate}\ \frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathsf{M} \qquad l, \overline{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

At SMT level:

$$\mathcal{T}\text{-}\textsc{Propagate}\ \frac{l \in \mathtt{Lits}(\Delta) \qquad \mathsf{M} \models_{\mathcal{T}} l \qquad l, \overline{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

If M entails some literal l in $\mathcal{T}$, extend it with l

SMT-level Rules

At SAT level:

$$\textsc{Explain}\ \frac{\mathsf{C} = \{l\} \cup C' \quad \{l_1, \ldots, l_n, \overline{l}\} \in \Delta \quad \overline{l}_1, \ldots, \overline{l}_n, \overline{l} \in \mathsf{M} \quad \overline{l}_1, \ldots, \overline{l}_n \prec_{\mathsf{M}} \overline{l}}{\mathsf{C} := \{l_1, \ldots, l_n\} \cup C'}$$

At SMT level:

$$\mathcal{T}\text{-}\textsc{Explain}\ \frac{\mathsf{C} = \{l\} \cup D \quad \overline{l}_1 \wedge \cdots \wedge \overline{l}_n \models_{\mathcal{T}} \overline{l} \quad \overline{l}_1, \ldots, \overline{l}_n \prec_{\mathsf{M}} \overline{l}}{\mathsf{C} := \{l_1, \cdots, l_n\} \cup D}$$

If the complement $\overline{l}$ of a literal in the conflict clause is entailed in $\mathcal{T}$ by some literals $\overline{l}_1, \ldots, \overline{l}_n$ at lower decision levels, derive a new conflict clause by resolution with $\{l_1, \ldots, l_n, \overline{l}\}$

CDCL Modulo Theories proof rules

$$\textsc{Decide}\ \frac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$$

$$\textsc{Fail}\ \frac{C \neq \mathtt{no} \qquad \bullet \notin M}{\mathsf{UNSAT}}$$

$$\textsc{Restart}\ \frac{}{M := M^{[0]} \qquad C := \mathtt{no}}$$

$$\textsc{Learn}\ \frac{D \text{ is a clause} \qquad \Delta \models D \qquad D \notin \Delta}{\Delta := \Delta \cup \{D\}}$$

$$\textsc{Forget}\ \frac{C = \mathtt{no} \qquad \Delta = \Delta' \cup \{C\} \qquad \Delta' \models C}{\Delta := \Delta'}$$

$$\mathcal{T}\text{-}\textsc{Propagate}\ \frac{l \in \mathtt{Lits}(\Delta) \qquad M \models_{\mathcal{T}} l \qquad l, \bar{l} \notin M}{M := M\, l}$$

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$$\textsc{Backjump}\ \frac{C = D \quad D = \{l_1, \ldots, l_n, l\} \quad \mathtt{lev}(\bar{l}_1), \ldots, \mathtt{lev}(\bar{l}_n) \leq i < \mathtt{lev}(\bar{l})}{M := M^{[i]} l \qquad C := \mathtt{no} \qquad \Delta := \Delta \cup \{D\}}$$

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Modeling the Very Lazy Theory Approach

$\mathcal{T}$-**Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier EUF example

$$\underbrace{g(a) \doteq c}_{1} \;\wedge\; \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) \doteq d}_{3} \;\wedge\; \underbrace{c \neq d}_{\overline{4}}$$

M	Δ	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by Propagate$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by Decide
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by $\mathcal{T}$-Conflict
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by Learn
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by Restart
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by Propagate$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by $\mathcal{T}$-Conflict
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	no	by Learn
	$\vdots$		
	UNSAT		by Fail

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) \doteq c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\bar{2}} \vee \underbrace{g(a) \doteq d}_{3} \quad \wedge \quad \underbrace{c \not\doteq d}_{\bar{4}}$$

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$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4,\ \bar{1} \vee \bar{3} \vee 4$	no	by Learn
	⋮		
	UNSAT		by Fail

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) \doteq c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\bar{2}} \vee \underbrace{g(a) \doteq d}_{3} \quad \wedge \quad \underbrace{c \not\doteq d}_{\bar{4}}$$

