CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

First-order Logic: Syntax and Semantics

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Motivation

Consider formalizing and reasoning about these sentences in propositional logic

English	PL
Every natural number is greater than 0	p
1 is a natural number not equal to 0	$\neg q$

What facts can we logically deduce? Only: $p \wedge \neg q$

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Propositional logic is often too coarse to express information
about individual objects and formalize correct deductions about them

We cannot deduce that 1 is greater than 0 from the two sentences above

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First-order Logic (FOL) allows us to (dis)prove the validity of sentences like the above or deduce their consequences (e.g., that 1 is greater than 0)

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In this case, we need a first-order language for number theory

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Intuitively, a first-order language has the following features:

- A sublanguage to denote individual things (numbers, people, colors, ...)
- A sublanguage to express properties of individuals and relations among them
- A sublanguage to denote groups of individuals with common features and ascribe them to specific individuals
- A way to quantify statements about individuals

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"Every positive integer number different from 1 is smaller than its square"

English	FOL language
generic number	x
the number 1	1
the square of x	$\mathrm{square}(x)$
"x is positive"	$\mathrm{positive}(x)$
"x is different from 1"	$x \neq 1$
"x is smaller than its square"	$x < \mathrm{square}(x)$
"every integer number"	$\forall x : \mathrm{Int}$

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Sentence above in FOL: $\forall x : \text{Int}. \, (\text{positive}(x) \land x \neq 1 \;\Rightarrow\; x < \text{square}(x))$

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Sentence above in FOL: $\forall x : \mathrm{Int}. (\mathrm{positive}(x) \land x \neq 1 \Rightarrow x < \mathrm{square}(x))$

The formula is true in the intended interpretation

Outline

- Syntax (ML 7.1-2)
- Semantics (ML 7.3)

ML presents a one-sorted first-order logic

We will use a many-sorted first-order logic

This makes it convenient to present Satisfiability Modulo Theories later

Note:
Many-sorted FOL is not more expressive than one-sorted FOL:
It is possible to faithfully encode the former in the latter

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> **Note:**
> Many-sorted FOL is not more expressive than one-sorted FOL:
> It is possible to faithfully encode the former in the latter
>
> However, using different sorts makes it more convenient to rule out non-sensical expressions

Symbols

Review: what does the syntax of a logic consist of?

Symbols + rules for combining them

First-order logic is an umbrella term for different *first-order languages*

The *symbols* of a first-order language consist of:

1. *Logical symbols* ($\Rightarrow$, $\top$, $\wedge$, $\neg$, (,))
2. *Signature*, $\Sigma := \langle \Sigma^S, \Sigma^F \rangle$, where:
   - Σ^S is a set of *sorts*: e.g., Real, Int, Set
   - Σ^F is a set of *function symbols*: e.g., $=$, $+$, $+_{[2]}$, $<$, $\emptyset$

Note: We consider symbols as *atomic* (not divisible further)

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- an *arity* n: a natural number denoting the number of arguments f takes
- a *rank* a $(n+1)$-tuple of sorts: $\mathrm{rank}(f) = \langle \sigma_1, \ldots, \sigma_n, \sigma_{n+1} \rangle$

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$\sigma_1, \ldots, \sigma_n$ are the *input sorts* of f and σ_{n+1} is the *output sort*

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We call function symbols a of arity 0 *constants* and say they have sort σ when $\mathrm{rank}(a) = \langle \sigma \rangle$

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We also assume an infinite set of *variable (symbols)* $x, y, \ldots$

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Example: In the first-order language of number theory

- Σ^S contains a sort Nat and Σ^F contains a function symbols $\mathbf{0}$, $\mathbf{1}$, $+$
- $\mathbf{0}$ and $\mathbf{1}$ have arity 0 and $\mathrm{rank}(0) = \mathrm{rank}(1) = \langle \mathsf{Nat} \rangle$
- $+$ has arity 2 and $\mathrm{rank}(+) = \langle \mathsf{Nat}, \mathsf{Nat}, \mathsf{Nat} \rangle$

Signature

We assume for every signature Σ that

- Σ^S includes a distinguished sort Bool
- Σ^F contains distinguished constants $\top$ and $\bot$ with $\mathrm{sort}(\bot) = \mathrm{sort}(\top) = \mathsf{Bool}$, and distinguished functions symbols $\doteq_\sigma$ with $\mathrm{rank}(\doteq_\sigma) = \langle \sigma, \sigma, \mathsf{Bool} \rangle$ for all $\sigma \in \Sigma^S$

There are two special kinds of function symbols:

- *Constant symbols*: *function symbols of 0 arity* (e.g., $\bot$, $\top$, π, John, 0)
- *Predicate symbols*: *function symbols of return sort* Bool (e.g., $\doteq_\sigma$, $<$)

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First-Order Languages: Examples

Recall that a first-order language is defined wrt a signature $\Sigma := \langle \Sigma^S, \Sigma^F \rangle$

Elementary Number Theory

- $\Sigma^S : \{ \text{Nat}, \text{Bool} \}$
- $\Sigma^F : \{ <, 0, S, +, \times, \doteq_{\text{Nat}} \} \cup \{ \top, \bot, \doteq_{\text{Bool}} \}$

where:

- $\mathrm{rank}(<) = \langle \text{Nat}, \text{Nat}, \text{Bool} \rangle$
- $\mathrm{rank}(0) = \langle \text{Nat} \rangle$
- $\mathrm{rank}(S) = \langle \text{Nat}, \text{Nat} \rangle$
- $\mathrm{rank}(+) = \mathrm{rank}(\times) = \langle \text{Nat}, \text{Nat}, \text{Nat} \rangle$

First-Order Languages: Examples

Set Theory

- Σ^S : $\{ \textsf{Set}, \textsf{Bool} \}$
- Σ^F : $\{ \in, \emptyset, \cup, \cap, \doteq_{\textsf{Set}} \} \cup \{ \top, \bot, \doteq_{\textsf{Bool}} \}$

where:

- $\mathrm{rank}(\emptyset) = \langle \textsf{Set} \rangle$
- $\mathrm{rank}(\cup) = \mathrm{rank}(\cap) = \langle \textsf{Set}, \textsf{Set}, \textsf{Set} \rangle$
- $\mathrm{rank}(\in) = \langle \textsf{Set}, \textsf{Set}, \textsf{Bool} \rangle$

First-Order Languages: Examples

Propositional logic formulas

- $\Sigma^S : \{ \mathsf{Bool} \}$
- $\Sigma^F : \{ \neg, \wedge, \vee, \dots, p_1, p_2, \dots \} \cup \{ \top, \bot, \doteq_{\mathsf{Bool}} \}$

where:

- $\mathrm{rank}(p_i) = \langle \mathsf{Bool} \rangle$
- $\mathrm{rank}(\neg) = \langle \mathsf{Bool}, \mathsf{Bool} \rangle$
- $\mathrm{rank}(\wedge) = \mathrm{rank}(\vee) = \langle \mathsf{Bool}, \mathsf{Bool}, \mathsf{Bool} \rangle$

Expressions

Recall that an expression is any finite sequence of symbols

Example

- $\forall x_1. ((<\ 0\ x_1) \Rightarrow (\neg \forall x_2. (<\ x_1\ x_2)))$
- $x_1 < \forall x_2))$
- $x_1 < x_2 \Rightarrow \forall x{:}\mathrm{Nat}.\, x > 0$

Most expressions are not well-formed

Expressions of interest in FOL are *terms* and *well-formed formulas (wffs)*

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Terms

Expressions built up from function symbols, variables, and parentheses ((,))

Formally, let $\mathcal{B}$ be the set of all variables and all constant symbols in some signature Σ

For each function symbol $f \in \Sigma^F$ of arity $n > 0$, we define a *term-building operation* $\mathcal{T}_f$:

$$\mathcal{T}_f(\varepsilon_1, \ldots, \varepsilon_n) := (f\ \varepsilon_1 \cdots \varepsilon_n)$$

Terms are expressions that are generated from $\mathcal{B}$ by $\mathcal{T} = \{\, \mathcal{T}_f \mid f \in \Sigma^F \,\}$

Examples of terms in the language of number theory:

✓ $(+\ x_2\ (S\ 0))$	✗ $(x_2 + 0)$	✓ $(+\ x_2\ \bot)$
✓ $(S\ (S\ (S\ (S\ 0))))$	✗ $(S\ 0\ 0)$	✓ $(S\ \bot)$
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Well-sorted terms

Not all well-formed terms are meaningful

We consider only terms that are *well-sorted* wrt a given signature Σ

Well-sortedness

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where

- $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is *sort context*, a set of sorted variables
- t is a well-formed term
- σ is a sort of Σ

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We formulate the notion of *well-sortedness* wrt Σ with a *sort system*, a proof system over sequents of the form $\Gamma \vdash t : \sigma$

$$\textsc{Var}\ \frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \textsc{Const}\ \frac{c \in \Sigma^F \quad \mathrm{rank}(c) = \langle \sigma \rangle}{\Gamma \vdash c : \sigma}$$

$$\textsc{Fun}\ \frac{f \in \Sigma^F \quad \mathrm{rank}(f) = \langle \sigma_1, \ldots, \sigma_n, \sigma \rangle \quad \Gamma \vdash t_1 : \sigma_1 \quad \cdots \quad \Gamma \vdash t_n : \sigma_n}{\Gamma \vdash (f\ t_1\ \cdots\ t_n) : \sigma}$$

A term t is *well-sorted* wrt Σ and *has sort* σ in a sort context Γ
if $\Gamma \vdash t : \sigma$ is derivable in the sort system above

Well-sortedness

We formulate the notion of *well-sortedness* wrt Σ with a *sort system*, a proof system over sequents of the form $\Gamma \vdash t : \sigma$

$$\textsc{Var}\ \frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \textsc{Const}\ \frac{c \in \Sigma^F \quad \mathrm{rank}(c) = \langle \sigma \rangle}{\Gamma \vdash c : \sigma}$$

$$\textsc{Fun}\ \frac{f \in \Sigma^F \quad \mathrm{rank}(f) = \langle \sigma_1, \ldots, \sigma_n, \sigma \rangle \quad \Gamma \vdash t_1 : \sigma_1 \quad \cdots \quad \Gamma \vdash t_n : \sigma_n}{\Gamma \vdash (f\ t_1\ \cdots\ t_n) : \sigma}$$

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Note: Every well-sorted term is also well-formed

Well-sorted terms example: Elementary number theory

Let $\Sigma^S = \{ \mathsf{Nat} \} \; (\cup \{ \mathsf{Bool} \})$ and $\Sigma^F = \{ 0, S, +, \times, <, \doteq_{\mathsf{Nat}} \} \; (\cup \{ \top, \bot, \doteq_{\mathsf{Bool}} \})$

- $\mathrm{rank}(0) = \langle \mathsf{Nat} \rangle$
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Are these well-formed terms also well-sorted in context $\Gamma = \{ x_1 : \mathsf{Bool}, x_2 : \mathsf{Nat}, x_3 : \mathsf{Nat} \}$?

1. $(+\ 0\ x_2)$ ✓
2. $(+\ (+\ 0\ x_1)\ x_2)$ ✗
3. $(S\ (+\ 0\ x_5))$ ✓
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Note: As a notational convention, we will use an infix notation for parentheses and common operators like $\doteq$, $<$, $+$ and so on

So we will often write $S(x_3) \doteq_{\mathsf{Nat}} S(0) + x_1$ instead of $(\doteq_{\mathsf{Nat}}\ (S\ x_3)\ (+\ (S\ 0)\ x_1))$

Σ-Formulas

Given a signature Σ, an *atomic Σ-formula* is any term that is a Σ-term t of sort Bool under some sort context Γ

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Examples: $(\doteq_{\mathsf{Nat}}\ 0\ (S\ 0))$, $(<\ (S\ x_3)\ (+\ (S\ 0)\ x_1))$

