CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

Proof systems for First-order Logic

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Outline

- Semantic arguments for FOL
- PCNF (ML 9.2) and Clausal Form
- First-order Resolution (ML 10)

Proofs in first-order logic

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Semantic arguments for FOL: propositional rules

(a) $$\frac{\mathcal{I} \models \neg\alpha}{\mathcal{I} \not\models \alpha}$$

(b) $$\frac{\mathcal{I} \not\models \neg\alpha}{\mathcal{I} \models \alpha}$$

(c) $$\frac{\mathcal{I} \models \alpha \wedge \beta}{\mathcal{I} \models \alpha, \mathcal{I} \models \beta}$$

(d) $$\frac{\mathcal{I} \not\models \alpha \wedge \beta}{\mathcal{I} \not\models \alpha \mid \mathcal{I} \not\models \beta}$$

(e) $$\frac{\mathcal{I} \models \alpha \vee \beta}{\mathcal{I} \models \alpha \mid \mathcal{I} \models \beta}$$

(f) $$\frac{\mathcal{I} \not\models \alpha \vee \beta}{\mathcal{I} \not\models \alpha, \mathcal{I} \not\models \beta}$$

(g) $$\frac{\mathcal{I} \models \alpha \Rightarrow \beta}{\mathcal{I} \not\models \alpha \mid \mathcal{I} \models \beta}$$

(h) $$\frac{\mathcal{I} \not\models \alpha \Rightarrow \beta}{\mathcal{I} \models \alpha, \mathcal{I} \not\models \beta}$$

(i) $$\frac{\mathcal{I} \models \alpha \quad \mathcal{I} \not\models \alpha}{\mathcal{I} \models \bot}$$

(k) $$\frac{\mathcal{I} \models \alpha \Leftrightarrow \beta}{\mathcal{I} \models \alpha, \mathcal{I} \models \beta \mid \mathcal{I} \not\models \alpha, \mathcal{I} \not\models \beta}$$

(j) $$\frac{\mathcal{I} \not\models \alpha \Leftrightarrow \beta}{\mathcal{I} \not\models \alpha, \mathcal{I} \models \beta \mid v \models \alpha, \mathcal{I} \not\models \beta}$$

Semantic arguments for FOL: quantifier rules

Notation: if v is a variable, ε is a term/formula, and t is a term, $\varepsilon[v \leftarrow t]$ denotes the term/formula obtained from ε by replacing every free occurrence of v in ε by t

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Examples:

$$x[x \leftarrow \mathsf{S}(y)] = \mathsf{S}(y) \qquad (x + y)[x \leftarrow y] = y + y$$

$$x[x \leftarrow \mathsf{S}(x)] = \mathsf{S}(x) \qquad (x \doteq y)[x \leftarrow 0] = 0 \doteq y$$

$$x[x \leftarrow y] = y \qquad (x \doteq x)[x \leftarrow \mathsf{S}(x)] = \mathsf{S}(x) \doteq \mathsf{S}(x)$$

$$(x \doteq y \lor x < y)[x \leftarrow \mathsf{S}(0)] = \mathsf{S}(0) \doteq y \lor \mathsf{S}(0) < y$$

$$(x \doteq y \lor \forall x.\, x < y)[x \leftarrow \mathsf{S}(y)] = \mathsf{S}(y) \doteq y \lor \forall x.\, x < y$$

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Quantifier rules

(m) $$\frac{\mathcal{I} \models \forall v{:}\sigma.\ \alpha}{\mathcal{I} \models \alpha[v \leftarrow t]}$$ for any term t of sort σ

(n) $$\frac{\mathcal{I} \not\models \exists v{:}\sigma.\ \alpha}{\mathcal{I} \not\models \alpha[v \leftarrow t]}$$ for any term t of sort σ

(o) $$\frac{\mathcal{I} \models \exists v{:}\sigma.\ \alpha}{\mathcal{I} \models \alpha[v \leftarrow k]}$$ for a fresh variable k of sort σ