M	Δ	C	rule
	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by PROPAGATE$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by DECIDE
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2 \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee 2 \vee 4$	by LEARN
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by RESTART
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by PROPAGATE$^+$
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4,\ \bar{1} \vee \bar{3} \vee 4$	no	by LEARN
	⋮		
	UNSAT		by FAIL

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$$\underbrace{g(a) \doteq c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\overline{2}} \vee \underbrace{g(a) \doteq d}_{3} \quad \wedge \quad \underbrace{c \not\doteq d}_{\overline{4}}$$

M	Δ	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by PROPAGATE^{+}
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by DECIDE
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by LEARN
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by RESTART
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by PROPAGATE^{+}
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	no	by LEARN
	⋮		
	UNSAT		by FAIL

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M	Δ	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by PROPAGATE$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by DECIDE
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by LEARN
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by RESTART
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by PROPAGATE$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	no	by LEARN
	⋮		
	UNSAT		by FAIL

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$$\underbrace{g(a) \doteq c}_{1} \quad\wedge\quad \underbrace{f(g(a)) \not\doteq f(c)}_{\bar{2}} \vee \underbrace{g(a) \doteq d}_{3} \quad\wedge\quad \underbrace{c \not\doteq d}_{\bar{4}}$$

M	Δ	C	rule
	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Decide**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2 \vee 4$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee 2 \vee 4$	by **Learn**
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \bar{4}\ \bar{2}\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \bar{4}\ \bar{2}\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4}\ \bar{2}\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4,\ \bar{1} \vee \bar{3} \vee 4$	no	by **Learn**
	⋮		
	UNSAT		by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) \doteq c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\overline{2}} \vee \underbrace{g(a) \doteq d}_{3} \quad \wedge \quad \underbrace{c \not\doteq d}_{\overline{4}}$$

M	Δ	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by Propagate$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by Decide
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by $\mathcal{T}$-Conflict
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by Learn
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by Restart
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by Propagate$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by $\mathcal{T}$-Conflict
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	no	by Learn
	$\vdots$		
	UNSAT		by Fail

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	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Decide**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2 \vee 4$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee 2 \vee 4$	by **Learn**
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \bar{4}\ \bar{2}\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \bar{4}\ \bar{2}\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4}\ \bar{2}\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4,\ \bar{1} \vee \bar{3} \vee 4$	no	by **Learn**
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$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by PROPAGATE^{+}
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by DECIDE
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by LEARN
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by RESTART
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by PROPAGATE^{+}
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by $\mathcal{T}$-CONFLICT
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	no	by LEARN
	⋮		
	UNSAT		by FAIL

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$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by Decide
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2 \vee 4$	by $\mathcal{T}$-Conflict
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee 2 \vee 4$	by Learn
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by Restart
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by Propagate$^+$
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-Conflict
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4,\ \bar{1} \vee \bar{3} \vee 4$	no	by Learn
	⋮		
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$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Decide**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2 \vee 4$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee 2 \vee 4$	by **Learn**
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4},\ \bar{1} \vee 2 \vee 4,\ \bar{1} \vee \bar{3} \vee 4$	no	by **Learn**
	$\vdots$		
	UNSAT		by **Fail**

A Better Lazy Approach

The very lazy approach can be improved considerably with

- an *on-line* SAT engine that accept new input clauses on the fly
- an *incremental and explicating $\mathcal{T}$-solver* that can

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M	Δ	C	rule
	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by PROPAGATE$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by DECIDE
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2$	by $\mathcal{T}$-CONFLICT
$1\ \bar{4}\ 2$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by BACKJUMP
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by PROPAGATE
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-CONFLICT
UNSAT			by FAIL

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M	Δ	C	rule
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$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by PROPAGATE$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by DECIDE
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$1\ \bar{4}\ 2$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by BACKJUMP
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by PROPAGATE
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-CONFLICT
UNSAT			by FAIL

A Better Lazy Approach

$$\underbrace{g(a) \doteq c}_{1} \quad \land \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\bar{2}} \lor \underbrace{g(a) \doteq d}_{3} \quad \land \quad \underbrace{c \not\doteq d}_{\bar{4}}$$

