CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

Propositional Logic Basics

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, **Emina Torlak** at the University of Washington, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Propositional Logic

- Syntax
- Semantics, Satisfiability, and Validity
- Proof by deduction

Automating Inference

Automated Reasoning tries to automated the process of *inference*:

deriving consequences of a given set of statements

In AR, both the given and the derived knowledge are expressed in a formal language

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Many (formal) logics have been developed and studied,
with various degrees of expressiveness and mechanizability

We will consider a couple in this course, starting with the most basic one:
Propositional Logic (PL)

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Defining features of formal logics

A *formal logic* is

- defined by its syntax and semantics
- equipped with one or more inference/proof systems

syntax: a set of symbols and rules for combining them to form sentences (formulas) of the logic

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Formalize natural language statements that can be either true or false (but not both)

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Basic sentences are called *atomic*

Examples:

1. $0 < 1$
2. Iowa City is in Iowa
3. $1 + 1 = 10$

Classical logics

Formalize natural language statements that can be either true or false (but not both)

More complex sentences are built from simpler ones via a small number of constructs

Examples:

1. If Iowa City is in Iowa then University Height is Iowa
2. $1 + 1 = 10$ or $1 + 1 \neq 10$

Truth of atomic sentences

Each proposition formalizes a statement that is either true or false

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- it is false, if we interpret 1 and 10 as integers in decimal notation (and $+$ as addition)
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Let α be a complex sentence built with a construct c from simpler sentences $\alpha_1, \ldots, \alpha_n$

The truth value of α is uniquely determined by

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$$1 + 1 = 5 \text{ or } 1 + 1 \neq 5$$

is true if at least one of $1 + 1 = 5$, $1 + 1 \neq 5$ is true

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Propositional Logic Syntax: symbols

The set of symbols, or *alphabet*, of propositional logic consists of

1. a set $\mathcal{B}$ of *atomic symbols* or *atoms*:
   - truth constants: $\top$ (for true), $\bot$ (for false)
   - propositional variables: $p, q, r, \ldots$
2. *logical symbols:* connectives (i.e., $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$), parentheses (i.e., (,))

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Note: We will use the same characters: '(' and ')' at three levels of discourse:

1. as part of propositional logic formulas, as in $(p \Rightarrow q)$
2. in mathematical notation, as in $f(x)$, $\log(a)$
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Do not confuse the three!

Propositional Logic Syntax: expressions

A sentence, or formula, is a finite sequence of symbols

- $(p \wedge q)$
- $((\neg p) \Rightarrow r)$

Not all sequences of symbols are formulas:

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- pq
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Propositional Logic Syntax: Formula-building operations

Consider the *formula-building operators* defined as follows for all formulas α and β:

- $\mathcal{E}_{\neg}(\alpha) = (\neg\alpha)$ (negation)
- $\mathcal{E}_{\wedge}(\alpha, \beta) = (\alpha \wedge \beta)$ (conjunction)
- $\mathcal{E}_{\vee}(\alpha, \beta) = (\alpha \vee \beta)$ (disjunction)
- $\mathcal{E}_{\Rightarrow}(\alpha, \beta) = (\alpha \Rightarrow \beta)$ (implication)
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The set $\mathcal{W}$ of *well-formed formulas*, or simply *formulas* or *wffs*, is the set of all sentences finitely-generated by the operators above from the atoms in $\mathcal{B}$

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In other words,

- every atom in $\mathcal{B}$ is a wff
- if α and β are wffs,
  so are the expressions generated from them by $\mathcal{E}_{\neg}$, $\mathcal{E}_{\wedge}$, $\mathcal{E}_{\vee}$, $\mathcal{E}_{\Rightarrow}$, and $\mathcal{E}_{\Leftrightarrow}$
- nothing else is a wff

Closed sets and generated sets

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Examples

- The set $\mathbb{N}$ of all natural numbers is closed under addition and multiplication but not negation
- The set $\mathbb{Z}$ of all integer numbers is closed under addition, multiplication, and negation
- The set $\mathbb{E}$ of all even integers is closed under addition, multiplication, and negation
- The set $\mathbb{O}$ of all odd integers is closed under multiplication and negation but not addition

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Examples

- The set $\mathbb{N}$ of all natural numbers is generated from $\{0, 1\}$ by $\{+\}$
- The set $\mathbb{Z}$ of all integer numbers is generated from $\{1\}$ by $\{+, -\}$
- The set $\mathbb{E}$ of all even integers is generated from $\{2\}$ by $\{+, -\}$
- The set $\mathbb{R}$ of all real number is generated from no sets of numbers[a]

[a] Generated sets are necessarily countable.

The Structural Induction Principle

Consider a set C generated from a set B by a set F of operators

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Example $\mathbb{Z}$ is inductive w.r.t. $\mathbb{N}$ (which is generated from $\{0, 1\}$ by $\{+\}$)

The Structural Induction Principle

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Note: S inductive w.r.t. C implies that $C \subseteq S$

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The argument goes like this:

1. Consider a set S whose elements all have property P
2. Show that S is inductive with respect to C

This proves that $C \subseteq S$ and thus all elements of C have property P

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We often use structural induction to prove properties about formulas

Structural Induction: Example

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Proof

Let $l(\alpha)$ be the number of left parentheses and
let $r(\alpha)$ be the number of right parentheses in an expression α

Let S be the set of all expressions α such that $l(\alpha) = r(\alpha)$

We wish to show that $\mathcal{W} \subseteq S$

This follows from the induction principle if we can show that S is inductive w.r.t. $\mathcal{W}$

Structural Induction: Example (cont.)

Base Case:

We must show that $\mathcal{B} \subseteq S$

Recall that $\mathcal{B}$ is the set of expressions consisting of a single propositional symbol

It is clear that for such expressions, $l(\alpha) = r(\alpha) = 0$

Structural Induction: Example (cont.)

Inductive Case:

We must show that S is closed under each formula-building operator

- $\mathcal{E}_\neg$
  Let $\alpha \in S$. We know that $\mathcal{E}_\neg(\alpha) = (\neg\alpha)$.
  It follows that $l(\mathcal{E}_\neg(\alpha)) = 1 + l(\alpha)$ and $r(\mathcal{E}_\neg(\alpha)) = 1 + r(\alpha)$.
  Since $\alpha \in S$, we know that $l(\alpha) = r(\alpha)$; it follows that $l(\mathcal{E}_\neg(\alpha)) = r(\mathcal{E}_\neg(\alpha))$, and thus $\mathcal{E}_\neg(\alpha) \in S$.

- $\mathcal{E}_\wedge$
  Let $\alpha, \beta \in S$. We know that $\mathcal{E}_\wedge(\alpha, \beta) = (\alpha \wedge \beta)$.
  Thus $l(\mathcal{E}_\wedge(\alpha, \beta)) = 1 + l(\alpha) + l(\beta)$ and $r(\mathcal{E}_\wedge(\alpha, \beta)) = 1 + r(\alpha) + r(\beta)$.
  As before, it follows from the inductive hypothesis that $\mathcal{E}_\wedge(\alpha, \beta) \in S$.

- The arguments for $\mathcal{E}_\vee$, $\mathcal{E}_\rightarrow$, and $\mathcal{E}_\leftrightarrow$ are analogous to the one for $\mathcal{E}_\wedge$.

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Notational conventions for formulas

- We fix a countably infinite set of propositional variables
  We typically use $p, q, r, p_1, p_2, p_3, \ldots$ to denote them

- We may omit outermost parentheses, e.g., write $p \wedge q$ instead of $(p \wedge q)$

- We may further omit parentheses by defining *order of operations (precedence)*:
  - Negation binds most strongly, with small as possible scope: $\neg p \wedge q$ means $((\neg p) \wedge q)$
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- We use $\alpha,\ \beta,\ \gamma,\ \varphi,\ \psi$ to denote arbitrary wffs

Propositional Logic: Compositional Semantics

The meaning of a wff α is a truth value: true or false

Given a mapping v from the propositional variables in α to $\{$ false, true $\}$,
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Let v be a variable assignment for all the propositional variables of $\mathcal{B}$

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For every $\alpha \in \mathcal{W}$, we will use the following statements interchangeably

- $v \models \alpha$
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Satisfiability of formulas

A wff α is *satisfiable*
if $\overline{v}(\alpha) = \text{true}$ for some interpretation v

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A wff α is *unsatisfiable*
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A set $U \subseteq \mathcal{W}$ is *(un)satisfiable*
if there is (no) interpretation v such that $\overline{v}(\alpha) = \text{true}$ for all $\alpha \in U$

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A set $U \subseteq \mathcal{W}$ *entails* or *logically implies* a wff β, written $U \models \beta$, if every satisfying assignment v for U satisfies β as well

We also say that U *entails* β and β is a *logical consequence* of U

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- If $\emptyset \models \alpha$, α is a *tautology* or is *valid* and write $\models \alpha$
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Note: We use $\models$ for two different relations:

1. satisfaction between a variable assignment and a formula ($\overline{v} \models \alpha$)
2. entailment between a set of formulas and a formula ($\{\alpha_1, \alpha_2, \ldots\} \models \alpha$)

Use context to disambiguate!