(p) $$\frac{\mathcal{I} \not\models \forall v{:}\sigma.\ \alpha}{\mathcal{I} \not\models \alpha[v \leftarrow k]}$$ for a fresh variable k of sort σ

Proof by deduction: Example 1

Consider signature Σ with $\Sigma^S = \{ \mathsf{A} \}$, $\Sigma^F = \{ P \}$, $\mathrm{rank}(P) = \langle \mathsf{A}, \mathsf{Bool} \rangle$, and all vars of sort A

Prove that $\exists x.\, P(x) \Rightarrow \exists y. P(y)$ is valid

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2. $\mathcal{I} \models \exists x.\, P(x)$ by (h) on 1
3. $\mathcal{I} \not\models \exists y.\, P(y)$ by (h) on 1
4. $\mathcal{I} \models P(x_0)$ by (o) on 2
5. $\mathcal{I} \not\models P(x_0)$ by (n) on 3
6. $\mathcal{I} \models \bot$ by (i) on 4, 5

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Proof by deduction: Example 2

Consider signature Σ with $\Sigma^S = \{ A \}$, $\Sigma^F = \{ P \}$, $\mathrm{rank}(P) = \langle A, \mathrm{Bool} \rangle$, and all vars of sort A

Prove that $\forall x.\, (P(x) \Rightarrow \exists y.\, P(y))$ is valid

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Proof by deduction: Example 3

Consider signature Σ with $\Sigma^S = \{ \mathsf{A} \}$, $\Sigma^F = \{ Q \}$, $\mathrm{rank}(Q) = \langle \mathsf{A}, \mathsf{A}, \mathsf{Bool} \rangle$, and all vars of sort A

Prove that $\exists x.\, \forall y.\, Q(x,y) \Rightarrow \forall y.\, \exists x.\, Q(x,y)$ is valid

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4. $\mathcal{I} \models \forall y.Q(x_0,y)$ by (o) on 2
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Refutation Soundness and Completeness

Theorem 1 (Soundness)

For all Σ-formulas α, if there is a closed derivation tree with root $\mathcal{I} \not\models \alpha$ then α is valid

Theorem 2 (Completeness)

For all Σ-formulas α without equality, if α is valid, then there is a closed derivation tree with root $\mathcal{I} \not\models \alpha$

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Termination?

Does the semantic argument method describe a decision procedure then?

No, for an invalid formula, the semantic argument proof system might not terminate

Intuition: Consider the invalid formula $\forall x.\ q(x, x)$

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There is no strategy that guarantees termination in all cases of invalid formulas

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FOL is only *semi-decidable*: you can always show validity algorithmically but not invalidity

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Prenex Normal Form (PNF)

For AR purposes, it is useful in FOL too impose syntactic restrictions on formulas

A Σ-formula α is in *prenex normal form* (PNF) if it has the form

$$Q_1 x_1. \cdots Q_n x_n. \beta$$

where each Q_i is a quantifier and β is a quantifier-free formula

Formula α above is in *prenex conjunctive normal form* (PCNF) if, in addition, β is in conjunctive normal form[1]

Example: The formula below is in PCNF

$$\forall y. \exists z. \Big(\overbrace{(\underbrace{p(f(y))}_{A_1} \lor \underbrace{q(z)}_{A_2})}^{C_1} \land \overbrace{(\underbrace{\neg q(z)}_{A_2} \lor \underbrace{q(x)}_{A_3})}^{C_2}\Big)$$

[1] If we treat every atomic formula of β as if it was a propositional variable

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Clausal Form

A Σ-formula is in *clausal form* if

1. it is in PCNF
2. it is closed (i.e., it has no free variables)
3. all of its quantifiers are universal

Exercise: Which of the following formulas are clausal form?

- $\forall y.\, \exists z.\, (p(f(y)) \land \neg q(y, z))$ ✗
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Clausal Form: transformation

Theorem 3 (Skolem's Theorem)

Any sentence can be transformed to an equi-satisfiable formula in clausal form.