M	Δ	C	rule
	$1,\ \bar{2} \lor 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \lor 3,\ \bar{4}$	no	by **Propagate**$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \lor 3,\ \bar{4}$	no	by **Decide**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \lor 3,\ \bar{4}$	$\bar{1} \lor 2$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4}\ 2$	$1,\ \bar{2} \lor 3,\ \bar{4}$	no	by **Backjump**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \lor 3,\ \bar{4}$	no	by **Propagate**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \lor 3,\ \bar{4}$	$\bar{1} \lor \bar{3} \lor 4$	by $\mathcal{T}$-**Conflict**
UNSAT			by **Fail**

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	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Decide**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4}\ 2$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Backjump**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-**Conflict**
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	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Decide**
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$1\ \bar{4}\ 2$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Backjump**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-**Conflict**
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M	Δ	C	rule
	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**$^{+}$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Decide**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2$	by $\mathcal{T}$-**Conflict**
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M	Δ	C	rule
	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**$^+$
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Decide**
$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2$	by $\mathcal{T}$-**Conflict**
$1\ \bar{4}\ 2$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Backjump**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-**Conflict**
UNSAT			by **Fail**

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$1\ \bar{4} \bullet \bar{2}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee 2$	by **$\mathcal{T}$-Conflict**
$1\ \bar{4}\ 2$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Backjump**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by **Propagate**
$1\ \bar{4}\ 2\ 3$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{1} \vee \bar{3} \vee 4$	by **$\mathcal{T}$-Conflict**
UNSAT			by **Fail**

Lazy Approach – Strategies

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is (propositionally) falsified by the current assignment M, apply **Conflict**
2. If M is $\mathcal{T}$-unsatisfiable, apply $\mathcal{T}$**-Conflict**
3. Apply **Fail** or **Explain**+**Learn**+**Backjump** as appropriate
4. Apply **Propagate**
5. Apply **Decide**

Depending on the cost of checking the $\mathcal{T}$-satisfiability of M,
Step (2) can be applied with lower frequency or priority

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Theory Propagation

With $\mathcal{T}$-**Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

With $\mathcal{T}$-**Propagate** and $\mathcal{T}$-**Explain**, it can also be used to guide the engine's search

$$\mathcal{T}\text{-}\textbf{Propagate}\ \frac{l \in \mathtt{Lits}(\Delta) \qquad \mathsf{M} \models_{\mathcal{T}} l \qquad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

$$\mathcal{T}\text{-}\textbf{Explain}\ \frac{\mathsf{C} = \{l\} \cup D \qquad \bar{l}_1 \wedge \cdots \wedge \bar{l}_n \models_{\mathcal{T}} \bar{l} \qquad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := \{l_1, \cdots, l_n\} \cup D}$$

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M	Δ	C	rule
	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by Propagate^{+}
$1\ \bar{4}\ 2$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by $\mathcal{T}$-Propagate (as $1 \models_{\mathcal{T}} 2$)
$1\ \bar{4}\ 2\ \bar{3}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	no	by $\mathcal{T}$-Propagate (as $1,\ \bar{4} \models_{\mathcal{T}} \bar{3}$)
$1\ \bar{4}\ 2\ \bar{3}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{2} \vee 3$	by Conflict
UNSAT			by Fail

$\mathcal{T}$-propagation eliminates search altogether in this case!
No applications of **Decide** are needed

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$$\underbrace{g(a) \doteq c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\overline{2}} \vee \underbrace{g(a) \doteq d}_{3} \quad \wedge \quad \underbrace{c \not\doteq d}_{\overline{4}}$$

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by $\mathcal{T}$-PROPAGATE (as $1 \models_{\mathcal{T}} 2$)
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by $\mathcal{T}$-PROPAGATE (as $1,\ \overline{4} \models_{\mathcal{T}} \overline{3}$)
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{2} \vee 3$	by CONFLICT
UNSAT			by FAIL

