AN OVERVIEW OF SATISFIABILITY MODULO THEORIES AND ITS APPLICATIONS

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The University of Iowa

ACKNOWLEDGMENTS

Many thanks to **Clark Barrett**, **Alberto Griggio**, **Liana Hadarean**, **Dejan Jovanovic**, and **Albert Oliveras** for contributing some of the material used in these slides.

Disclaimer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a small and highly incomplete sample. Apologies to all authors whose work is not cited.

INTRODUCTION

Historically:

Automated logical reasoning achieved through uniform theorem-proving procedures for First Order Logic

(e.g., resolution, superposition, and tableaux calculi)

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Limited success:

Uniform proof producedure for FOL are not always the best compromise between expressiveness and efficiency

Last 20 years: R&D has focused on

- expressive enough decidable fragments of various logics
- incorporating domain-specific reasoning, e.g., on:
    - temporal reasoning
    - arithmetic reasoning
    - equality reasoning
    - reasoning about certain data structures (arrays, lists, finite sets, ...)
- combining specialized reasoners modularly

Two successful examples of this trend:

SAT: propositional formalization, Boolean reasoning

+ high degree of efficiency
– expressive (all NP-complete problems) but involved encodings

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- $-$ expressive (all NP-complete problems) but involved encodings

SMT: first-order formalization, Boolean + domain-specific reasoning

- $+$ improves expressivity and scalability
- $-$ some (but acceptable) loss of efficiency

This tutorial: an overview of SMT and its applications

Determining the satisfiability of a logical formula wrt some combination T of background theories

Example

$$n > 3 * m + 1 \;\wedge\; (\, f(n) \leq \; head\,(l_1) \;\vee\; l_2 = \; f(n) :: l_1 \,)$$

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THE BASIC SMT PROBLEM

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Linear Arithm.
(LIA)

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Equality (EUF)

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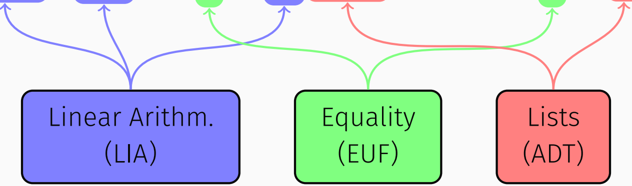

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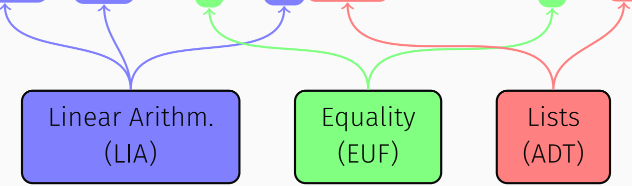

SMT formulas are formulas in
many-sorted FOL with built-in symbols

SMT SOLVERS

Are highly efficient tools for the SMT problem based on specialized logic engines

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Are changing the way people solve problems in Computer Science and beyond:

- instead of building a special-purpose tool
- translate problem into a logical formula
- use an SMT solver as backend reasoner

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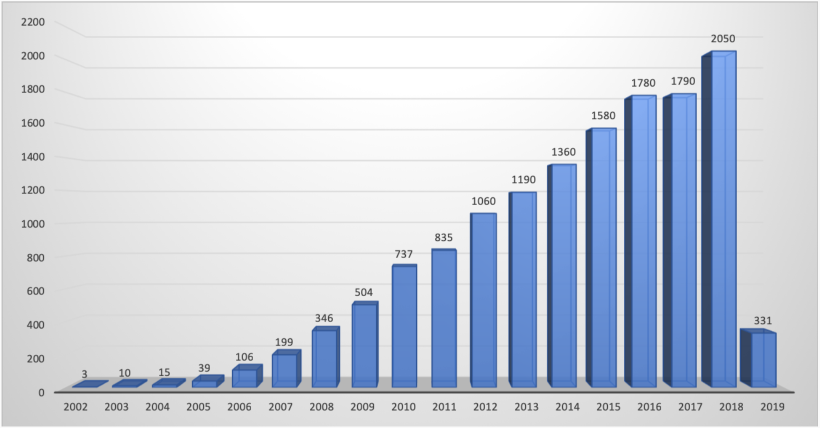
- use an SMT solver as backend reasoner

Not only easier, often better

THE EXPLOSION OF SMT

Google Scholar

“Satisfiability Modulo Theories” OR “SMT Solver”



	Citations	Google Scholar Hits
Z3	5,068[1]	7,870
CVC Lite, CVC 3, 4	1,560[2]	2,030
Yices 1, 2	972[3]	2,430
MathSat 3, 4, 5	628[4]	1,010

[1] [DMB08], 2018 ETAPS Test of Time Award to Z3 developers
[2] [BT07, BT07, BCD+11]
[3] [DdM06b, Dut14]
[4] [BBC+05b, BCF+08, CGSS13]

Model Checking

- (in)finite-state systems
- hybrid systems
- abstraction refinement
- state invariant generation
- interpolation

Type Checking

- dependent types
- semantic subtyping
- type error localization

Program Analysis

- symbolic execution
- program verification
- verification in separation logic
- (non-)termination
- loop invariant generation
- procedure summaries
- race analysis
- concurrency errors detection

Software Synthesis

- syntax-guided function synthesis
- automated program repair
- synthesis of reactive systems
- synthesis of self-stabilizing systems
- network schedule synthesis

MORE APPLICATIONS OF SMT

Security

- automated exploit generation
- protocol debugging
- protocol verification
- analysis of access control policies
- run-time monitoring