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We define the following formula-building operations, denoted $\mathcal{F}$:

$$\mathcal{F}_\vee(\alpha, \beta) := (\alpha \vee \beta) \qquad \mathcal{F}_\wedge(\alpha, \beta) := (\alpha \wedge \beta) \qquad \mathcal{F}_\neg(\alpha) := (\neg \alpha)$$

$$\mathcal{F}_\Rightarrow(\alpha, \beta) := (\alpha \Rightarrow \beta) \qquad \mathcal{F}_\Leftrightarrow(\alpha, \beta) := (\alpha \Leftrightarrow \beta)$$

$\mathcal{E}_{x,\sigma}(\alpha) := (\exists x : \sigma.\ \alpha)$ for each var x and sort $\sigma \in \Sigma^S$

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The set of *well-formed formulas* is the set of expressions generated from the atomic Σ-formulas by $\mathcal{F}$

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Each $\exists x : \sigma$ is an *existential quantifier*
Each $\forall x : \sigma$ is a *universal quantifier*

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We simplify the notation as in PL by

- forgoing parentheses around top-level formulas — e.g., $(x \doteq y) \vee ((y \doteq z) \vee (x \doteq z))$
- forgoing parenths around atomic formulas in infix form — e.g., $x \doteq y \vee (y \doteq z \vee x \doteq z)$
- treating the binary connectives as n-ary and right associative — e.g., $x \doteq y \vee y \doteq z \vee x \doteq z$

Σ-Formulas: Examples

Let $\Sigma = \langle \Sigma^S := \{\mathsf{Nat}\}, \Sigma^F := \{\, 0, S, +, \times, <, \doteq_{\mathsf{Nat}} \,\}\rangle$ a x_i be variables for all i

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Which of the following formulas (with atomic subformulas in infix form) are well-formed?

1. $(\doteq_{\mathrm{Nat}}\ (+\ x_1\ 0)\ x_2)$ ✓
2. $(\doteq_{\mathrm{Nat}}\ (+\ x_1\ 0)\ x_2) \Rightarrow \bot$ ✓
3. $(+\ 0\ x_3) \wedge (<\ \ 0\ (S\ 0))$ ✗
4. $\forall\, x_3 : Nat.\, (+\ (+\ 0\ x_3)\ x_2)$ ✗
5. $\forall\, x_3 : Bool.\, (\doteq_{\mathrm{Nat}}\ (+\ 0\ x_3)\ x_2)$ ✓
6. $\neg\exists\, x_0 : Nat.\, (<\ \ 0\ x_0\ (S\ 0))$ ✗

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Note: Formula (5) is well-formed but not well-sorted

To know which formulas are well-sorted we need to extend our sort system to the logical operators

Well-sorted formulas

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$$\textsc{Bconst}\ \frac{c \in \{\top, \bot\}}{\Gamma \vdash c : \mathsf{Bool}} \qquad \textsc{Not}\ \frac{\Gamma \vdash \alpha : \mathsf{Bool}}{\Gamma \vdash (\neg\alpha) : \mathsf{Bool}}$$

$$\textsc{Conn}\ \frac{\Gamma \vdash \alpha : \mathsf{Bool} \quad \Gamma \vdash \beta : \mathsf{Bool} \quad \bowtie \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}}{\Gamma \vdash (\alpha \bowtie \beta) : \mathsf{Bool}}$$

$$\textsc{Quant}\ \frac{\Gamma[x : \sigma] \vdash \alpha : \mathsf{Bool} \quad \sigma \in \Sigma^S \quad Q \in \{\forall, \exists\}}{\Gamma \vdash (Q\,x : \sigma.\, \alpha) : \mathsf{Bool}}$$

$\Gamma[x : \sigma]$ is a context that assigns sort σ to x and is otherwise identical to Γ

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A formula α is *well-sorted* wrt Σ in a sort context Γ
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We call α a Σ-*formula*

Exercise

Draw two Venn Diagram that illustrate the relations between

A: terms

B: well-formed terms

C: well-sorted terms

D: well-sorted atomic formulas

and between

D: well-sorted atomic formulas

E: well-formed formulas

F: well-sorted formulas

Notational conventions for formulas

From now on, to improve readability:

- We will use the infix notation for logical operators and function symbols typically written in that notation ($\doteq_\sigma$, $<$, $+$, $\ldots$)
- Finally, we will omit the sort symbol in equalities and quantifiers when it is clear from the context or not important:

  Example: $\forall x_1 .\, \forall y_1 .\, x_1 \doteq x_2$ instead of $\forall x{:}\sigma_1 .\, \forall x_2{:}\sigma_2 .\, x_1 \doteq x_2$
- We may also omit parentheses by defining *precedence*:
  - Same precedence for propositional connectives as in propositional logic
  - Quantifiers have the highest precedence after $\neg$

    Example: $\neg\forall x.\, (p\, x) \wedge (q\, x)$ abbreviates $(\neg(\forall x.\, (p\, x))) \wedge (q\, x))$
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  Example: $\forall x.\, p(r(x)) \wedge q(x)$ instead of $\forall x.\, (p\, (r\, x)) \wedge (q\, x)$

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Free and Bound Variables

A variable x may occur free in a Σ-formula α or not

We formalize that by defining inductively the *set $\mathcal{FV}$ of free variables* of α

$$\mathcal{FV}(\alpha) := \begin{cases} \{\, x \mid x \text{ is a var in } \alpha \,\} & \text{if } \alpha \text{ is atomic} \\ \mathcal{FV}(\beta) & \text{if } \alpha = \neg\beta \\ \mathcal{FV}(\beta) \cup \mathcal{FV}(\gamma) & \text{if } \alpha = \beta \bowtie \gamma \text{ with } \bowtie \in \{\, \wedge, \vee, \Rightarrow, \Leftrightarrow \,\} \\ \mathcal{FV}(\beta) \setminus \{\, v \,\} & \text{if } \alpha = Q\, v : \sigma.\ \beta \text{ with } Q \in \{\, \forall, \exists \,\} \end{cases}$$

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Examples: Let x, y, z be variables