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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by $\mathcal{T}$-**Propagate** (as $1 \models_{\mathcal{T}} 2$)
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$1\ \bar{4}\ 2\ \bar{3}$	$1,\ \bar{2} \vee 3,\ \bar{4}$	$\bar{2} \vee 3$	by **Conflict**
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$$\underbrace{g(a) \doteq c}_{1} \quad \land \quad \underbrace{f(g(a)) \not\doteq f(c)}_{\bar{2}} \lor \underbrace{g(a) \doteq d}_{3} \quad \land \quad \underbrace{c \not\doteq d}_{\bar{4}}$$

M	Δ	C	rule
	$1,\ \bar{2} \lor 3,\ \bar{4}$	no	
$1\ \bar{4}$	$1,\ \bar{2} \lor 3,\ \bar{4}$	no	by **Propagate**$^{+}$
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$1\ \bar{4}\ 2\ \bar{3}$	$1,\ \bar{2} \lor 3,\ \bar{4}$	$\bar{2} \lor 3$	by **Conflict**
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Theory Propagation Features

- With *exhaustive* theory propagation, every assignment M is $\mathcal{T}$-satisfiable
(since $M\,l$ is $\mathcal{T}$-unsatisfiable iff $M \models_{\mathcal{T}} \bar{l}$)

- For theory propagation to be effective in practice, it needs specialized theory solvers

- For some theories, e.g., difference logic, detecting $\mathcal{T}$-entailed literals is cheap and so exhaustive theory propagation is extremely effective

- For others, e.g., the theory of equality, detecting $\mathcal{T}$-entailed equalities is cheap but detecting $\mathcal{T}$-entailed disequalities is quite expensive

- If $\mathcal{T}$-**Propagate** is not applied exhaustively, $\mathcal{T}$-**Conflict** is needed to repair $\mathcal{T}$-unsatisfiable assignments

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Theory Propagation Exercise

$$\underbrace{a \doteq b}_{1} \quad \wedge \quad \underbrace{a \doteq c}_{2} \vee \underbrace{c \doteq b}_{3} \quad \wedge \quad \underbrace{a \not\doteq b}_{\bar{1}} \vee \underbrace{f(a) \not\doteq f(c)}_{\bar{4}} \quad \wedge \quad \underbrace{c \not\doteq b}_{\bar{3}} \vee \underbrace{g(a) \doteq g(c)}_{5}$$

$$\Delta_0 \;:=\; 1,\; 2 \vee 3,\; \bar{1} \vee \bar{4},\; \bar{3} \vee 5$$

Theory Propagation Exercise

Scenario 1: propagating only $\mathcal{T}$-entailed equalities (no disequalities)

$$\underbrace{a \doteq b}_{1} \quad \wedge \quad \underbrace{a \doteq c}_{2} \vee \underbrace{c \doteq b}_{3} \quad \wedge \quad \underbrace{a \not\doteq b}_{\bar{1}} \vee \underbrace{f(a) \not\doteq f(c)}_{\bar{4}} \quad \wedge \quad \underbrace{c \not\doteq b}_{\bar{3}} \vee \underbrace{g(a) \doteq g(c)}_{5}$$

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M	Δ	C	rule
	Δ_0	no	
$1\ \overline{4}$	Δ_0	no	by PROPAGATE$^+$
$1\ \overline{4}\ \bullet\ 2$	Δ_0	no	by DECIDE
$1\ \overline{4}\ \bullet\ 2$	Δ_0	$\overline{2} \vee 4$	by $\mathcal{T}$-CONFLICT (as $2, \overline{4} \models_{\mathcal{T}} \bot$)
$1\ \overline{4}\ \overline{2}$	$\Delta_0,\ \overline{2} \vee 4$	no	by BACKJUMP
$1\ \overline{4}\ \overline{2}\ 3$	$\Delta_0,\ \overline{2} \vee 4$	no	by PROPAGATE
$1\ \overline{4}\ \overline{2}\ 3$	$\Delta_0,\ \overline{2} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by $\mathcal{T}$-CONFLICT (as $1, \overline{3}, \overline{4} \models_{\mathcal{T}} \bot$)
UNSAT			by FAIL