The high level transformation strategy is the following:

$$\text{Sentence} \longrightarrow \text{PNF} \longrightarrow \text{PCNF} \longrightarrow \text{Clausal Form}$$

Running example: $(\forall x.(p(x) \Rightarrow q(x))) \Rightarrow (\forall x.p(x) \Rightarrow \forall x.q(x))$

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Any sentence can be transformed to an equi-satisfiable formula in clausal form.

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II: Transforming into PCNF

Transforming a PNF to a logically equivalent PCNF is straightforward

We apply the distributive laws from propositional logic

$$\exists x.\, \forall z.\, \exists y.\, ((p(x) \wedge \neg q(x)) \vee (\neg p(y) \vee q(z)))$$

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III: Transforming into Clausal Form (Skolemization)

$$\exists x.\, \forall z.\, \exists y.\, ((p(x) \vee \neg p(y) \vee q(z)) \wedge (\neg q(x) \vee \neg p(y) \vee q(z)))$$

For every existential quantifier $\exists v$ in the PCNF, let $u_1, \ldots, u_n$ be the universally quantified variables preceding $\exists v$,

1. introduce a fresh function symbol f_v with arity n and $\langle \mathrm{sort}(u_1), \ldots, \mathrm{sort}(u_n), \mathrm{sort}(v) \rangle$
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For the formula above, introduce nullary function (i.e., a constant) symbol f_x and unary function symbol f_y for $\exists x$ and $\exists y$, respectively

$$\forall z.\, ((p(f_x) \lor \neg p(f_y(z)) \lor q(z)) \land (\neg q(f_x) \lor \neg p(f_y(z)) \lor q(z))$$

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Note: Technically, the resulting formula is no longer a Σ-formula, but a Σ_E-formula, where $\Sigma_E^S = \Sigma^S$ and $\Sigma_E^F = \Sigma^F \cup \bigcup_v \{ f_v \}$

Clausal forms as clause sets

As with propositional logic, we can write a formula in clausal form unambiguously as a set of clauses

Example:

$$\forall z.\, ((p(f(z)) \lor \neg p(g(z)) \lor q(z)) \land (\neg q(f(z)) \lor \neg p(g(z)) \lor q(z)))$$

can be written as

$$\Delta := \{\, \{p(f(z)), \neg p(g(z)), q(z)\}, \{\neg q(f(z)), \neg p(g(z)), q(z)\}\,\}$$

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Traditionally, theorem provers for FOL use the latter version of the clausal form

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A resolution-based proof system for PL

Recall: The satisfiability proof system consisting of the rules below is sound, complete and terminating for clause sets in PL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{ p \}) \cup (C_2 \setminus \{ \neg p \}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{ C \}}$$

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Consider the FOL clause set below where x, z are variables and a is a constant symbol

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Example: $C_1 : \{\neg P(z), Q(z)\}$ $C_2 : \{P(a)\}$ $C_3 : \{\neg Q(x)\}$

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$\{\,\}$	$\{\, C_1, C_2, C_3, C_4, C_5{:}\{Q(a)\}\,\}$	by **Resolve** on C_2, C_4
$\{\,\}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6{:}\{\neg Q(a)\}\,\}$	by **Inst** on C_3 with $x \leftarrow a$
$\{\,\}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6, C_7{:}\{\}\,\}$	by **Resolve** on C_5, C_6
	UNSAT	by **Unsat** on C_7

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$\{\ \}$	$\{\, C_1, C_2, C_3, C_4{:}\{\neg P(a), Q(a)\} \,\}$	by Inst on C_1 with $z \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5{:}\{Q(a)\} \,\}$	by Resolve on C_2, C_4
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6{:}\{\neg Q(a)\} \,\}$	by Inst on C_3 with $x \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6, C_7{:}\{\} \,\}$	by Resolve on C_5, C_6
	UNSAT	by Unsat on C_7