- $\mathcal{FV}(x) = \{\, x \,\}$ (provided x has sort Bool)
- $\mathcal{FV}(x < S(0) + y) = \{\, x, y \,\}$
- $\mathcal{FV}(x < S(0) + y \ \wedge\ x \doteq z) = \mathcal{FV}(x < S(0) + y) \cup \mathcal{FV}(x \doteq z) = \{\, x, y \,\} \cup \{\, x, z \,\} = \{\, x, y, z \,\}$
- $\mathcal{FV}(\forall x : \text{Nat}.\, x < S(0) + y) = \mathcal{FV}(x < S(0) + y) \setminus \{\, x \,\} = \{\, x, y \,\} \setminus \{\, x \,\} = \{\, y \,\}$

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A variable x *occurs free* in a Σ-formula α if $x \in \mathcal{FV}(\alpha)$

For $\alpha = Q\, v : \sigma.\ \beta$, we say that v is *bound* in α

The *scope* of x in α is the subformula β

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A Σ-formula α is *closed*, or is a (Σ-)*sentence*, if $\mathcal{FV}(\alpha) = \emptyset$

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This can be confusing, so we typically rename the bound variables of a formula so that they are distinct from its free variables (e.g., $x < x \Rightarrow \forall y : \mathrm{Nat}.\ 0 < y$)

FOL Semantics

Recall: The syntax of a first-order language is defined wrt a *signature* $\Sigma := \langle \Sigma^S, \Sigma^F \rangle$ where:

- Σ^S is a set of *sorts*
- Σ^F is a set of *function symbols*

In propositional logic, the truth of a formula depends on the meaning of its variables

In first-order logic, the truth of a Σ-formula depends on:

1. the meaning of each sort symbol σ
2. the meaning of each function symbol f
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Semantics

Let α be a Σ-formula and let Γ be a sorting context that includes α's free variables

The truth of α is determined by *interpretations* $\mathcal{I}$ of Σ and Γ consisting of:

1. an interpretation $\sigma^{\mathcal{I}}$ of each $\sigma \in \Sigma^S$ as a nonempty set, the *domain* of σ
2. an interpretation $f^{\mathcal{I}}$ of each $f \in \Sigma^F$ of rank $\langle \sigma_1, \ldots, \sigma_n, \sigma_{n+1} \rangle$ as a total n-ary function from $\sigma_1^{\mathcal{I}} \times \cdots \times \sigma_n^{\mathcal{I}}$ to $\sigma_{n+1}^{\mathcal{I}}$
3. an interpretation $x^{\mathcal{I}}$ of each $x : \sigma \in \Gamma$ as an element of $\sigma^{\mathcal{I}}$

Note: We consider only interpretations $\mathcal{I}$ such that

- $\mathsf{Bool}^{\mathcal{I}} = \{ \mathsf{true}, \mathsf{false} \}$, $\quad \bot^{\mathcal{I}} = \mathsf{false}$, $\quad \top^{\mathcal{I}} = \mathsf{true}$
- for all $\sigma \in \Sigma^S$, $=_{\sigma}^{\mathcal{I}}$ maps its two arguments to true iff they are identical

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Semantics: Example

Consider a signature $\Sigma = \langle \Sigma^S, \Sigma^F \rangle$ for a fragment of set theory with non-set elements:

$\Sigma^S = \{\mathsf{Elem}, \mathsf{Set}\}$, $\Sigma^F = \{\emptyset, \sqsubseteq\}$, $\mathrm{rank}(\emptyset) = \langle \mathsf{Set} \rangle$, $\mathrm{rank}(\sqsubseteq) = \langle \mathsf{Elem}, \mathsf{Set}, \mathsf{Bool} \rangle$

$\Gamma = \{\, e_i : \mathsf{Elem} \mid i \geq 0 \,\} \cup \{\, s_i : \mathsf{Set} \mid i \geq 0 \,\}$

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A possible interpretation $\mathcal{I}$ of Σ, Γ:

1. $\mathsf{Elem}^{\mathcal{I}} = \mathbb{N}$, the natural numbers
2. $\mathsf{Set}^{\mathcal{I}} = 2^{\mathbb{N}}$, all sets of natural numbers
3. $\emptyset^{\mathcal{I}} = \{\}$
4. for all $n \in \mathbb{N}$ and $s \subseteq \mathbb{N}$, $\quad \sqsubseteq^{\mathcal{I}}(n, s) = \mathsf{true}$ iff $n \in s$
5. for $i = 0, 1, \ldots,$ $\quad e_i{}^{\mathcal{I}} = i$ and $s_i{}^{\mathcal{I}} = [0, i] = \{\, 0, 1, \ldots, i \,\}$

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Another interpretation $\mathcal{I}$ of Σ, Γ:

1. $\mathsf{Elem}^{\mathcal{I}} = \mathsf{Set}^{\mathcal{I}} = \mathbb{N}$, the natural numbers
2. $\emptyset^{\mathcal{I}} = 0$
3. for all $m, n \in \mathbb{N}$, $\sqsubseteq^{\mathcal{I}}(m, n) = \mathsf{true}$ iff m is divisible by n
4. for $i = 0, 1, \ldots$, $e_i^{\mathcal{I}} = i$ and $s_i^{\mathcal{I}} = 2$

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There is an infinity of interpretations of Σ, Γ !

Term Semantics

Interpretations are analogous to a variable assignments in propositional logic

We define how to determine the truth value of a Σ-formula in an interpretation $\mathcal{I}$ in FOL in analogy to how to determine the truth value of a formula under a variable assignment v in PL

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The first step is to extend $\mathcal{I}$ by structural induction to an interpretation $\overline{\mathcal{I}}$ for well-sorted terms

$$t^{\overline{\mathcal{I}}} = \begin{cases} t^{\mathcal{I}} & \text{if } t \text{ is a constant of } \Sigma \text{ or a variable} \\ f^{\mathcal{I}}(t_1^{\overline{\mathcal{I}}}, \ldots, t_n^{\overline{\mathcal{I}}}) & \text{if } t = (f\ t_1\ \cdots\ t_n) \end{cases}$$

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Example:

$\Sigma^S = \{ \text{Pers} \}$, $\Sigma^f = \{ \text{pa}, \text{ma}, \text{mar} \}$, $\Gamma = \{ x{:}\text{Pers}, y{:}\text{Pers}, \ldots \}$,
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Consider $\mathcal{I}$ such that