Theory Propagation Exercise

Scenario 2: propagating $\mathcal{T}$-entailed equalities and disequalities

$$\underbrace{a \doteq b}_{1} \quad \wedge \quad \underbrace{a \doteq c}_{2} \vee \underbrace{c \doteq b}_{3} \quad \wedge \quad \underbrace{a \not\doteq b}_{\bar{1}} \vee \underbrace{f(a) \not\doteq f(c)}_{\bar{4}} \quad \wedge \quad \underbrace{c \not\doteq b}_{\bar{3}} \vee \underbrace{g(a) \doteq g(c)}_{5}$$

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M	Δ	C	rule
	Δ_0	no	
$1\ \bar{4}$	Δ_0	no	by **Propagate**$^+$
$1\ \bar{4}\ \bar{2}$	$\Delta_0,\ \bar{2} \vee 4$	no	by $\mathcal{T}$-**Propagate** (as $1, \bar{4} \models_{\mathcal{T}} \bar{2}$)
$1\ \bar{4}\ \bar{2}\ 3$	$\Delta_0,\ \bar{2} \vee 4$	no	by **Propagate**
$1\ \bar{4}\ \bar{2}\ 3$	$\Delta_0,\ \bar{2} \vee 4$	$\bar{1} \vee \bar{3} \vee 4$	by $\mathcal{T}$-**Conflict** (as $1, 3, \bar{4} \models_{\mathcal{T}} \bot$)
UNSAT			by **Fail**

Modeling Modern Lazy SMT Solvers

At the core, current lazy SMT solvers are implementations of the proof system with rules:

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) $\mathcal{T}$-**Conflict**, $\mathcal{T}$-**Propagate**, $\mathcal{T}$-**Explain**

(3) **Learn**, **Forget**, **Restart**

Basic CDCL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2)

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Correctness

Irreducible state: state to which no Basic CDCL Modulo Theories rules apply (updated terminology)
Execution: a (single-branch) derivation tree starting with $M = \emptyset$ and $C = \texttt{no}$
Exhausted execution: execution ending in an irreducible state

Theorem 2 (Strong Termination)

Every execution in which (i) LEARN/FORGET are applied only finitely many times and (ii) RESTART is applied with increased periodicity is finite.

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Irreducible state: state to which no Basic CDCL Modulo Theories rules apply (updated terminology)
Execution: a (single-branch) derivation tree starting with $M = \emptyset$ and $C = \texttt{no}$
Exhausted execution: execution ending in an irreducible state

Theorem 2 (Strong Termination)

Every execution in which (i) LEARN/FORGET are applied only finitely many times and (ii) RESTART is applied with increased periodicity is finite.

Lemma 3

Every exhausted execution ends with either $C = no$ or UNSAT.

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Theorem 3 (Refutation Soundness)

For every exhausted execution starting with $\Delta = \Delta_0$ and ending with UNSAT, the clause set Δ_0 is $\mathcal{T}$-unsatisfiable.

Correctness

Irreducible state: state to which no Basic CDCL Modulo Theories rules apply (updated terminology)
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Theorem 2 (Strong Termination)

Every execution in which (i) **Learn/Forget** *are applied only finitely many times and (ii)* **Restart** *is applied with increased periodicity is finite.*

Theorem 3 (Refutation Soundness)

For every exhausted execution starting with $\Delta = \Delta_0$ and ending with UNSAT, *the clause set Δ_0 is $\mathcal{T}$-unsatisfiable.*

Theorem 4 (Refutation Completeness)

For every exhausted execution starting with $\Delta = \Delta_0$ and ending with $\mathsf{C} = no$, the clause set Δ_0 is $\mathcal{T}$-satisfiable; specifically, M is $\mathcal{T}$-satisfiable and $\mathsf{M} \models_{\mathrm{p}} \Delta_0$.