Satisfiability vs. validity

Satisfiability and validity are dual concepts:

a wff α is valid iff $\neg\alpha$ is unsatisfiable

Consequence:

If we have a procedure that can check satisfiability, then we can also check validity, and vice versa

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Examples

p, q propositional variables α, β, γ formulas

- $p,\ p \Rightarrow q,\ p \vee \neg q,\ (p \Rightarrow q) \Rightarrow p$ are all satisfiable
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- $\top$ is valid and $\bot$ is unsatisfiable
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Implication ($\Rightarrow$) vs. logical implication ($\models$)

The two concepts are semantically related:

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Proof: Exercise

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because

$$\alpha \equiv \beta \quad \text{iff} \quad \alpha \models \beta \text{ and } \beta \models \alpha$$

and

$$\models \alpha \Leftrightarrow \beta \quad \text{iff} \quad \models \alpha \Rightarrow \beta \text{ and } \models \beta \Rightarrow \alpha$$

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Note: $\alpha \models \beta$ and $\alpha \equiv \beta$ are mathematical statements, *not formulas*

Defining One Operator in Terms of Another

A binary connective $\circ$ over wffs is *defined from* a set of connectives *C*
if for all wffs α and β, $\alpha \circ \beta \equiv \gamma$,
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Example: defining $\vee, \wedge, \Leftrightarrow$ from $\{\neg, \Rightarrow\}$

- $\alpha \wedge \beta \;\equiv\; \neg(\alpha \Rightarrow \neg\beta)$
- $\alpha \vee \beta \;\equiv\; \neg\alpha \Rightarrow \beta$
- $\alpha \Leftrightarrow \beta \;\equiv\; (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha) \;\equiv\; \neg((\alpha \Rightarrow \beta) \Rightarrow \neg(\beta \Rightarrow \alpha))$

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The connectives $\vee, \wedge, \Rightarrow, \Leftrightarrow$ can be defined from $\neg$ and one of $\vee, \wedge, \Rightarrow, \Leftrightarrow$

Why do we care about this?

- To simplify arguments by structural induction
- Many algorithms are defined over normal forms using a specified subset of connectives

Decision Procedure in Propositional Logic

Let $U \in \mathcal{W}$

A *decision procedure* for U is a terminating procedure[1] that takes wffs as input and for each input α returns

yes if $\alpha \in U$ no if $\alpha \notin U$

This course: We consider decision procedures for validity/satisfiability, hence U will the set of valid/satisfiable formulas

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Basic Decision Procedures for Validity/Satisfiability

Two fundamental strategies for deciding validity/satisfiability:

- *Search-based procedures*:
  search the space of possible interpretations of the given wff
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SAT solvers (covered later) interleave search and deduction

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The Truth-table Method

In PL, it is possible to enumerate all the interpretations, e.g., with *truth tables*

Example: is $\alpha := (p \wedge q) \Rightarrow (p \vee \neg q)$ a valid formula?

Writing 0 for false and 1 for true, for conciseness:

p	q	$p \wedge q$	$\neg q$	$p \vee \neg q$	α
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Drawbacks?

- Need to evaluate a formula for each of 2^n possible interpretations
  This can be memory efficient but is runtime inefficient
- Works because the number of interpretations of a formula is finite

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Proof by deduction

Informally, a *proof system* consists of a set of *proof rules*

A proof rule consists of:

- *premises* (or antecedents): facts that must hold for the rule apply
- *conclusions* (or consequents): facts deduced/derived from applying the rule

$$\frac{P_1 \quad \cdots \quad P_n}{C_{1,1}, \ldots, C_{1,n_1} \;\Big|\; \cdots \;\Big|\; C_{m,1}, \ldots, C_{m,n_m}}$$

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Commas indicate derivation of multiple conclusions

Pipes indicate alternative conclusions (giving rise to *proof branches*)

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Examples:

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta} \qquad \frac{\alpha \qquad \alpha \Rightarrow \beta}{\beta} \qquad \frac{\alpha \Leftrightarrow \beta}{\alpha,\ \beta \mid \neg\alpha, \neg\beta}$$

Proof by deduction: semantic arguments

Premises and conclusions can be anything

including satisfiability assertions about some interpretation v

$$\frac{v \models \neg\alpha}{v \not\models \alpha}$$

$$\frac{v \not\models \neg\alpha}{v \models \alpha}$$

$$\frac{v \models \alpha \wedge \beta}{v \models \alpha,\ v \models \beta}$$

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To prove that a wff α is valid:

- Assume α is not valid, i.e., there is a interpretation v such that $v \not\models \alpha$
- Apply semantic arguments in the form of previous proof rules
- In the presence of multi-conclusion rules, proof evolves as a tree
  A proof tree branch is *closed* if it ends with $v \models \bot$, and is *open* otherwise
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Proof by deduction: example

Prove $\alpha = p \wedge \neg q$ is valid or find a falsifying interpretation

(a) $\dfrac{v \models \neg\alpha}{v \not\models \alpha}$

(b) $\dfrac{v \not\models \neg\alpha}{v \models \alpha}$

(c) $\dfrac{v \models \alpha \wedge \beta}{v \models \alpha, v \models \beta}$

(d) $\dfrac{v \not\models \alpha \wedge \beta}{v \not\models \alpha \mid v \not\models \beta}$

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1. $v \not\models p \wedge \neg q$ (assumption)
   - 1.1 $v \not\models p$ (by (d) on 1)
   - 1.2 $v \not\models \neg q$ (by (d) on 1)
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Falsifying interpretations v:

- Branch 1.1: $\{p \mapsto \text{false}, q \mapsto \text{true/false}\}$
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there is at least a v that falsifies α
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   - 1.2 $v \not\models \neg q$ (by (d) on 1)
     - 1.2.1 $v \models q$ (by (b) on 1.2)

Falsifying interpretations v:

- Branch 1.1:
  $\{p \mapsto \text{false}, q \mapsto \text{true/false}\}$
- Branch 1.2:
  $\{p \mapsto \text{true/false}, q \mapsto \text{true}\}$

there is at least a v that falsifies α
hence α is invalid

Proof by deduction: example

Prove $\alpha = p \wedge \neg q$ is valid or find a falsifying interpretation

(a) $\dfrac{v \models \neg\alpha}{v \not\models \alpha}$

(b) $\dfrac{v \not\models \neg\alpha}{v \models \alpha}$

(c) $\dfrac{v \models \alpha \wedge \beta}{v \models \alpha, v \models \beta}$

(d) $\dfrac{v \not\models \alpha \wedge \beta}{v \not\models \alpha \mid v \not\models \beta}$

(e) $\dfrac{v \models \alpha \vee \beta}{v \models \alpha \mid v \models \beta}$

(f) $\dfrac{v \not\models \alpha \vee \beta}{v \not\models \alpha, v \not\models \beta}$

(g) $\dfrac{v \models \alpha \Rightarrow \beta}{v \not\models \alpha \mid v \models \beta}$

(h) $\dfrac{v \not\models \alpha \Rightarrow \beta}{v \models \alpha, v \not\models \beta}$

(i) $\dfrac{v \models \alpha \quad v \not\models \alpha}{v \models \bot}$

(k) $\dfrac{v \models \alpha \Leftrightarrow \beta}{v \models \alpha, v \models \beta \mid v \not\models \alpha, v \not\models \beta}$

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- Branch 1.1: $\{p \mapsto \text{false}, q \mapsto \text{true}/\text{false}\}$
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Some useful tautologies

- Associative and Commutative laws
  - $\wedge$, $\vee$, and $\Leftrightarrow$
- Distributive laws
  - $\alpha \wedge (\beta \vee \gamma) \Leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
  - $\alpha \vee (\beta \wedge \gamma) \Leftrightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
- Negation
  - $\neg\neg\alpha \Leftrightarrow \alpha$
  - $\neg(\alpha \Rightarrow \beta) \Leftrightarrow (\alpha \wedge \neg\beta)$
  - $\neg(\alpha \Leftrightarrow \beta) \Leftrightarrow (\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$
- De Morgan's laws
  - $\neg(\alpha \wedge \beta) \Leftrightarrow (\neg\alpha \vee \neg\beta)$
  - $\neg(\alpha \vee \beta) \Leftrightarrow (\neg\alpha \wedge \neg\beta)$
- Implication
  - $(\alpha \Rightarrow \beta) \Leftrightarrow (\neg\alpha \vee \beta)$
- Excluded Middle
  - $\alpha \vee \neg\alpha$
- Contradiction
  - $\neg(\alpha \wedge \neg\alpha)$
- Contraposition
  - $(\alpha \Rightarrow \beta) \Leftrightarrow (\neg\beta \Rightarrow \neg\alpha)$
- Exportation
  - $((\alpha \wedge \beta) \Rightarrow \gamma) \Leftrightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma))$