Compilers

- compilation validation
- optimization of arithmetic computations

Software Engineering

- system model consistency
- design analysis
- test case generation
- verification of ATL transformations
- semantic search for code reuse
- interactive (software) requirements prioritization
- generating instances of meta-models
- behavioral conformance of web services

Planning

- motion planning
- nonlinear PDDL planning

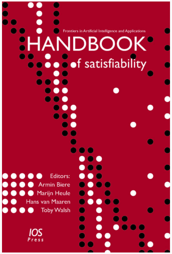
Machine Learning

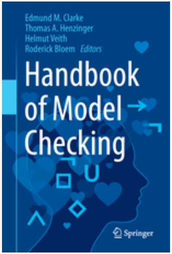
- verification of deep NNs

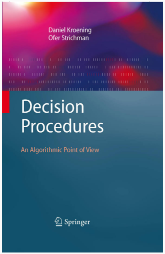
Business

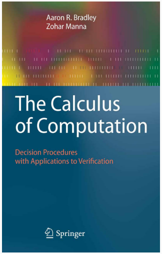
- verification of business rules
- spreadsheet debugging

Handbook chapters and books [BSST09, BT18, BM07, KS08]









Online

- SMT-LIB at `http://smt-lib.org`
- SMT-COMP at `http://smt-comp.org`

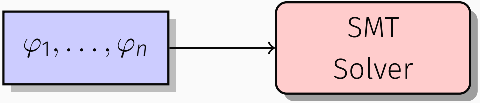
FUNCTIONALITY

LEGEND

v	value — i.e., distinguished variable-free term
$\varphi[\mathbf{x}]$	formula with free vars from $\mathbf{x} = (x_1, \ldots, x_n)$
$\varphi[\mathbf{x} \mapsto \mathbf{v}]$	formula obtained by replacing free occurrences of variables from $\mathbf{x}$ in φ with corresponding values from $\mathbf{v} = (v_1, \ldots, v_n)$
$\mathbf{x} = \mathbf{v}$	$x_1 = v_1 \wedge \cdots \wedge x_n = v_n$
$\mathbf{z} \subseteq \mathbf{x}$	every element of $\mathbf{z}$ occurs in $\mathbf{x}$
$M \models \varphi$	model M satisfies formula φ
$\varphi \models_T \psi$	formula φ entails formula ψ in theory T

SMT SOLVER BASIC FUNCTIONALITY

Background theory T



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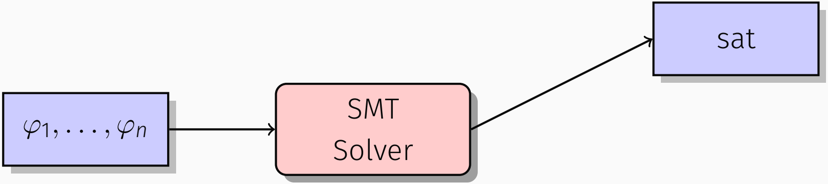

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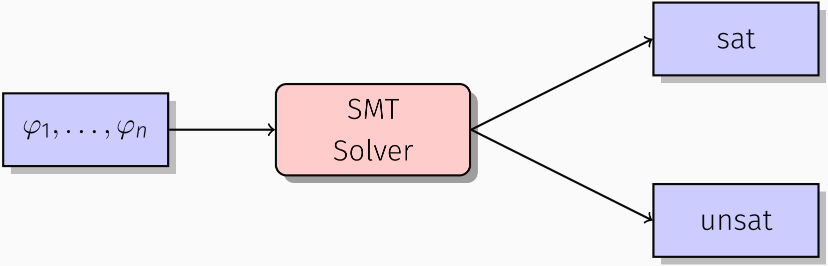

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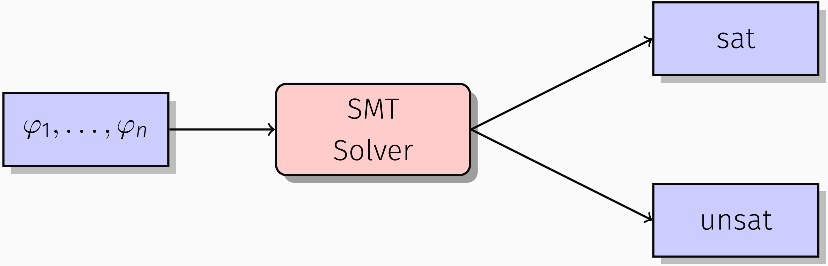

sat/unsat: there is a/no model M of T such that

$$M \models \varphi_1 \wedge \cdots \wedge \varphi_n$$

SMT SOLVER BASIC FUNCTIONALITY

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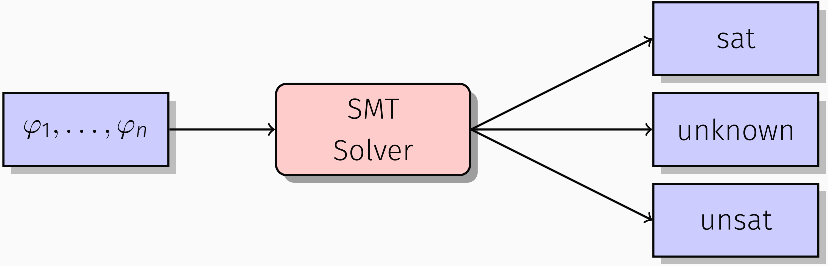
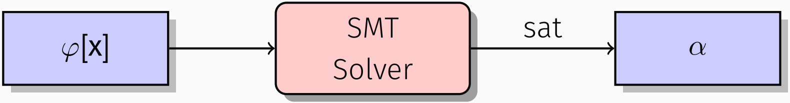


sat/unsat: there is a/no model M of T such that

$$M \models \varphi_1 \land \cdots \land \varphi_n$$

unknown: inconclusive — because of resource limits or incompleteness

Background theory T



α is a *satisfying assignment* for $\mathbf{x} = (x_1, \ldots, x_n)$:

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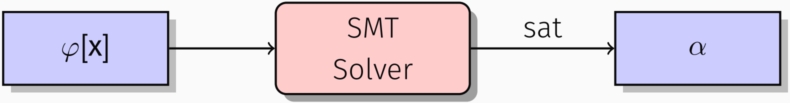
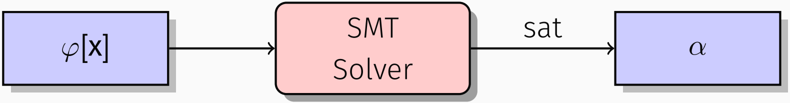


α is a *satisfying assignment* for $\mathbf{x} = (x_1, \ldots, x_n)$:

1. $\alpha = \{x_1 \mapsto v_1, \ldots, x_n \mapsto v_n\}$ for some values $\mathbf{v} = (v_1, \ldots, v_n)$
2. $M \models \varphi[\mathbf{x} \mapsto \mathbf{v}]$ for some model M of T

SMT SOLVER OUTPUT: SATISFYING ASSIGNMENTS

Background theory T



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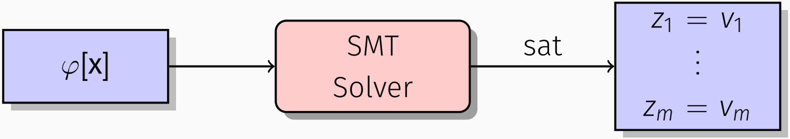
2. $M \models \varphi[\mathbf{x} \mapsto \mathbf{v}]$ for some model M of T

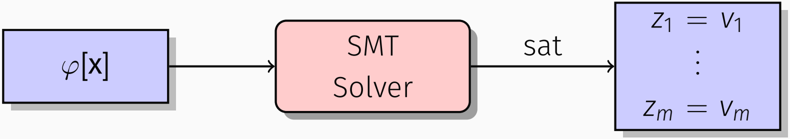
Note.
$\mathbf{x}$ may consist of first- and second-order variables
(aka, *uninterpreted constants and function symbols*)

Background theory T



$\mathbf{z} = \mathbf{v}$ is a *backbone* for φ:

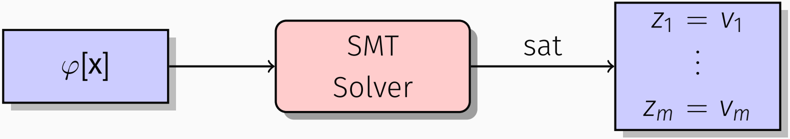
Background theory T



$\mathbf{z} = \mathbf{v}$ is a *backbone* for φ:

1. $\mathbf{z} \subseteq \mathbf{x}$
2. $\varphi \models_T \mathbf{z} = \mathbf{v}$
3. $\mathbf{z}$ is maximal (or largish)

Background theory T



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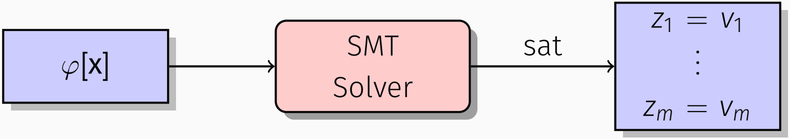
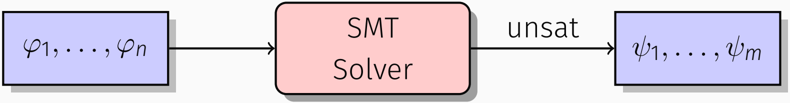


$\mathbf{z} = \mathbf{v}$ is a *sat core* for φ:

1. $\mathbf{z} \subseteq \mathbf{x}$
2. $\mathbf{y} = \mathbf{x} \setminus \mathbf{z}$
3. $\forall \mathbf{y}\,(\varphi \wedge \mathbf{z} = \mathbf{v})$ is satisfiable in T
4. $\mathbf{z}$ is minimal (or smallish)

Background theory T



$\psi_1, \ldots, \psi_m$ is a *unsat core* of $\{\varphi_1, \ldots, \varphi_n\}$:

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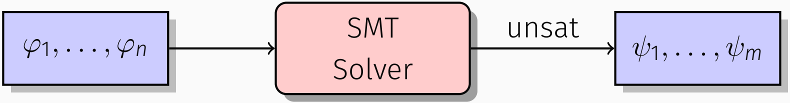

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1. $\{\psi_1, \ldots, \psi_m\} \subseteq \{\varphi_1, \ldots, \varphi_n\}$
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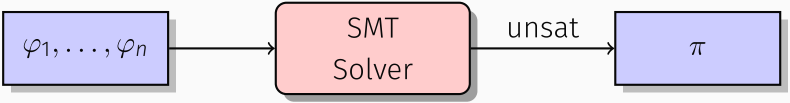
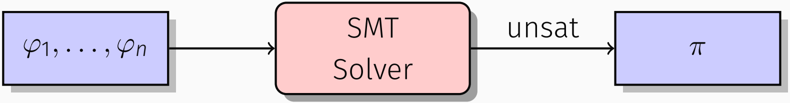
Background theory T



π is a checkable *proof object* for $\{\varphi_1, \ldots, \varphi_n\}$:

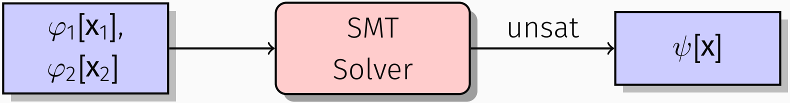
Background theory T



π is a checkable *proof object* for $\{\varphi_1, \ldots, \varphi_n\}$:

1. π is a proof term in some formal proof system
2. π expresses a refutation of $\{\varphi_1, \ldots, \varphi_n\}$
3. π can be efficiently checked by an external proof checker

Background theory T



ψ is a logical *interpolant* of φ_1 and φ_2:

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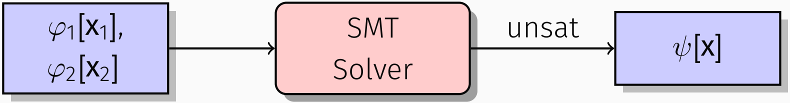
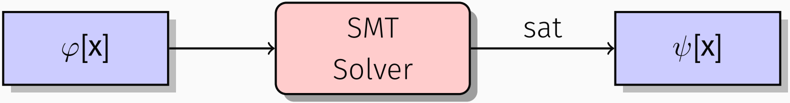


ψ is a logical *interpolant* of φ_1 and φ_2:

1. $\varphi_1 \models_T \psi$ and $\psi \models_T \neg\varphi_2$
2. $\mathsf{x} = \mathsf{x}_1 \cap \mathsf{x}_2$

EXTENDED FUNCTIONALITY: PRIME IMPLICATE COMPUTATION

Background theory T



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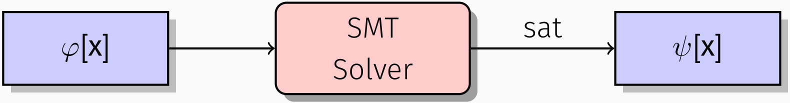
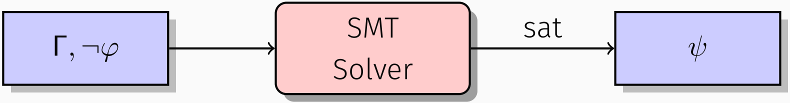
Background theory T



ψ is a *prime implicate* of φ:

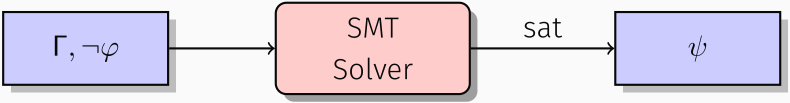
1. ψ is a disjunction of literals
2. $\varphi \models_T \psi$
3. there is no disjunction of literals $\psi' \notin \{\varphi, \psi\}$ s.t. $\varphi \models_T \psi'$ and $\psi' \models_T \psi$

Background theory T



ψ is an *abduction hypothesis* for φ wrt Γ:

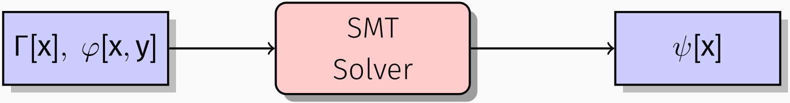
Background theory T



ψ is an *abduction hypothesis* for φ wrt Γ:

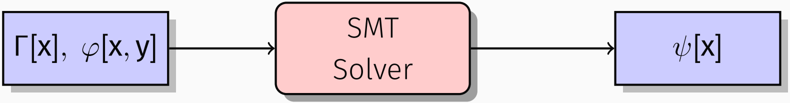
1. Γ, ψ is satisfiable in T
2. $\Gamma, \psi \models_T \varphi$
3. ψ is maximal, e.g., with respect to $\models_T$
   (if ψ' satisfies 1 and 2 and $\psi \models_T \psi'$ then $\psi' \models_T \psi$)

Background theory T



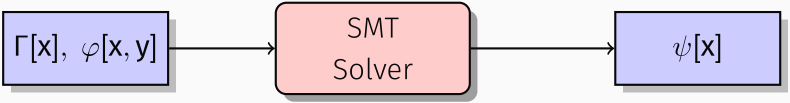
EXTENDED FUNCTIONALITY: QUANTIFIER ELIMINATION

Background theory T



ψ is a *projection* of φ over **y** with respect to Γ:

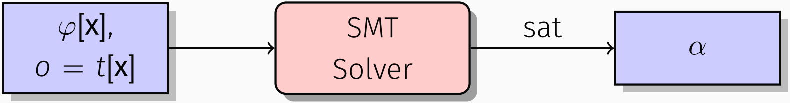
Background theory T



ψ is a *projection* of φ over $\mathbf{y}$ with respect to Γ:

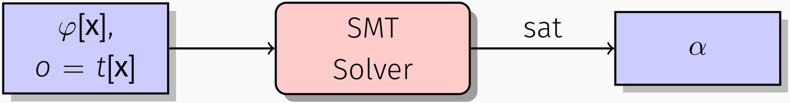
1. $\Gamma \models_T \psi \Leftrightarrow \exists \mathbf{y}\, \varphi$

Background theory T



α is a *an optimal assignment* for φ:

Background theory T



α is a *an optimal assignment* for φ:

1. $\alpha = \{x_1 \mapsto v_1, \ldots, x_n \mapsto v_n\}$ for some *values* $v_1, \ldots, v_n$
2. $M \models \varphi[\mathbf{x} \mapsto \mathbf{v}]$ for some model M of T
3. α minimizes/maximizes *objective* o

BACKGROUND THEORIES

Uninterpreted Funs	$x = y \Rightarrow f(x) = f(y)$
Integer/Real Arithmetic	$2x + y = 0 \land 2x - y = 4 \rightarrow x = 1$
Floating Point Arithmetic	$x + 1 \neq NaN \land x < \infty \Rightarrow x + 1 > x$
Bit-vectors	$4 \cdot (x \gg 2) = x \mathbin{\&} {\sim}3$
Strings and RegExs	$x = y \cdot z \land z \in ab^* \Rightarrow \lvert x \rvert > \lvert y \rvert$
Arrays	$i = j \Rightarrow \mathrm{store}(a, i, x)[j] = x$
Algebraic Data Types	$x \neq Leaf \Rightarrow \exists\, l, r : Tree(\alpha).\ \exists\, a : \alpha.\ x = Node(l, a, r)$
Finite Sets	$e_1 \in x \land e_2 \in x \setminus e_1 \Rightarrow \exists y, z : Set(\alpha).\ \lvert y \rvert = \lvert z \rvert \land x = y \cup z \land y \neq \emptyset$
Finite Relations	$(x, y) \in r \land (y, z) \in r \Rightarrow (x, z) \in r \bowtie s$

EQUALITY AND UNINTERPRETED FUNCTIONS (EUF) [NO80, NO07]

Simplest first-order theory with equality, applications of uninterpreted functions, and variables of uninterpreted sorts

For all sorts σ, σ' and function symbols $f : \sigma \to \sigma'$

$$\begin{aligned}
\text{Reflexivity:} &\quad \forall x : \sigma.\ x = x \\
\text{Symmetry:} &\quad \forall x : \sigma.\ x = y \Rightarrow y = x \\
\text{Transitivity:} &\quad \forall x, y : \sigma.\ x = y \land y = z \Rightarrow x = z \\
\text{Congruence:} &\quad \forall \mathbf{x}, \mathbf{y} : \sigma.\ \mathbf{x} = \mathbf{y} \Rightarrow f(\mathbf{x}) = f(\mathbf{y})
\end{aligned}$$

Example

$$f(f(f(a))) = b \qquad g(f(a), b) = a \qquad f(a) \neq a$$

Operates over sorts $\mathsf{Array}(\sigma_i, \sigma_e)$, σ_i, σ_e and function symbols

$$\_[\_] : \mathsf{Array}(\sigma_i, \sigma_e) \times \sigma_i \to \sigma_e$$
$$\mathsf{store} : \mathsf{Array}(\sigma_i, \sigma_e) \times \sigma_i \times \sigma \to \mathsf{Array}(\sigma_i, \sigma_e)$$

For any index sort σ_i and element sort σ_e

Read-Over-Write-1: $\forall a, i, e.\ \mathsf{store}(a, i, e)[i] = e$

Read-Over-Write-2: $\forall a, i, j, e.\ i \neq j \Rightarrow \mathsf{store}(a, i, e)[j] = a[j]$

Extensionality: $\forall a, b, i.\ a \neq b \Rightarrow \exists i.\ a[i] \neq b[i]$

Example

$$\mathsf{store}(\mathsf{store}(a, i, a[j]), j, a[i]) = \mathsf{store}(\mathsf{store}(a, j, a[i]), i, a[j])$$

Restricted fragments, over the reals or the integers, support efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$ [BBC+05a]
- Difference constraints: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
- Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06a]
- Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM+09, ZM10, JdM12]

Family of user-definable theories

Example

$$
\begin{aligned}
\text{Color} &:= \text{red} \mid \text{green} \mid \text{blue} \\
\text{List}(\alpha) &:= \text{nil} \mid (\text{head} : \alpha) :: (\text{tail} : \text{List}(\alpha))
\end{aligned}
$$

$$
\begin{aligned}
\text{Distinctiveness:}\quad & \forall h, t.\ \text{nil} \neq h :: t \\
\text{Exhaustiveness:}\quad & \forall l.\ l = \text{nil} \vee \exists h, t.\ h :: t \\
\text{Injectivity:}\quad & \forall h_1, h_2, t_1, t_2. \\
& \qquad h_1 :: t_1 = h_2 :: t_2 \Rightarrow h_1 = h_2 \wedge t_1 = t_2 \\
\text{Selectors:}\quad & \forall h, t.\ \text{head}(h :: t) = h \wedge \text{tail}(h :: t) = t \\
\text{(Non-circularity:}\quad & \forall l, x_1, \ldots, x_n.\ l \neq x_1 :: \cdots :: x_n :: l)
\end{aligned}
$$

- Strings and regular expressions [KGG+09, LRT+14]
- Floating point arithmetic [BDG+14, ZWR14]
- Finite sets with cardinality [BRBT16]
- Finite relations [MRTB17]
- Transcendental Functions [GKC13]
- Ordinary differential equations [GKC13]
- ...

APPLICATIONS

To check the reachability of a class S of *bad states* for a *system model* M:

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4. Encode S as a T-formula $B[\mathbf{x}]$
5. Find a k such that $I[\mathbf{x}_0] \wedge R[\mathbf{x}_0, \mathbf{x}_1] \wedge \cdots \wedge R[\mathbf{x}_{k-1}, \mathbf{x}_k] \wedge B[\mathbf{x}_k]$ is satisfiable in T

To check the invariance of a *state property* S
for a *system model* M:

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4. Encode S as a T-formula $P[\mathbf{x}]$
5. Prove that $P[\mathbf{x}]$ holds in all reachable states of $(I[\mathbf{x}], R[\mathbf{x}, \mathbf{x}'])$

Example (Parametric Resettable Counter)