CDCL($\mathcal{T}$) Architecture

The approach formalized so far can be implemented with a simple architecture originally named DPLL($\mathcal{T}$) but currently known as *CDCL($\mathcal{T}$)*

$$\text{CDCL}(\mathcal{T}) \;=\; \text{CDCL}(X) \text{ engine} \;+\; \mathcal{T}\text{-solver}$$

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CDCL(X):

- Very similar to a SAT solver, enumerates Boolean models
- Not allowed: pure literal rule (and other SAT specific optimizations)
- Required: incremental addition of clauses
- Desirable: partial model detection

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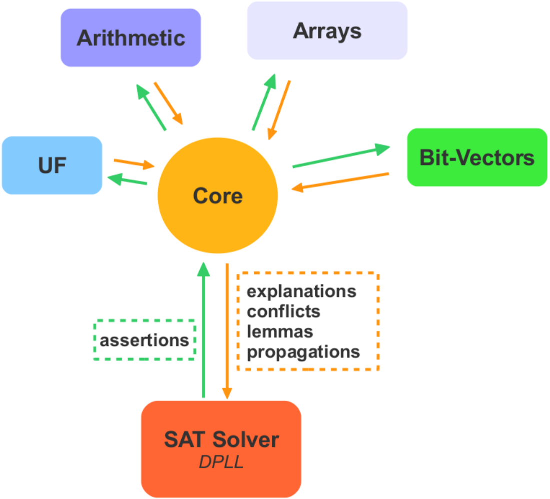
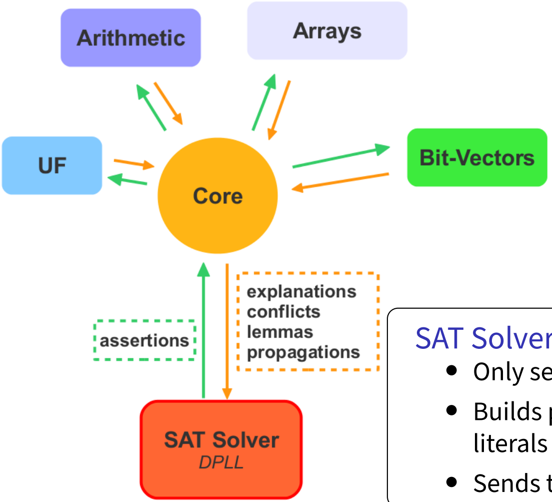
$\mathcal{T}$-solver:

- Checks the $\mathcal{T}$-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of $\mathcal{T}$-unsatisfiability/propagation
- Must be incremental and backtrackable

Typical SMT solver architecture



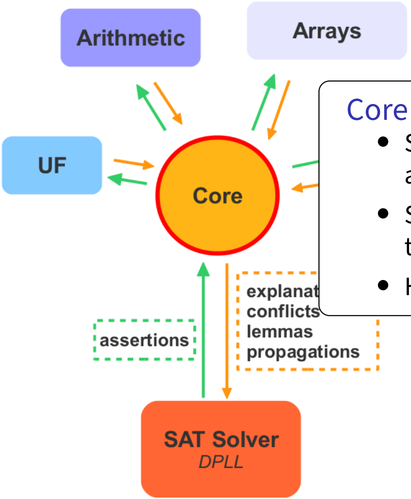
Typical SMT solver architecture



SAT Solver

- Only sees *Boolean skeleton* of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as *assertions*

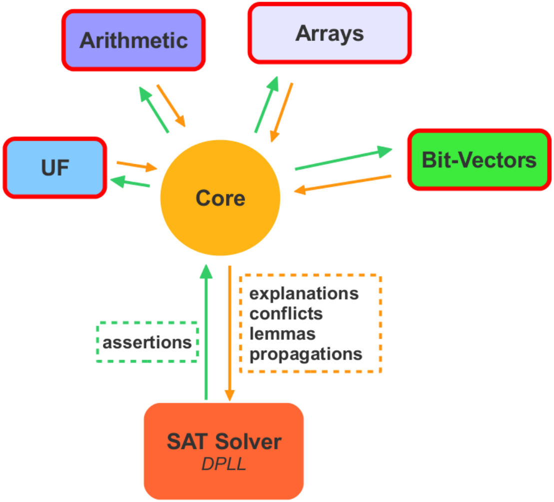
Typical SMT solver architecture



Core

- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

Typical SMT solver architecture



Theory Solvers

- Check $\mathcal{T}$-satisfiability of sets of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation