SMT-based Unbounded Model Checking with IC3 and Approximate QE

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Acknowledgements

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Modeling Computational Systems

Software or hardware systems can be often represented as a **state transition system** $\mathcal{M} = (S, I, T, \mathcal{L})$ where

- $S$ is a set of **states**, the **state space**
- $I \subseteq S$ is a set of **initial states**
- $T \subseteq S \times S$ is a (right-total) **transition relation**
- $\mathcal{L}: S \rightarrow 2^\mathcal{P}$ is a **labeling function** where $\mathcal{P}$ is a set of **state predicates**

Typically, the state predicates denote variable-value pairs $x = v$
Model Checking

Software or hardware systems can be often represented as a state transition system \( \mathcal{M} = (S, I, T, L) \)

\( \mathcal{M} \) can be seen as a model both

1. in an engineering sense:
   - an abstraction of the real system

   and

2. in a mathematical logic sense:
   - a Kripke structure in some modal logic
Model Checking

The functional properties of a computational system can be expressed as *temporal* properties

- for a suitable model \( \mathcal{M} = (S, I, T, L) \) of the system
- in a suitable temporal logic
Model Checking

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Two main classes of properties:

- \textit{Safety properties}: nothing bad ever happens
- \textit{Liveness properties}: something good eventually happens
Invariance Model Checking

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Two main classes of properties:

- *Safety properties*: nothing bad ever happens
- *Liveness properties*: something good eventually happens

Safety checking can be reduced to invariance checking
Basic Terminology

Let $M = (S, I, T, L)$ be a transition system

The set $R$ of *reachable states (of $M$)* is the smallest subset of $S$ such that

1. $I \subseteq R$ (initial states are reachable)
2. $(R \bowtie T) \subseteq R$ ($T$-successors of reachable states are reachable)
Basic Terminology

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The set $\mathcal{R}$ of reachable states (of $\mathcal{M}$) is the smallest subset of $S$ such that

1. $I \subseteq \mathcal{R}$ (initial states are reachable)
2. $(\mathcal{R} \rhd T) \subseteq \mathcal{R}$ ($T$-successors of reachable states are reachable)

A state property $\mathcal{P} \subseteq S$ is invariant (for $\mathcal{M}$) iff $\mathcal{R} \subseteq \mathcal{P}$

![Invariant and not invariant states](image)

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Checking Invariance

In principle, to check that $\mathcal{P}$ is invariant for $\mathcal{M}$ it suffices to

1. compute $\mathcal{R}$ and
2. check that $\mathcal{R} \subseteq \mathcal{P}$
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Alternatively, we can represent $\mathcal{M}$ symbolically and use

- BDD-based methods, if $\mathcal{S}$ is finite,
- automata-based methods,
- abstract interpretation methods, or
- logic-based methods
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Logic-based Model Checking

Applicable if we can encode

\[ M = (S, I, T, L) \]

in some classical logic \( L \) with decidable entailment \( \models_L \) for some large enough class of formulas in \( L \)

\((\varphi \models_L \psi \text{ iff } \varphi \land \neg \psi \text{ is unsatisfiable in } L)\)
Logic-based Model Checking

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in some classical logic \( \mathbb{L} \) with decidable entailment \( \models_{\mathbb{L}} \) for some large enough class of formulas in \( \mathbb{L} \)

\[ (\varphi \models_{\mathbb{L}} \psi \text{ iff } \varphi \land \neg \psi \text{ is unsatisfiable in } \mathbb{L}) \]

Some (reasonable) additional requirements on \( \mathbb{L} \) are needed
Requirements on $\mathbb{L}$

$\mathbb{L} = (\Sigma, F, A, \models_{\mathbb{L}}, V)$ with

- $\Sigma$, a many-sorted first-order signature with equality
- $F$, language of $\Sigma$-formulas closed under all Boolean operators and quantifiers
- $A$, a single $\Sigma$-structure with decidable satisfiability for quantifier-free formulas
Requirements on $L$

$L = (\Sigma, F, \mathcal{A}, \models_L, \mathcal{V})$ with

- $\models_L$, same as entailment in $\mathcal{A}$

- $\mathcal{V}$, set of \textit{values} in $\mathcal{A}$, variable-free terms with unique interpretation in $\mathcal{A}$