A resolution-based proof system for FOL

Resolve $$\frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

Clash $$\frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}}$$

Inst $$\frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

Unsat $$\frac{\{\} \in \Delta}{\text{UNSAT}}$$

Sat $$\frac{\text{No other rules apply}}{\text{SAT}}$$

Example: $C_1 : \{\neg P(z), Q(z)\} \quad C_2 : \{P(a)\} \quad C_3 : \{\neg Q(x)\}$

Φ	Δ	
$\{\ \}$	$\{\, C_1, C_2, C_3 \,\}$	
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4{:}\{\neg P(a), Q(a)\} \,\}$	by **Inst** on C_1 with $z \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5{:}\{Q(a)\} \,\}$	by **Resolve** on C_2, C_4
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6{:}\{\neg Q(a)\} \,\}$	by **Inst** on C_3 with $x \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6, C_7{:}\{\} \,\}$	by **Resolve** on C_5, C_6
	UNSAT	by **Unsat** on C_7

A resolution-based proof system for FOL

Resolve $\dfrac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{ p \}) \cup (C_2 \setminus \{ \neg p \}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{ C \}}$

Clash $\dfrac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{ C \} \quad \Phi := \Phi \cup \{ C \}}$ **Inst** $\dfrac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$

Unsat $\dfrac{\{\} \in \Delta}{\text{UNSAT}}$ **Sat** $\dfrac{\text{No other rules apply}}{\text{SAT}}$

Example: $C_1 : \{\neg P(z), Q(z)\} \quad C_2 : \{P(a)\} \quad C_3 : \{\neg Q(x)\}$

Φ	Δ	
$\{ \}$	$\{ C_1, C_2, C_3 \}$	
$\{ \}$	$\{ C_1, C_2, C_3, C_4{:}\{\neg P(a), Q(a)\} \}$	by **Inst** on C_1 with $z \leftarrow a$
$\{ \}$	$\{ C_1, C_2, C_3, C_4, C_5{:}\{Q(a)\} \}$	by **Resolve** on C_2, C_4
$\{ \}$	$\{ C_1, C_2, C_3, C_4, C_5, C_6{:}\{\neg Q(a)\} \}$	by **Inst** on C_3 with $x \leftarrow a$
$\{ \}$	$\{ C_1, C_2, C_3, C_4, C_5, C_6, C_7{:}\{\} \}$	by **Resolve** on C_5, C_6
	UNSAT	by **Unsat** on C_7

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

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Φ	Δ	
$\{\ \}$	$\{\, C_1, C_2, C_3 \,\}$	
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4{:}\{\neg P(a), Q(a)\} \,\}$	by **Inst** on C_1 with $z \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5{:}\{Q(a)\} \,\}$	by **Resolve** on C_2, C_4
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6{:}\{\neg Q(a)\} \,\}$	by **Inst** on C_3 with $x \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6, C_7{:}\{\} \,\}$	by **Resolve** on C_5, C_6
	UNSAT	by **Unsat** on C_7

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

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$$\textsc{Unsat}\ \frac{\{\} \in \Delta}{\text{UNSAT}} \qquad \textsc{Sat}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

Example: $C_1 : \{\neg P(z), Q(z)\} \quad C_2 : \{P(a)\} \quad C_3 : \{\neg Q(x)\}$

Φ	Δ	
$\{\ \}$	$\{\, C_1, C_2, C_3 \,\}$	
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4{:}\{\neg P(a), Q(a)\} \,\}$	by **Inst** on C_1 with $z \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5{:}\{Q(a)\} \,\}$	by **Resolve** on C_2, C_4
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6{:}\{\neg Q(a)\} \,\}$	by **Inst** on C_3 with $x \leftarrow a$
$\{\ \}$	$\{\, C_1, C_2, C_3, C_4, C_5, C_6, C_7{:}\{\} \,\}$	by **Resolve** on C_5, C_6
	UNSAT	by **Unsat** on C_7