System

Vars

input pos int, n_0
input bool r
int c, n

Initialization

c := 1
n := n_0

Transitions

n' := n
c' := if (r' or c = n)
then 1
else c + 1

Property

c <= n

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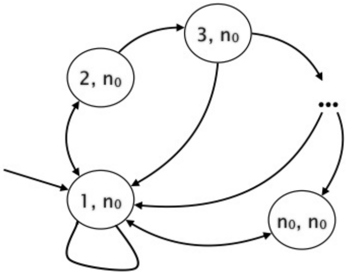

The transition relation contains infinitely many instances of the schema above, one for each $n_0 > 0$

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then 1
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Property

c <= n

Encoding in T = LIA

$$
\begin{aligned}
\mathbf{x} &:= (c, n, r, n_0) \\
I[\mathbf{x}] &:= c = 1 \\
&\quad \land\ n = n_0 \\
R[\mathbf{x}, \mathbf{x}'] &:= n' = n \\
&\quad \land\ (\neg r' \land c \neq n \lor c' = 1) \\
&\quad \land\ (r' \lor c = n \lor c' = c + 1) \\
P[\mathbf{x}] &:= c \leq n
\end{aligned}
$$

$$M = (I[\mathsf{x}], R[\mathsf{x}, \mathsf{x}'])$$

$M = (I[\mathbf{x}], R[\mathbf{x}, \mathbf{x}'])$

To prove $P[x]$ invariant for M it suffices
to show that it is *inductive* for M,
i.e.,

(1)	$I[\mathbf{x}] \models_T P[\mathbf{x}]$	(base case)
	and	
(2)	$P[\mathbf{x}] \wedge R[\mathbf{x}, \mathbf{x}'] \models_T P[\mathbf{x}']$	(inductive step)

$M = ($

To pro
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i.e.,

Problem: Not all invariants are inductive

For the parametric resettable counter,
$P := c \leq n + 1$ is invariant but (2) is falsifiable

e.g., by $(c, n, r) = (4, 3, \mathit{false})$ and $(c, n, r)' = (5, 3, \mathit{false})$

(1)	$I[\mathbf{x}] \models_T P[\mathbf{x}]$	(base case)
	and	
(2)	$P[\mathbf{x}] \wedge R[\mathbf{x}, \mathbf{x}'] \models_T P[\mathbf{x}']$	(inductive step)

(1) $I[\mathsf{x}] \models_T P[\mathsf{x}]$ (2) $P[\mathsf{x}] \land R[\mathsf{x}, \mathsf{x}'] \models_T P[\mathsf{x}']$

Various approaches:

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Various approaches:

Strengthen P: find a property Q such that $Q[\mathbf{x}] \models_T P[\mathbf{x}]$ and prove Q inductive
(ex., interpolation-based MC, IC3, CHC)

$$(1)\quad I[\mathsf{x}] \models_T P[\mathsf{x}] \qquad\qquad (2)\quad P[\mathsf{x}] \wedge R[\mathsf{x}, \mathsf{x}'] \models_T P[\mathsf{x}']$$

Various approaches:

Strengthen P: find a property Q such that $Q[\mathsf{x}] \models_T P[\mathsf{x}]$ and prove Q inductive
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Strengthen R: find an auxiliary invariant $Q[\mathsf{x}]$ and use $Q[\mathsf{x}] \wedge R[\mathsf{x}, \mathsf{x}'] \wedge Q[\mathsf{x}']$ instead of $R[\mathsf{x}, \mathsf{x}']$
(ex:, Houdini, invariant sifting)

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Various approaches:

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(ex:, Houdini, invariant sifting)

Lengthen R: Consider increasingly longer R-paths
$R[\mathbf{x}_0, \mathbf{x}_1] \wedge \cdots \wedge R[\mathbf{x}_{k-1}, \mathbf{x}_k] \wedge R[\mathbf{x}_k, \mathbf{x}_{k+1}]$
(ex:, k-induction)

Example

```
void swap(int* a, int* b) {
  *a = *a + *b;
  *b = *a - *b;
  *a = *a - *b;
}
```

Check if the swap is correct:

- Heap: $\mathit{Array}(BV_{32}) \mapsto BV_{32}$
- Update heap line by line
- Check that
  a* = old(b*) and b* = old(a*)

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Check if the swap is correct:

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- Update heap line by line
- Check that
  a* = old(b*) and b* = old(a*)

$$h_1 = \text{store}(h_0, a, h_0[a] +_{32} h_0[b])$$
$$h_2 = \text{store}(h_1, b, h_1[a] -_{32} h_1[b])$$
$$h_3 = \text{store}(h_2, a, h_2[a] -_{32} h_2[b])$$
$$\neg(h_3[a] = h_0[b] \wedge h_3[b] = h_0[a])$$

Example

```c
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Check if the swap is correct:

- Heap: $Array(BV_{32}) \mapsto BV_{32}$
- Update heap line by line
- Check that
  a* = old(b*) and b* = old(a*)
- Incorrect: aliasing

SMT solver solution

$$a \mapsto 0, \quad b \mapsto 0$$
$$h_0[0] \mapsto 1, \quad h_1[0] \mapsto 2$$
$$h_2[0] \mapsto 0, \quad h_3[0] \mapsto 0$$

$$\neg(h_3[a] = h_0[b] \land h_3[b] = h_0[a])$$

Example (Binary Search)