- Quantifier-free formulas \textit{satisfied by values}:
  
  for all qffs $F[x] \in F$ satisfiable in $\mathcal{A}$, there is a $v \in \mathcal{V}$ such that $F[v]$ is true in $\mathcal{A}$
Examples of \L

Any modular combination of the logics of

- Boolean formulas (with variables belonging to a single Boolean sort)
- linear integer, rational or floating point arithmetic
- fixed size bit vectors
- algebraic data types
- strings
- finite sets

... with a suitable choice of function and predicate symbols
Logical encodings of transitions systems

\[ \mathcal{M} = (S, I, T, \mathcal{L}) \]

\[ X: \text{set of variables} \quad V: \text{values in } \mathbb{L} \]

\text{Not.}: \text{if } x = (x_1, \ldots, x_n) \text{ and } s = (v_1, \ldots, v_n), \quad \phi[s] := \phi[v_1/x_1, \ldots, v_n/x_n]
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- states \( s \in \mathcal{S} \) encoded as \( n \)-tuples of \( \mathbb{V}^n \)
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- states \( s \in S \) encoded as \( n \)-tuples of \( V^n \)
- \( I \) encoded as a formula \( I[x] \) with free variables \( x \) such that

\[ s \in I \iff \models_{\mathbb{L}} I[s] \]
Logical encodings of transitions systems

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\( X \): set of variables \( \mathcal{V} \): values in \( \mathbb{L} \)

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- states \( s \in S \) encoded as \( n \)-tuples of \( \mathcal{V}^n \)
- \( \mathcal{I} \) encoded as a formula \( I[x] \) with free variables \( x \) such that \( s \in \mathcal{I} \) iff \( \models_{\mathcal{L}} I[s] \)
- \( \mathcal{T} \) encoded as a formula \( T[x, x'] \) such that \( \models_{\mathcal{L}} T[s, s'] \) for all \( (s, s') \in \mathcal{T} \)
Logical encodings of transitions systems

\[ M = (S, \mathcal{I}, \mathcal{T}, \mathcal{L}) \]

\( X \): set of variables \( V \): values in \( \mathbb{L} \)

Not.: if \( x = (x_1, \ldots, x_n) \) and \( s = (v_1, \ldots, v_n) \), \( \phi[s] := \phi[v_1/x_1, \ldots, v_n/x_n] \)

- states \( s \in S \) encoded as \( n \)-tuples of \( V^n \)
- \( \mathcal{I} \) encoded as a formula \( I[x] \) with free variables \( x \) such that

\[
\text{s} \in \mathcal{I} \text{ iff } \models_{\mathbb{L}} I[s]
\]

- \( \mathcal{T} \) encoded as a formula \( T[x, x'] \) such that

\[
\models_{\mathbb{L}} T[s, s'] \text{ for all } (s, s') \in \mathcal{T}
\]

- State properties encoded as formulas \( P[x] \)
Strongest Inductive Invariant

The *strongest inductive invariant (for $\mathcal{M}$ in $\mathcal{L}$)* is a formula $R[x]$ such that $\models_{\mathcal{L}} R[s]$ iff $s \in \mathcal{R}$.
The **strongest inductive invariant (for $\mathcal{M}$ in $\mathbb{L}$)** is a formula $R[x]$ such that $\models_\mathbb{L} R[s]$ iff $s \in R$

Suppose we can compute $R$ from $I$ and $T$. Then,

checking that a property $P[x]$ is invariant for $\mathcal{M}$ reduces to checking that $R[x] \models_\mathbb{L} P[x]$
Strongest Inductive Invariant

The *strongest inductive invariant (for \( M \) in \( L \))* is a formula \( R[x] \) such that \( \models_L R[s] \) iff \( s \in R \)

**Problem:** \( R \) may be very expensive or impossible to compute, or not even representable in \( L \)
Strongest Inductive Invariant

The strongest inductive invariant (for $\mathcal{M}$ in $\mathcal{L}$) is a formula $R[x]$ such that $\models_\mathcal{L} R[s]$ iff $s \in R$

**Problem:** $R$ may be very expensive or impossible to compute, or not even representable in $\mathcal{L}$

**One Strategy:** *Property-Directed Reachability*. Try to construct an over-approximation $\hat{R}$ of $R$ that entails $P$ in $\mathcal{L}$
Property Directed Reachability

Two main methods:

- Interpolation-based model checking [McMillan’03]

- Incremental Construction of Inductive Clauses for Indubitable Correctness (IC3) [Bradley’10]
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- Interpolation-based model checking [McMillan’03]

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Note: PDR is used typically to refer to IC3
IC3’s Main Idea

Given $M = (I[x], T[x, x'])$ and $P[x]$, construct $\hat{R}$ incrementally

Maintain list $\hat{R}_0 \hat{R}_1 \cdots \hat{R}_i \hat{R}_{i+1}$ where

- $\hat{R}_0 = \{I[x]\}$
- $\hat{R}_{i+1} = \{P[x]\}$

- for each $i = 1, \ldots, k$
  - $\hat{R}_i$ is a set of one-state formulas over $x$
  - $\hat{R}_i$ over-approximates the states reachable in $i$-steps
  - $\hat{R}_i$ under-approximates $\hat{R}_{i+1}$
IC3’s Main Idea
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\[ I \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq P \]

\[ \hat{R}_1 \subseteq \hat{R}_2 \]

\[ \mathcal{I} \]

\[ \mathcal{R} \]

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IC3’s Main Idea
Recursively Refining $\hat{R}_i$
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Suppose there are $s, s'$ s.t. $\hat{R}_k[s] \land T[s, s'] \land \neg P[s']$ is satisfiable
Recursively Refining $\hat{R}_i$

Find $B[x]$ s.t. $B[s]$ is satisfiable and $B[x], T[x, x'] \models I \neg P[x']$
If $\hat{R}_{k-1}[x], T[x, x'] \models_{\mathbb{L}} \neg B[x']$ let $\hat{R}_{k} := \hat{R}_{k-1} \cup \{\neg B\}$
Else there are $s, s'$ s.t. $\hat{R}_{k-1}[s] \land T[s, s'] \land \neg P[s']$ is satisfiable. Refine $\hat{R}_{k-1}$
Frame Sequences

IC3 constructs (initial segments of) sequences \((R_i)_{i \geq 0}\) of frames, sets of one-state formulas, satisfying the following

Frame Conditions

1. \(R_0 = \{I\}\)
2. \(R_i \supseteq R_{i+1}\) for all \(i > 0\)
3. \(R_i \supseteq \{P\}\) for all \(i > 0\)
4. \(R_i[x] \land T[x, x'] \models L R_{i+1}[x']\) for all \(i \geq 0\)
Extension of a Formula

The *extension* of an $m$-state formula $F[y_1, \ldots, y_m]$ of $\mathbb{L}$ is the following subset of $S^m$:

$$
[F] \overset{\text{def}}{=} \{ (s_1, \ldots, s_m) \in S^m \mid F[s_1, \ldots, s_m] \text{ is satisfiable in } \mathbb{L} \}
$$

**Note:** I will sometimes identify a state formula $F$ with its extension $[F]$.
Properties of Frame Sequences

Frame Conditions

(1) $R_0 = I$

(2) $R_i \supseteq R_{i+1}$ for all $i > 0$

(3) $R_i \supseteq \{P\}$ for all $i > 0$

(4) $R_i[x] \land T[x, x'] \models_{\text{IL}} R_{i+1}[x']$ for all $i \geq 0$
Properties of Frame Sequences

Frame Conditions

(1) $R_0 = I$

(2) $R_i \supseteq R_{i+1}$ for all $i > 0$

(3) $R_i \supseteq \{ P \}$ for all $i > 0$

(4) $R_i[x] \land T[x, x'] \models_{L} R_{i+1}[x']$ for all $i \geq 0$

Lemma 1 [Soundness] Suppose $(R_i)_{i \geq 0}$ satisfies the frame conditions and $R_0[x] \models_{L} P[x]$. If there is an $i > 0$ such that $R_i = R_{i+1}$, then $P$ is invariant.
Properties of Frame Sequences

Frame Conditions

(1) $R_0 = I$
(2) $R_i \supseteq R_{i+1}$ for all $i > 0$
(3) $R_i \supseteq \{P\}$ for all $i > 0$
(4) $R_i[x] \land T[x, x'] \models L R_{i+1}[x']$ for all $i \geq 0$