A resolution-based proof system for FOL

Resolve $\dfrac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{p\}) \cup (C_2 \setminus \{\neg p\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{C\}}$

Clash $\dfrac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{C\} \quad \Phi := \Phi \cup \{C\}}$ **Inst** $\dfrac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$

Unsat $\dfrac{\{\} \in \Delta}{\text{UNSAT}}$ **Sat** $\dfrac{\text{No other rules apply}}{\text{SAT}}$

Example: $C_1 : \{\neg P(z), Q(z)\} \quad C_2 : \{P(a)\} \quad C_3 : \{\neg Q(x)\}$

Φ	Δ	
$\{\}$	$\{C_1, C_2, C_3\}$	
$\{\}$	$\{C_1, C_2, C_3, C_4{:}\{\neg P(a), Q(a)\}\}$	by **Inst** on C_1 with $z \leftarrow a$
$\{\}$	$\{C_1, C_2, C_3, C_4, C_5{:}\{Q(a)\}\}$	by **Resolve** on C_2, C_4
$\{\}$	$\{C_1, C_2, C_3, C_4, C_5, C_6{:}\{\neg Q(a)\}\}$	by **Inst** on C_3 with $x \leftarrow a$
$\{\}$	$\{C_1, C_2, C_3, C_4, C_5, C_6, C_7{:}\{\}\}$	by **Resolve** on C_5, C_6
	UNSAT	by **Unsat** on C_7

A resolution-based proof system for FOL

Resolve $$\frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p\,\}) \cup (C_2 \setminus \{\, \neg p\,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C\,\}}$$

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Inst $$\frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

Unsat $$\frac{\{\} \in \Delta}{\text{UNSAT}}$$

Sat $$\frac{\text{No other rules apply}}{\text{SAT}}$$

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

$$\textsc{Clash}\ \frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}} \qquad \textsc{Inst}\ \frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

$$\textsc{Unsat}\ \frac{\{\} \in \Delta}{\text{UNSAT}} \qquad \textsc{Sat}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

This system is refutation-sound and complete for FOL clause sets without equality:

- If a clause set Δ_0 is unsatisfiable, there is a derivation of UNSAT from Δ_0

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

$$\textsc{Clash}\ \frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}} \qquad \textsc{Inst}\ \frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

$$\textsc{Unsat}\ \frac{\{\} \in \Delta}{\text{UNSAT}} \qquad \textsc{Sat}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

This system is refutation-sound and complete for FOL clause sets without equality:

- If a clause set Δ_0 is unsatisfiable, there is a derivation of UNSAT from Δ_0

The system is also solution-sound:

- There is a derivation of SAT from Δ_0 only if Δ_0 is satisfiable

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

$$\textsc{Clash}\ \frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}} \qquad \textsc{Inst}\ \frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

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This system is refutation-sound and complete for FOL clause sets without equality:

- If a clause set Δ_0 is unsatisfiable, there is a derivation of UNSAT from Δ_0

The system is not, and cannot be, terminating:

- if Δ_0 is satisfiable, it is possible for **Sat** to never apply

A resolution-based proof system for FOL

Resolve $$\frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{p\}) \cup (C_2 \setminus \{\neg p\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{C\}}$$

Clash $$\frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{C\} \quad \Phi := \Phi \cup \{C\}}$$

Inst $$\frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

Unsat $$\frac{\{\} \in \Delta}{\text{UNSAT}}$$

Sat $$\frac{\text{No other rules apply}}{\text{SAT}}$$

Note: This proof system is challenging to implement efficiently because **Inst** is not constrained enough

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

$$\textsc{Clash}\ \frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}} \qquad \textsc{Inst}\ \frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

$$\textsc{Unsat}\ \frac{\{\} \in \Delta}{\mathsf{UNSAT}} \qquad \textsc{Sat}\ \frac{\text{No other rules apply}}{\mathsf{SAT}}$$