$\mathsf{ma}^{\mathcal{I}} = \{\, \mathrm{Jim} \mapsto \mathrm{Jill},\ \mathrm{Joe} \mapsto \mathrm{Jen},\ \ldots \}$, $\mathsf{pa}^{\mathcal{I}} = \{\, \mathrm{Jim} \mapsto \mathrm{Joe},\ \mathrm{Jill} \mapsto \mathrm{Jay},\ \ldots \}$,
$\mathsf{mar}^{\mathcal{I}} = \{\, (\mathrm{Jill}, \mathrm{Joe}) \mapsto \mathsf{true},\ (\mathrm{Joe}, \mathrm{Jill}) \mapsto \mathsf{true},\ (\mathrm{Jill}, \mathrm{Jill}) \mapsto \mathsf{false},\ \ldots \}$, $x^{\mathcal{I}} = \mathrm{Jim}$, $y^{\mathcal{I}} = \mathrm{Joe}$

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$$\begin{aligned}(\mathsf{pa}\ (\mathsf{ma}\ x))^{\overline{\mathcal{I}}} &= \mathsf{pa}^{\mathcal{I}}((\mathsf{ma}\ x)^{\overline{\mathcal{I}}}) = \mathsf{pa}^{\mathcal{I}}(\mathsf{ma}^{\mathcal{I}}(x^{\overline{\mathcal{I}}})) = \mathsf{pa}^{\mathcal{I}}(\mathsf{ma}^{\mathcal{I}}(x^{\mathcal{I}})) \\ &= \mathsf{pa}^{\mathcal{I}}(\mathsf{ma}^{\mathcal{I}}(\mathrm{Jim})) = \mathsf{pa}^{\mathcal{I}}(\mathrm{Jill}) = \mathrm{Jay}\end{aligned}$$

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$\mathsf{ma}^{\mathcal{I}} = \{ \mathrm{Jim} \mapsto \mathrm{Jill},\ \mathrm{Joe} \mapsto \mathrm{Jen},\ \ldots \}$, $\mathsf{pa}^{\mathcal{I}} = \{ \mathrm{Jim} \mapsto \mathrm{Joe},\ \mathrm{Jill} \mapsto \mathrm{Jay},\ \ldots \}$,
$\mathsf{mar}^{\mathcal{I}} = \{ (\mathrm{Jill},\ \mathrm{Joe}) \mapsto \mathsf{true},\ (\mathrm{Joe},\ \mathrm{Jill}) \mapsto \mathsf{true},\ (\mathrm{Jill},\ \mathrm{Jill}) \mapsto \mathsf{false},\ \ldots \}$, $x^{\mathcal{I}} = \mathrm{Jim}$, $y^{\mathcal{I}} = \mathrm{Joe}$

$$\begin{aligned} (\mathsf{mar}\ (\mathsf{ma}\ x)\ y)^{\overline{\mathcal{I}}} &= \mathsf{mar}^{\mathcal{I}}((\mathsf{ma}\ x)^{\overline{\mathcal{I}}},\ y^{\overline{\mathcal{I}}}) = \mathsf{mar}^{\mathcal{I}}(\mathsf{ma}^{\mathcal{I}}(x^{\overline{\mathcal{I}}}),\ y^{\mathcal{I}}) = \mathsf{mar}^{\mathcal{I}}(\mathsf{ma}^{\mathcal{I}}(x^{\mathcal{I}}),\ \mathrm{Joe}) \\ &= \mathsf{mar}^{\mathcal{I}}(\mathsf{ma}^{\mathcal{I}}(\mathrm{Jim}),\ \mathrm{Joe}) = \mathsf{mar}^{\mathcal{I}}(\mathrm{Jill},\ \mathrm{Joe}) = \mathsf{true} \end{aligned}$$

Formula Semantics

We further extend $\overline{\mathcal{I}}$ to well-sorted non-atomic formulas by structural induction as follows:

- $(\neg\alpha)^{\overline{\mathcal{I}}} = \text{true}$ iff $\alpha^{\overline{\mathcal{I}}} = \text{false}$
- $(\alpha \wedge \beta)^{\overline{\mathcal{I}}} = \text{true}$ iff $\alpha^{\overline{\mathcal{I}}} = \beta^{\overline{\mathcal{I}}} = \text{true}$
- $(\alpha \vee \beta)^{\overline{\mathcal{I}}} = \text{true}$ iff $\alpha^{\overline{\mathcal{I}}} = \text{true}$ or $\beta^{\overline{\mathcal{I}}} = \text{true}$
- $(\alpha \Rightarrow \beta)^{\overline{\mathcal{I}}} = \text{true}$ iff $\alpha^{\overline{\mathcal{I}}} = \text{false}$ or $\beta^{\overline{\mathcal{I}}} = \text{true}$
- $(\alpha \Leftrightarrow \beta)^{\overline{\mathcal{I}}} = \text{true}$ iff $\alpha^{\overline{\mathcal{I}}} = \beta^{\overline{\mathcal{I}}}$
- $(\exists x : \sigma.\, \alpha)^{\overline{\mathcal{I}}} = \text{true}$ iff $\alpha^{\overline{\mathcal{I}}[x \mapsto a]} = \text{true}$ for some $a \in \sigma^{\mathcal{I}}$
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where $\overline{\mathcal{I}}[x \mapsto a]$ denotes the interpretation that maps x to a and is otherwise identical to $\overline{\mathcal{I}}$

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Entailment, validity

Let Φ be a set of Σ-formulas. We write $\mathcal{I} \models \Phi$ to mean that $\mathcal{I} \models \alpha$ for every $\alpha \in \Phi$

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Exercise Suppose that $\Sigma^S = \{ \mathsf{A} \}$, $\Sigma^F = \{ \mathsf{p}, \mathsf{q} \}$, $\mathrm{rank}(\mathsf{p}) = \langle \mathsf{A}, \mathsf{Bool} \rangle$, $\mathrm{rank}(\mathsf{q}) = \langle \mathsf{A}, \mathsf{A}, \mathsf{Bool} \rangle$, and all variables v_i have sort A. Do the following entailment actually hold?