```
//@assume 0 <= n <= |a| &&
//        foreach i in [0..n-2]. a[i] <= a[i+1]
//@ensure (0 <= res ==> a[res] = k) &&
//        (res < 0 ==> foreach i in [0..n-1]. a[i] != k)
int BinarySearch(int[] a, int n, int k) {
  int l = 0;  int h = n;
  while (l < h) { // Find middle value
    //@invariant 0 <= low < high <= len <= |a| &&
    //           foreach i in [0..low-1]. a[i]<k &&
    //           foreach i in [high..len-1]. a[i] > k
    int m = l + (h - l) / 2;  int v = a[m];
    if (k < v) { l = m + 1; } else if (v < k) { h = m; }
    else { return m; }
  }
  return -1;
}
```

Example adapted from [dMB10]

Example (Binary Search)

```
//
//
//
//
i
}
```

Main approach

1. Compile source and annotations to program in Dijkstra's core language:

$$\begin{aligned} S, T \quad ::= \quad & x = t \mid \textbf{havoc}\ x \mid \textbf{assert}\ \varphi \mid \textbf{assume}\ \varphi \mid \\ & S; T \mid S \ [] \ T \end{aligned}$$

2. Convert core program to SMT using the weakest liberal precondition transformer wp:

$$wp(x = t, \varphi) = \varphi\{x \mapsto t\} \qquad wp(\textbf{assert}\ \psi, \varphi) = \psi \wedge \varphi$$
$$wp(\textbf{assume}\ \psi, \varphi) = \psi \Rightarrow \varphi \qquad wp(\textbf{havoc}\ x, \varphi) = \forall x\ \varphi$$
$$wp(S; T, \varphi) = wp(S, wp(T, \varphi))$$
$$wp(S \ [] \ T, \varphi) = wp(S, \varphi) \wedge wp(T, \varphi)$$

Example adapted from [dMB10]

Example (Binary Search)

$$pre = 0 \leq n \leq |a| \wedge \forall i : \mathrm{Int}\ \ 0 \leq i \wedge i \leq n-2 \Rightarrow a[i] \leq a[i+1]$$

$$\begin{aligned} post = {} & (0 \leq res \Rightarrow a[res] = k) \wedge \\ & (res < 0 \Rightarrow \forall i : \mathrm{Int}\ 0 \leq i \wedge i \leq n-1 \Rightarrow a[i] \neq k) \end{aligned}$$

$$\begin{aligned} inv = {} & 0 \leq l \wedge l \leq h \wedge h \leq n \wedge n \leq |a| \wedge \\ & \forall i : \mathrm{Int}\ \ 0 \leq i \wedge i \leq l-1 \Rightarrow a[i] < k \wedge \\ & \forall i : \mathrm{Int}\ \ h \leq i \wedge i \leq n-1 \Rightarrow a[i] > k \end{aligned}$$

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$$\begin{aligned} & pre \land \neg \mathbf{let}\ l = 0,\ h = n\ \mathbf{in}\ inv \land \forall : \mathrm{Int}\ l, h.\ \ inv \Rightarrow \\ & \quad (\neg(l < h) \Rightarrow post\{res \mapsto -1\}) \land \\ & \quad (l < h \Rightarrow \mathbf{let}\ m = l + (h-l)/2,\ v = a[m]\ \mathbf{in} \\ & \qquad (k < v \Rightarrow inv\{l \mapsto m+1\}) \land \\ & \qquad (\neg(k < v) \land v < k \Rightarrow inv\{n \mapsto m\}) \land \\ & \qquad (\neg(k < v) \land \neg(v < k) \Rightarrow post\{res \mapsto m\})) \end{aligned}$$

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$$inv = 0 \le l \wedge l \le h \wedge h \le n \wedge n \le |a| \wedge$$

$\forall i : \mathrm{Int}\ \ 0 \le$

$\forall i : \mathrm{Int}\ \ h \le$

SMT solver answer: Unsatisfiable

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Synthesis

- Synthesize a function that satisfies a given high-level specification
- Already used extensively for hardware systems
- Particularly challenging for software

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Recent interest

- Major new efforts by several research groups
- New syntax-guided synthesis (SyGuS) format
- SyGuS competition started in 2014
- New technique: Refutation-Based Synthesis in SMT [RDK+15]

Formalization in second-order logic

- Let $P[f, \mathbf{x}]$ be a property (specification) for a function f over some variables $\mathbf{x} = (x_1, x_2)$
- The synthesis problem is to determine the satisfiability of

$$\exists f.\, \forall \mathbf{x}.\, P[f, \mathbf{x}]$$

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Example

Maximum of 2 values

$$P[f, \boldsymbol{x}] = f(\boldsymbol{x}) \geq x_1 \wedge f(\boldsymbol{x}) \geq x_2 \wedge (f(\boldsymbol{x}) = x_1 \vee f(\boldsymbol{x}) = x_2)$$

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Problem: SMT only understands first-order logic

Single-invocation properties

- Every occurrence of f is of the form $f(\mathbf{x})$