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- Negatio
  - $\neg\neg$
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These tautologies can be proven with semantic arguments

Semantic arguments for satisfiability

The previous proof system was used to prove a formula is valid

It can also be used to prove that a formula α is unsatisfiable:

1. Again by contradiction, start with the assertion $\mathcal{I} \models \alpha$
2. Try to derive a proof tree T whose branches are all closed

Such a tree proves that α is unsatisfiable

If T has an open branch B where no (more) rules apply
then α is satisfiable with an interpretation $\mathcal{I}$ constructible from B

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Deductive systems

A *deductive system* $\mathscr{D}$ is a proof system equipped with a distinguished set of tautologies (*axioms*)

A *proof* in $\mathscr{D}$ for a wff α_n is a sequence of formulas $S = (\alpha_1, \ldots, \alpha_n)$ where each α_i is

- either an axiom
- or the result of an application of a rule of $\mathscr{D}$ to previous formulas in S

In that case, α_n is *provable* or a *theorem* in $\mathscr{D}$, written as $\vdash \alpha_n$

For $U \subseteq W$, we write $U \vdash \alpha$ to denote that α can be proved in $\mathscr{D}$ from the axioms and the formulas in U

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A *deductive system* $\mathscr{D}$ is a proof system equipped with a distinguished set of tautologies (*axioms*)

A *proof* in $\mathscr{D}$ for a wff α_n is a sequence of formulas $S = (\alpha_1, \ldots, \alpha_n)$ where each α_i is

- either an axiom
- or the result of an application of a rule of $\mathscr{D}$ to previous formulas in S

In that case, α_n is *provable* or a *theorem* in $\mathscr{D}$, written as $\vdash \alpha_i$

For $U \subseteq \mathcal{W}$, we write $U \vdash \alpha$ to denote that α can be proved in $\mathscr{D}$ from the axioms and the formulas in U

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Hilbert System $\mathscr{H}_2$

A consistent, sound and complete deductive system for propositional logic

Axiom schemas (α, β, γ are arbitrary wffs):

A1: $\vdash \alpha \Rightarrow (\beta \Rightarrow \alpha)$

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Proofs in $\mathscr{H}_2$

Proofs can be complicated, even for trivial formulas (or formula schemas)

Example: Prove $\varphi \Rightarrow \varphi$

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Proofs in $\mathscr{H}_2$

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Solution:

Introduce *derived* proof rules, additional rules whose conclusion can be proved from their premises using no derived proof rules

Derived Rules in $\mathscr{H}_2$

$$\frac{}{U \cup \{\alpha\} \vdash \alpha} \text{(assumption)}$$

$$\frac{U \vdash \neg\beta \Rightarrow \neg\alpha}{U \vdash \alpha \Rightarrow \beta} \text{(contrapositive)}$$

$$\frac{U \vdash \alpha \Rightarrow \beta \qquad U \vdash \beta \Rightarrow \gamma}{U \vdash \alpha \Rightarrow \gamma} \text{(transitivity)}$$

$$\frac{U \vdash \alpha \Rightarrow (\beta \Rightarrow \gamma)}{U \vdash \beta \Rightarrow (\alpha \Rightarrow \gamma)} \text{(exchange of antecedent)}$$

$$\frac{U \cup \{\alpha\} \vdash \beta}{U \vdash \alpha \Rightarrow \beta} \text{(deduction)}$$

$$\frac{U \vdash \neg\neg\alpha}{U \vdash \alpha} \text{(double negation 1)}$$

$$\frac{U \vdash \alpha}{U \vdash \neg\neg\alpha} \text{(double negation 2)}$$

$$\frac{U \vdash \neg\alpha \Rightarrow \bot}{U \vdash \alpha} \text{(reductio ad absurdum)}$$