1. $\forall v_1 . \, \mathsf{p}(v_1) \models \mathsf{p}(v_2)$ ✓
2. $\mathsf{p}(v_1) \models \forall v_1 . \, \mathsf{p}(v_1)$ ✗
3. $\forall v_1 . \, \mathsf{p}(v_1) \models \exists v_2 . \, \mathsf{p}(v_2)$ ✓
4. $\exists v_2 . \, \forall v_1 . \, \mathsf{q}(v_1, v_2) \models \forall v_1 . \, \exists v_2 . \, \mathsf{q}(v_1, v_2)$ ✓
5. $\forall v_1 . \, \exists v_2 . \, \mathsf{q}(v_1, v_2) \models \exists v_2 . \, \forall v_1 . \, \mathsf{q}(v_1, v_2)$ ✗
6. $\models \exists v_1 . \, (\mathsf{p}(v_1) \Rightarrow \forall v_2 . \, \mathsf{p}(v_2))$ ✓

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3. $\forall v_1 .\, \mathsf{p}(v_1) \models \exists v_2 .\, \mathsf{p}(v_2)$ ✓
4. $\exists v_2 .\, \forall v_1 .\, \mathsf{q}(v_1, v_2) \models \forall v_1 .\, \exists v_2 .\, \mathsf{q}(v_1, v_2)$ ✓
5. $\forall v_1 .\, \exists v_2 .\, \mathsf{q}(v_1, v_2) \models \exists v_2 .\, \forall v_1 .\, \mathsf{q}(v_1, v_2)$ ✗
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Exercise Suppose that $\Sigma^S = \{ A \}$, $\Sigma^F = \{ p, q \}$, $\mathrm{rank}(p) = \langle A, \mathrm{Bool} \rangle$, $\mathrm{rank}(q) = \langle A, A, \mathrm{Bool} \rangle$, and all variables v_i have sort A. Do the following entailment actually hold?

1. $\forall v_1 . p(v_1) \models p(v_2)$ ✓
2. $p(v_1) \models \forall v_1 . p(v_1)$ ✗
3. $\forall v_1 . p(v_1) \models \exists v_2 . p(v_2)$ ✓
4. $\exists v_2 . \forall v_1 . q(v_1, v_2) \models \forall v_1 . \exists v_2 . q(v_1, v_2)$ ✓
5. $\forall v_1 . \exists v_2 . q(v_1, v_2) \models \exists v_2 . \forall v_1 . q(v_1, v_2)$ ✗
6. $\models \exists v_1 . (p(v_1) \Rightarrow \forall v_2 . p(v_2))$ ✓

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4. $\exists\, v_2 .\, \forall\, v_1 .\, \mathsf{q}(v_1, v_2) \models \forall\, v_1 .\, \exists\, v_2 .\, \mathsf{q}(v_1, v_2)$ ✓
5. $\forall\, v_1 .\, \exists\, v_2 .\, \mathsf{q}(v_1, v_2) \models \exists\, v_2 .\, \forall\, v_1 .\, \mathsf{q}(v_1, v_2)$ ✗
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3. $\forall\, v_1 .\, \mathsf{p}(v_1) \models \exists\, v_2 .\, \mathsf{p}(v_2)$ ✓
4. $\exists\, v_2 .\, \forall\, v_1 .\, \mathsf{q}(v_1, v_2) \models \forall\, v_1 .\, \exists\, v_2 .\, \mathsf{q}(v_1, v_2)$ ✓
5. $\forall\, v_1 .\, \exists\, v_2 .\, \mathsf{q}(v_1, v_2) \models \exists\, v_2 .\, \forall\, v_1 .\, \mathsf{q}(v_1, v_2)$ ✗
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Exercise

Let α be a Σ-formula and let Γ be a sorting context that includes α's free variables

The truth of α is determined by *interpretations* $\mathcal{I}$ of Σ and Γ consisting of:

1. an interpretation $\sigma^{\mathcal{I}}$ of each $\sigma \in \Sigma^S$ as a nonempty set, the *domain* of σ
2. an interpretation $f^{\mathcal{I}}$ of each $f \in \Sigma^F$ of rank $\langle \sigma_1, \ldots, \sigma_n, \sigma_{n+1} \rangle$ as a total n-ary function from $\sigma_1^{\mathcal{I}} \times \cdots \times \sigma_n^{\mathcal{I}}$ to $\sigma_{n+1}^{\mathcal{I}}$
3. an interpretation $x^{\mathcal{I}}$ of each $x : \sigma \in \Gamma$ as an element of $\sigma^{\mathcal{I}}$

Consider the signature where

$$\Sigma^S = \{ \sigma \}, \Sigma^F = \{ Q, \doteq_\sigma \}, \Gamma = \{ x : \sigma, y : \sigma \}, \mathrm{rank}(Q) = \langle \sigma, \sigma, \mathsf{Bool} \rangle$$

For each of the following Σ-formulas, describe an interpretation that satisfies it

1. $\forall x{:}\sigma.\, \forall y{:}\sigma.\, x \doteq y$
2. $\forall x{:}\sigma.\, \forall y{:}\sigma.\, Q(x, y)$
3. $\forall x{:}\sigma.\, \exists y{:}\sigma.\, Q(x, y)$