Lemma 2 [Termination, non-invariant case] If $P$ is not invariant, there is a $k \geq 0$ such that for all frame sequences $(R_i)_{i \geq 0}$ satisfying the frame conditions, $\llbracket R_k \rrbracket$ contains a $k$-reachable error state.
Properties of Frame Sequences

Frame Conditions

1. \( R_0 = I \)
2. \( R_i \supseteq R_{i+1} \) for all \( i > 0 \)
3. \( R_i \supseteq \{ P \} \) for all \( i > 0 \)
4. \( R_i[x] \land T[x, x'] \models_{\mathbb{L}} R_{i+1}[x'] \) for all \( i \geq 0 \)

Lemma 2 [Termination, non-invariant case] If \( P \) is not invariant, there is a \( k \geq 0 \) such that for all frame sequences \( (R_i)_{i \geq 0} \) satisfying the frame conditions, \( \llbracket R_k \rrbracket \) contains a \( k \)-reachable error state.

Lemma 3 [Termination, invariant case] If \( \llbracket P \rrbracket \) is finite, there is no frame sequence \( (R_i)_{i \geq 0} \) satisfying the frame conditions such that \( \llbracket R_i \rrbracket \subsetneq \llbracket R_{i+1} \rrbracket \) for all \( i \geq 0 \).
The IC3 Procedure: Our Version

Defined by \( \text{verify}(R_0 \ R_1) \)

where

\[
R_0 = \{I\}, \quad R_1 = \{P\}
\]
\[
I[x] \models_{\mathbb{L}} P[x]
\]
\[
I[x], \ T[x, x'] \models_{\mathbb{L}} P[x']
\]

Require: \( R_{i-1}[x] \land T[x, x'] \models_{\mathbb{L}} R_i[x'] \)

for \( i = 1, \ldots, k \) with \( R_k = P \)

1: function \( \text{verify}(R_0 \cdots R_k) \)
2: let \( R_0 \cdots R_k = \text{strengthen}(R_0 \cdots R_k) \) in
3: let \( R_0 \cdots R_k = \text{propagate}(R_0, R_1 \cdots R_k) \) in
4: \( \text{verify}(R_0 \cdots R_k \{P\}) \)
Backward Pass

**Require:** \( R_{i-1}[x] \land T[x, x'] \models L R_i[x'] \) for \( i = 1, \ldots, k \)

**Ensure:** \( R_{i-1}[x] \land T[x, x'] \models L R_i[x'] \) for \( i = 1, \ldots, k + 1 \)

with \( R_{k+1} = \{P\} \)

1: function strengthen\((R_0 \cdots R_k)\)
2: if \( R_k[x] \land T[x, x'] \models L P[x'] \) then
3: \( R_0 \cdots R_k \)
4: else
5: let \( B = \) generalize\((R_k, \neg P)\) in
6: let \( R_0 \cdots R_k = \) block\((R_0 \cdots R_{k-1}, \{B\}, R_k)\) in
7: strengthen\((R_0 \cdots R_k)\)

**Not.** \( A :: R \) denotes \( \{A\} \cup R \)
Blocking Bad States (simplified)

Require: $v \in [R_i]$, $v$ reaches $\neg P$ in $k - i + 1$ steps for each $v \in [B]$, $B \in Q_j$, $i = j, \ldots, k$

Invariant: $R_{i-1}[x] \land T[x, x'] \models L_R_i[x']$ for $i = 1, \ldots, k$

1: function block($R_0 \cdots R_{j-1}$, $(Q_j, R_j) \cdots (Q_k, R_k)$)

2: let $B \in Q_j$, $Q_j = Q_j \setminus \{B\}$ in

3: if $\neg B[x] \land R_{j-1}[x] \land T[x, x'] \models L \neg B[x']$ then

4: let $R_0 \cdots R_k = R_0 (\neg B :: R_1) \cdots (\neg B :: R_j) R_{j+1} \cdots R_k$ in

5: if $Q_j \neq \emptyset$ then

6: block($R_0 \cdots R_{j-1}$, $(Q_j, R_j)(B :: Q_{j+1}, R_{j+1}) \cdots (B :: Q_k, R_k)$)