Automated theorem provers for FOL use instead a more sophisticated **Resolve** rule

where two literals in different clauses are instantiated directly, and only as needed, to make them complementary (see ML Chap. 10)

A resolution-based proof system for FOL

Resolve $$\frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{p\}) \cup (C_2 \setminus \{\neg p\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{C\}}$$

Clash $$\frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{C\} \quad \Phi := \Phi \cup \{C\}}$$

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Unsat $$\frac{\{\} \in \Delta}{\text{UNSAT}}$$

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Automated theorem provers for FOL use instead a more sophisticated **Resolve** rule

where two literals in different clauses are instantiated directly, and only as needed, to make them complementary (see ML Chap. 10)

Example: $\{P(x,y), Q(a, f(y))\}$, $\{\neg Q(z, f(b)), R(g(z))\}$ resolve to $\{P(x,b), R(g(a))\}$

A resolution-based proof system for FOL

Resolve $$\frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

Clash $$\frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}}$$

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Unsat $$\frac{\{\} \in \Delta}{\text{UNSAT}}$$

Sat $$\frac{\text{No other rules apply}}{\text{SAT}}$$

Problem: How do we prove the unsatisfiability of these clause sets?

$$\{\, \{x \doteq y\}, \{\neg(y \doteq x)\} \,\} \quad \{\, \{x \doteq y\}, \{y \doteq z\}, \{\neg(x \doteq z)\} \,\} \quad \{\, \{x \doteq y\}, \{\neg(f(x) \doteq f(y))\} \,\}$$

A resolution-based proof system for FOL

Resolve $$\frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

Clash $$\frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}}$$

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We need specialized rules for equality reasoning!

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

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$$\textsc{Unsat}\ \frac{\{\} \in \Delta}{\text{UNSAT}} \qquad \textsc{Sat}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

Another Problem: How to we prove the unsatisfiability of these clause sets?

$$\{\, \{x < x\} \,\} \quad \{\, \{x < y\}, \{y < z\}, \{\neg(x < z)\} \,\} \quad \{\, \{\neg(x + y \doteq y + x)\} \,\} \quad \{\, \{\neg(x + 0 \doteq x)\} \,\}$$

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$$

$$\textsc{Clash}\ \frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}} \qquad \textsc{Inst}\ \frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

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The thing is: each of these clause set is actually satisfiable in FOL!

A resolution-based proof system for FOL

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{ p \}) \cup (C_2 \setminus \{ \neg p \}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{ C \}}$$

$$\textsc{Clash}\ \frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{ C \} \quad \Phi := \Phi \cup \{ C \}} \qquad \textsc{Inst}\ \frac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$$

$$\textsc{Unsat}\ \frac{\{\} \in \Delta}{\text{UNSAT}} \qquad \textsc{Sat}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

Another Problem: How to we prove the unsatisfiability of these clause sets?

$$\{ \{x < x\} \} \quad \{ \{x < y\}, \{y < z\}, \{\neg(x < z)\} \} \quad \{ \{\neg(x + y \doteq y + x)\} \} \quad \{ \{\neg(x + 0 \doteq x)\} \}$$

The thing is: each of these clause set is actually satisfiable in FOL!

However, they are unsatisfiable in the theory of arithmetic

A resolution-based proof system for FOL

Resolve $\dfrac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\, p \,\}) \cup (C_2 \setminus \{\, \neg p \,\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{\, C \,\}}$

Clash $\dfrac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{\, C \,\} \quad \Phi := \Phi \cup \{\, C \,\}}$

Inst $\dfrac{C \in \Delta \quad v \in \mathcal{FV}(C) \quad \mathrm{sort}(t) = \mathrm{sort}(v)}{\Delta := \Delta \cup \{C[v \leftarrow t]\}}$

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We need proof systems for satisfiability modulo theories