From English to FOL: Examples 1

1. There is a natural number that is smaller than any other natural number
   $\exists x{:}\mathrm{Nat}.\ \forall y{:}\mathrm{Nat}.\ (x \doteq y \lor x < y)$
2. For every natural number there is a greater one $\forall x{:}\mathrm{Nat}.\ \exists y{:}\mathrm{Nat}.\ x < y$
3. Two natural numbers are equal only if their respective successors are equal
   $\forall x{:}\mathrm{Nat}.\ \forall y{:}\mathrm{Nat}.\ (x \doteq y \Rightarrow S(x) \doteq S(y))$
4. Two natural numbers are equal if their respective successors are equal
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5. No two distinct natural numbers have the same successor
   $\forall x{:}\mathrm{Nat}.\ \forall y{:}\mathrm{Nat}.\ (\neg(x \doteq y) \Rightarrow \neg(S(x) \doteq S(y)))$
6. There are at least two natural numbers smaller than 3
   $\exists x{:}\mathrm{Nat}.\ \exists y{:}\mathrm{Nat}.\ (\neg(x \doteq y) \land (x < S(S(S(0)))) \land (y < S(S(S(0)))))$
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3. Two natural numbers are equal only if their respective successors are equal
   $\forall x{:}\mathrm{Nat}.\ \forall y{:}\mathrm{Nat}.\ (x \doteq y \Rightarrow \mathrm{S}(x) \doteq \mathrm{S}(y))$
4. Two natural numbers are equal if their respective successors are equal
   $\forall x{:}\mathrm{Nat}.\ \forall y{:}\mathrm{Nat}.\ (\mathrm{S}(x) \doteq \mathrm{S}(y) \Rightarrow x \doteq y)$
5. No two distinct natural numbers have the same successor
   $\forall x{:}\mathrm{Nat}.\ \forall y{:}\mathrm{Nat}.\ (\neg(x \doteq y) \Rightarrow \neg(\mathrm{S}(x) \doteq \mathrm{S}(y)))$
6. There are at least two natural numbers smaller than 3
   $\exists x{:}\mathrm{Nat}.\ \exists y{:}\mathrm{Nat}.\ (\neg(x \doteq y) \land (x < \mathrm{S}(\mathrm{S}(\mathrm{S}(0)))) \land (y < \mathrm{S}(\mathrm{S}(\mathrm{S}(0)))))$
7. There is no largest natural number $\neg\exists x{:}\mathrm{Nat}.\ \forall y{:}\mathrm{Nat}.\ (y \doteq x \lor y < x)$

From English to FOL: Examples 1

1. There is a natural number that is smaller than any other natural number
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From English to FOL: Examples 2

1. Everyone has a father and a mother $\forall x{:}\mathrm{Pers}.\ \exists y{:}\mathrm{Pers}.\ \exists z{:}\mathrm{Pers}.\ (y \doteq \mathrm{pa}(x) \wedge z \doteq \mathrm{ma}(x))$
2. The married relation is symmetric $\forall x{:}\mathrm{Pers}.\ \forall y{:}\mathrm{Pers}.\ (\mathrm{mar}(x, y) \Rightarrow \mathrm{mar}(y, x))$
3. No one can be married to themselves $\forall x{:}\mathrm{Pers}.\ \neg\mathrm{mar}(x, x)$
4. Not all people are married $\neg\forall x{:}\mathrm{Pers}.\ \exists y{:}\mathrm{Pers}.\ \mathrm{mar}(x, y)$
5. Some people have a father and a mother who are not married to each other
   $\exists x{:}\mathrm{Pers}.\ \neg\mathrm{mar}(\mathrm{ma}(x), \mathrm{pa}(x))$
6. You cannot marry more than one person
   $\forall x{:}\mathrm{Pers}.\ \forall y{:}\mathrm{Pers}.\ \forall z{:}\mathrm{Pers}.\ (\mathrm{mar}(x, y) \wedge \mathrm{mar}(x, z) \Rightarrow y \doteq z)$
7. Some people are not mothers $\exists x{:}\mathrm{Pers}.\forall y{:}\mathrm{Pers}.\ \neg(x \doteq \mathrm{ma}(y))$
8. Nobody can be both a father and a mother $\forall x{:}\mathrm{Pers}.\ \neg\exists y{:}\mathrm{Pers}.\ \neg\exists z{:}\mathrm{Pers}.\ (x \doteq \mathrm{pa}(y) \wedge z \doteq \mathrm{ma}(z))$
9. You can't be your own father or father's father $\forall x{:}\mathrm{Pers}.\ \neg(x \doteq \mathrm{pa}(x) \vee x \doteq \mathrm{pa}(\mathrm{pa}(x)))$
10. Some people are childless $\exists x{:}\mathrm{Pers}.\ \forall y{:}\mathrm{Pers}.\ (\neg(x \doteq \mathrm{pa}(y)) \wedge \wedge(x \doteq \mathrm{ma}(y)))$

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7. Some people are not mothers $\exists x{:}\mathrm{Pers}.\forall y{:}\mathrm{Pers}.\ \neg(x \doteq \mathrm{ma}(y))$
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7. Some people are not mothers $\exists x{:}\mathrm{Pers}.\forall y{:}\mathrm{Pers}.\ \neg(x \doteq \mathrm{ma}(y))$
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2. The married relation is symmetric $\forall x{:}\mathrm{Pers}.\ \forall y{:}\mathrm{Pers}.\ (\mathrm{mar}(x,y) \Rightarrow \mathrm{mar}(y,x))$
3. No one can be married to themselves $\forall x{:}\mathrm{Pers}.\ \neg\mathrm{mar}(x,x)$
4. Not all people are married $\neg\forall x{:}\mathrm{Pers}.\ \exists y{:}\mathrm{Pers}.\ \mathrm{mar}(x,y)$
5. Some people have a father and a mother who are not married to each other
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Invariance of term values

Consider a signature Σ, a Σ-context Γ, and two Σ-interpretations $\mathcal{I}$ and $\mathcal{J}$ that agree on the sorts and symbols of Σ.

Lemma 1
If $\mathcal{I}$ and $\mathcal{J}$ also agree on the variables of a Σ-term t with variables in Γ, then $t^{\overline{\mathcal{I}}} = t^{\overline{\mathcal{J}}}$.

Proof.
By structural induction on t.

- If t is a variable or a constant, then $t^{\overline{\mathcal{I}}} = t^{\mathcal{I}}$, $t^{\overline{\mathcal{J}}} = t^{\mathcal{J}}$.

  Since $t^{\mathcal{I}} = t^{\mathcal{J}}$ by assumption, we have that $t^{\overline{\mathcal{I}}} = t^{\mathcal{I}} = t^{\mathcal{J}} = t^{\overline{\mathcal{J}}}$.