Using derived rules in $\mathscr{H}_2$

With the deduction rule, the proof of $\alpha \Rightarrow \alpha$ becomes trivial

1. $\{ \alpha \} \vdash \alpha$ (by assumption)
2. $\vdash \alpha \Rightarrow \alpha$ (by deduction on 1)

This is because we front-load the proof burden in proving that the assumption and the deduction rule are derived rules

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$$\frac{}{U \vdash (\neg\beta \Rightarrow \neg\alpha) \Rightarrow (\alpha \Rightarrow \beta)} \text{(A3)}$$

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Soundness of rules in $\mathscr{H}_2$

Example 2: prove $(\varphi \Rightarrow \neg\varphi) \Rightarrow \neg\varphi$

1. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \neg\neg\varphi$ (assumption)
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5. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \varphi \Rightarrow (\neg\varphi \Rightarrow \bot)$ (Ex. 1)
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5. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \varphi \Rightarrow (\neg\varphi \Rightarrow \bot)$ (Ex. 1)
6. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \neg\varphi \Rightarrow \bot$ (MP 2, 5)
7. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \bot$ (MP 4, 6)
8. $\{\varphi \Rightarrow \neg\varphi\} \vdash \neg\neg\varphi \Rightarrow \bot$ (deduction 7)
9. $\{\varphi \Rightarrow \neg\varphi\} \vdash \neg\varphi$ (reductio ad absurdum 8)
10. $\vdash (\varphi \Rightarrow \neg\varphi) \Rightarrow \neg\varphi$ (deduction 9)

$$\frac{U \vdash \alpha \qquad U \vdash \alpha \Rightarrow \beta}{U \vdash \beta} \text{ (modus ponens)}$$

Soundness of rules in $\mathscr{H}_2$

Example 2: prove $(\varphi \Rightarrow \neg\varphi) \Rightarrow \neg\varphi$

1. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \neg\neg\varphi$ (assumption)
2. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \varphi$ (double negation 1)
3. $\{\varphi \Rightarrow \neg\varphi, \neg\neg\varphi\} \vdash \varphi \Rightarrow \neg\varphi$ (assumption)
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Soundness of rules in $\mathscr{H}_2$

A proof rule

$$\frac{U_1 \vdash \alpha_1 \quad \cdots \quad U_n \vdash \alpha_n}{V \vdash \beta}$$

is *sound* if $V \models \beta$ whenever $U_1 \models \alpha_1, \ldots, U_n \models \alpha_n$

Theorem: Axioms 1–3, modus ponens, and all the derived rules of $\mathscr{H}_2$ are sound

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Theorem: Axioms 1–3, modus ponens, and all the derived rules of $\mathscr{H}_2$ are sound

All rules of $\mathscr{H}_2$ are sound

$$\frac{\vdash \alpha \qquad \vdash \alpha \Rightarrow \beta}{\vdash \beta} \text{ (modus ponens)}$$

$$\frac{}{U \cup \{\alpha\} \vdash \alpha} \text{ (assumption)}$$

$$\frac{U \cup \{\alpha\} \vdash \beta}{U \vdash \alpha \Rightarrow \beta} \text{ (deduction)}$$

$$\frac{U \vdash \neg\beta \Rightarrow \neg\alpha}{U \vdash \alpha \Rightarrow \beta} \text{ (contrapositive)}$$

$$\frac{U \vdash \neg\neg\alpha}{U \vdash \alpha} \text{ (double negation 1)}$$

$$\frac{U \vdash \alpha \Rightarrow \beta \qquad U \vdash \beta \Rightarrow \gamma}{U \vdash \alpha \Rightarrow \gamma} \text{ (transitivity)}$$

$$\frac{U \vdash \alpha}{U \vdash \neg\neg\alpha} \text{ (double negation 2)}$$

$$\frac{U \vdash \alpha \Rightarrow (\beta \Rightarrow \gamma)}{U \vdash \beta \Rightarrow (\alpha \Rightarrow \gamma)} \text{ (exchange of antecedent)}$$

$$\frac{U \vdash \neg\alpha \Rightarrow \bot}{U \vdash \alpha} \text{ (reductio ad absurdum)}$$

Alternative proof systems

Another way to define a proof system is to

- include more logical connectives and
- have lots of proof rules

Proofs can be simpler to (manually) carry out in such a system

However, it becomes harder to prove properties about the proof system

Either way, Hilbert-style proof systems are difficult to automate

We will focus on proof systems more similar to semantic arguments

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