  - Previous example is single-invocation
  - Not single-invocation: $\forall \mathbf{x}.\, f(x_1, x_2) = f(x_2, x_1)$
- When the synthesis property is single-invocation, it can written as $\exists f.\, \forall \mathbf{x}.\, P[f(\mathbf{x}), \mathbf{x}]$

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Note that:

$$\exists f.\, \forall \mathbf{x}.\, P[f(\mathbf{x}), \mathbf{x}] \tag{1}$$

is equivalent to

$$\forall \mathbf{x}.\, \exists y\, P[y, \mathbf{x}] \tag{2}$$

because (1) is the Skolemization of (2) which is first-order!

Proving the validity of

$$\forall \boldsymbol{x}.\, \exists y\, P[y, \boldsymbol{x}]$$

Proving the validity of

$$\forall \boldsymbol{x}.\, \exists y\, P[y, \boldsymbol{x}]$$

is equivalent to proving the unsatisfiability of

$$\exists \boldsymbol{x}.\, \forall y\, \neg P[y, \boldsymbol{x}]$$

or the unsatisfiability of

$$\forall y\, \neg P[y, \boldsymbol{c}]$$

for some fresh constants $\boldsymbol{c}$

How does an SMT solver show that

$$\forall y\, \neg P[y, \mathbf{c}] \text{ is unsatisfiable?}$$

How does an SMT solver determine that

$$\forall y\, \neg P[y, \mathbf{c}] \text{ is unsatisfiable?}$$

SMT solvers use heuristic instantiation [GBT07, GdM09, RTGK13] to produce a set of unsatisfiable quantifier-free formulas:

$$\{\neg P[t_1[\mathbf{c}], \mathbf{c}], \neg P[t_2[\mathbf{c}], \mathbf{c}], \ldots, \neg P[t_n[\mathbf{c}], \mathbf{c}]\}$$

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This also gives a constructive solution to the original synthesis problem:

$$f = \lambda \mathbf{x}.\, ite(P[t_1[\mathbf{x}], \mathbf{x}], t_1[\mathbf{x}], (\cdots ite(P[t_{n-1}[\mathbf{x}], \mathbf{x}], t_{n-1}[\mathbf{x}], t_n[\mathbf{x}]) \cdots))$$

Example

Schedule n jobs, each composed of m consecutive tasks, on m machines.

Schedule in 8 time slots

$d_{i,j}$	Mach. 1	Mach. 2
Job 1	2	1
Job 2	3	1
Job 3	2	3

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$$(t_{1,1} \geq 0) \wedge (t_{1,2} \geq t_{1,1} + 2) \wedge (t_{1,2} + 1 \leq 8)$$
$$(t_{2,1} \geq 0) \wedge (t_{2,2} \geq t_{2,1} + 3) \wedge (t_{2,2} + 1 \leq 8)$$
$$(t_{3,1} \geq 0) \wedge (t_{3,2} \geq t_{3,1} + 2) \wedge (t_{3,2} + 3 \leq 8)$$
$$((t_{1,1} \geq t_{2,1} + 3) \vee (t_{2,1} \geq t_{1,1} + 2))$$
$$((t_{1,1} \geq t_{3,1} + 2) \vee (t_{3,1} \geq t_{1,1} + 2))$$
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$(t_{1,1} \geq 0) \wedge (t_{1,2} \geq t_{1,1} + 2) \wedge (t_{1,2} + 1 \leq 8)$

SMT solver solution

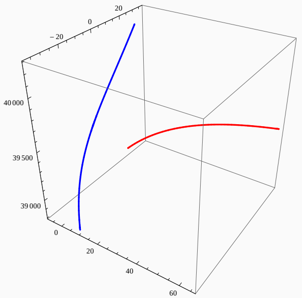
$t_{1,1} \mapsto 5, \quad t_{1,2} \mapsto 7$

$t_{2,1} \mapsto 2, \quad t_{2,2} \mapsto 6$

$t_{3,1} \mapsto 0, \quad t_{3,2} \mapsto 3$

$((t_{1,2} \geq t_{3,2} + 3) \vee (t_{3,2} \geq t_{1,2} + 1))$

$((t_{2,2} \geq t_{3,2} + 3) \vee (t_{3,2} \geq t_{2,2} + 1))$



$$H = 5\,nm \qquad V = 1000\,ft \qquad 0 \le t \le \frac{1}{20}h$$

$$|T_1^z(t) - T_2^z(t)| \le V$$

$$(T_1^x(t) - T_2^x(t))^2 + (T_1^y(t) - T_2^y(t))^2 \le H^2$$

$$T_1^x(t) = 3.2484 + 270.7t + 433.12t^2 - 324.83999t^3$$

$$T_1^y(t) = 15.1592 + 108.28t + 121.2736t^2 - 649.67999t^3$$

$$T_1^z(t) = 38980.8 + 5414t - 21656t^2 + 32484t^3$$

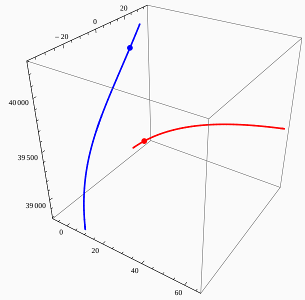
$$T_2^x(t) = 1.0828 - 135.35t + 234.9676t^2 2 + 3248.4t^3$$

$$T_2^y(t) = 18.40759 - 230.6364t - 121.2736t^2 - 649.67999t^3$$

$$T_2^z(t) = 40280.15999 - 10828t + 24061.9816t^2 - 32484t^3$$

Example from [NM12]

AIRCRAFT TRAJECTORY CONFLICT DETECTION



$$T_1^x(t) \ldots$$
$$T_1^y(t \ldots$$
$$T_1^z(t \ldots$$

SMT solver solution

$$t \mapsto \frac{319}{16384} \approx 0.019470215$$

$$T_2^x(t) = 1.0828 - 135.35t + 234.9676t^2 2 + 3248.4t^3$$
$$T_2^y(t) = 18.40759 - 230.6364t - 121.2736t^2 - 649.67999t^3$$
$$T_2^z(t) = 40280.15999 - 10828t + 24061.9816t^2 - 32484t^3$$

$$H = 5\,nm \qquad V = 1000\,ft \qquad 0 \leq t \leq \frac{1}{20}h$$
$$|T_1^z(t) - T_2^z(t)| \leq V$$
$$(T_1^x(t) - T_2^x(t))^2 + (T_1^y(t) - T_2^y(t))^2 \leq H^2$$

Example from [NM12]

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