- If $t = (f\ t_1 \cdots t_n)$ with $n > 1$, then $f^{\mathcal{I}} = f^{\mathcal{J}}$ by assumption and $t_i^{\overline{\mathcal{I}}} = t_i^{\overline{\mathcal{J}}}$ for $i = 1, \ldots, n$ by induction hypothesis.

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Consider a signature Σ, a Σ-context Γ, and two Σ-interpretations $\mathcal{I}$ and $\mathcal{J}$ that agree on the sorts and symbols of Σ.

Theorem 2

If $\mathcal{I}$ and $\mathcal{J}$ also agree on the free variables of a Σ-formula α with free variables in Γ, then $\alpha^{\overline{\mathcal{I}}} = \alpha^{\overline{\mathcal{J}}}$.

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- If α is $\neg\beta$ or $\alpha_1 \bowtie \alpha_2$ with $\bowtie \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$, the result follows from the inductive hypothesis.

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Proof.

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- If $\alpha = Q\, s{:}\sigma.\ \beta$ with $Q \in \{ \forall, \exists \}$. Then $\mathcal{FV}(\beta) = \mathcal{FV}(\alpha) \cup \{ x \}$.

  For any d in $\sigma^{\mathcal{I}}$, $\mathcal{I}[x \mapsto d]$ and $\mathcal{J}[x \mapsto d]$ agree on x by construction and on $\mathcal{FV}(\alpha)$ by assumption. The result follows from the inductive hypothesis and the semantics of $\forall$ and $\exists$.

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Note: The theorem implies that the interpretation of formula α is independent from the values assigned to variables that do not occur free in α.

Invariance of truth values

Consider a signature Σ, a Σ-context Γ, and two Σ-interpretations $\mathcal{I}$ and $\mathcal{J}$ that agree on the sorts and symbols of Σ.

Theorem 2

If $\mathcal{I}$ and $\mathcal{J}$ also agree on the free variables of a Σ-formula α with free variables in Γ, then $\alpha^{\overline{\mathcal{I}}} = \alpha^{\overline{\mathcal{J}}}$.

Note: The theorem implies that the interpretation of formula α is independent from the values assigned to variables that do not occur free in α.

Corollary 3

The truth value of sentences is independent from how variables are interpreted.

The Deduction Theorem of FOL

Consider a signature Σ

Theorem 4

For all Σ-formulas α and β, we have that $\alpha \models \beta$ iff $\models \alpha \Rightarrow \beta$

Proof.

$\Rightarrow$) We argue that every Σ interpretation $\mathcal{I}$ satisfies $\gamma := \alpha \Rightarrow \beta$. If $\mathcal{I}$ falsifies α, then it trivially satisfies γ. If, instead, $\mathcal{I}$ satisfies α, then, since $\alpha \models \beta$, it must satisfy β as well. Hence, it satisfies γ.

$\Leftarrow$) We argue that every Σ-interpretation $\mathcal{I}$ that satisfies α satisfies β as well. Any such interpretation must indeed satisfy β; otherwise, it would falsify $\alpha \Rightarrow \beta$, against the assumption that $\models \alpha \Rightarrow \beta$. □

Corollary 5

For all Σ-formulas α and β, we have that $\alpha \equiv \beta$ iff $\models \alpha \Leftrightarrow \beta$

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The Free Variables Theorem 1

Consider a signature Σ and a Σ-context Γ

Let Φ be a set of Σ-formulas, let α be Σ-formula with free variables from Γ, and let $x \in \mathcal{FV}(\alpha)$ where $x : \sigma \in \Gamma$.

Theorem 6

Suppose x occurs free in no formulas of Φ. Then, $\Phi \models \alpha$ iff $\Phi \models \forall x{:}\sigma.\, \alpha$

Proof.

$\Rightarrow$) Let $\mathcal{I}$ be any interpretation that satisfies Φ. Since x does not occur free in any formula of Φ we can conclude that $\mathcal{I}[x \mapsto a] \models \Phi$ for all $a \in \sigma^{\mathcal{I}}$. Since $\Phi \models \alpha$, we have that $\mathcal{I}[x \mapsto a] \models \alpha$ for all $a \in \sigma^{\mathcal{I}}$. But then $\mathcal{I} \models \forall x{:}\sigma.\, \alpha$ by definition of $\forall$. Hence, every interpretation that satisfies Φ also satisfies $\forall x{:}\sigma.\, \alpha$, that is, $\Phi \models \forall x{:}\sigma.\, \alpha$.

$\Leftarrow$) Let $\mathcal{I}$ be any interpretation that satisfies Φ. By assumption $\mathcal{I} \models \forall x{:}\sigma.\, \alpha$. This implies that $\mathcal{I} \models \alpha$ regardless of what $x^{\mathcal{I}}$ is. Hence $\Phi \models \alpha$. $\square$

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The Free Variables Theorem 2

Consider a signature Σ and a Σ-context Γ

Let β be Σ-formula, let α be a Σ-formula with free variables from Γ, and let $x \in \mathcal{FV}(\alpha)$ where $x : \sigma \in \Gamma$.

Theorem 7

Suppose x does not occur free in β. Then, $\alpha \models \beta$ iff $\exists x{:}\sigma.\, \alpha \models \beta$

Proof.

$\Rightarrow$) Let $\mathcal{I}$ be any interpretation that satisfies $\exists x{:}\sigma.\, \alpha$. This means that $\mathcal{I}[x \mapsto a] \models \alpha$ for some $a \in \sigma^{\mathcal{I}}$. By assumption, $\mathcal{I}[x \mapsto a]$ satisfies β as well. Since x does not occur free in β, changing the value assigned to x does not matter. It follows that $\mathcal{I} \models \beta$. Since $\mathcal{I}$ was arbitrary, this shows that $\exists x{:}\sigma.\, \alpha \models \beta$.

$\Leftarrow$) Let $\mathcal{I}$ be any interpretation that satisfies α. Then, trivially, $\mathcal{I} \models \exists x{:}\sigma.\, \alpha$. By assumption, $\mathcal{I} \models \beta$. Since $\mathcal{I}$ was arbitrary, we can conclude that $\alpha \models \beta$. $\square$

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