7: else if $j = k$ then $R_0 \cdots R_k$

8: else block($R_0 \cdots R_j$, $(B :: Q_{j+1}, R_{j+1}) \cdots (B :: Q_k, R_k)$)

9: else

10: let $\bar{B} = \text{generalize}(R_{j-1} \land C_j, B)$ in

11: block($R_0 \cdots R_{j-2}$, $\{\bar{B}\}, R_{j-1}) (Q_j, R_j) \cdots (Q_k, R_k)$)
The IC3 Procedure

Require: \( 0 \leq j < k \)

Invariant: \( R_{i-1}[x] \land T[x, x'] \models_L R_i[x'] \) for \( i = 1, \ldots, k \)

1: function \( \text{propagate}(R_0 \cdots R_j, R_{j+1} \cdots R_k) \)

2: \( \text{if} \left( \begin{array}{c}
\text{there is } C \in R_j \setminus R_{j+1} \text{ s.t. } \\
R_j[x] \land T[x, x'] \models_L C[x']
\end{array} \right) \text{ then} 

3: \( \text{propagate}(R_0 \cdots R_j, (C :: R_{j+1}) \cdots R_k) \)

4: \( \text{else if } R_j = R_{j+1} \text{ then} 

5: \text{raise Success} 

6: \( \text{else if } j + 1 < k \text{ then} 

7: \( \text{propagate}(R_0 \cdots R_{j+1}, R_{j+2} \cdots R_k) \)

8: \( \text{else} 

9: \( R_0 \cdots R_k \)
The IC3 Procedure

Require:  \([F[x] \land T[x, x'] \land B[x']] \neq \emptyset\)

1: function generalize\((F, B)\)
2: let \((s, s') \in \{[F[x] \land T[x, x'] \land B[x']]\} in
3: let \(\bar{B}[x] = \text{extrapolate}(s, s', F[x] \land T[x, x'] \land B[x'])\) in
4: if \(I[x], \bar{B}[x] \models \bot\) then
5: \(\bar{B}[x]\)
6: else
7: raise Counterexample
Key Point of non-Boolean IC3

The critical component in generalizing IC3 beyond propositional logic is *extrapolate*

*extrapolate* encapsulates IC3’s idea of generalizing *induction counterexamples*

Producing lemmas that eliminate whole sets of induction counterexamples is crucial for refining the frame sequence

Eliminating these states one by one is either impractical or even impossible

It is imperative to find a *finite* number of lemmas that eliminate all induction counterexamples from a frame
Generalizing Induction Counterexamples

The set of all induction counterexamples in a frame $F$ wrt bad states $B'$ has an exact and compact representation:

$$G[x] := \exists x'(F[x] \land T[x, x'] \land B[x'])$$
Generalizing Induction Conterexamples

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Quantifier elimination, when possible at all, can be very expensive (e.g., doubly exponential in the length of $x'$).
Generalizing Induction Counterexamples

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**Our approach:** compute quantifier-free under-approximations of $G$ driven by specific counterexamples.
Our Approach

Additional requirement: $L$ has quantifier elimination
Our Approach

Additional requirement: \( L \) has quantifier elimination

Given \( E[x, x'] := F[x] \land T[x, x'] \land B'[x'] \) and \( (s, s') \in \llbracket E \rrbracket \),
Our Approach

Additional requirement: \( L \) has quantifier elimination

Given \( E[x, x'] := F[x] \land T[x, x'] \land B'[x'] \) and \((s, s') \in \llbracket E \rrbracket\),

Step 1 Extract from \( E \) a conjunction \( H[x, x'] \) of literals s.t.

\[
(s, s') \in \llbracket H \rrbracket \quad \text{and} \quad H[x, x'] \models_L F[x] \land T[x, x'] \land B'[x']
\]
Our Approach

Additional requirement: $\mathbb{L}$ has quantifier elimination

Given $E[x, x'] := F[x] \land T[x, x'] \land B'[x']$ and $(s, s') \in \llbracket E \rrbracket$,

Step 1 Extract from $E$ a conjunction $H[x, x']$ of literals s.t.

$$(s, s') \in \llbracket H \rrbracket \text{ and } H[x, x'] \models \llbracket F[x] \land T[x, x'] \land B'[x'] \rrbracket$$

Step 2 Compute a conjunction $B[x]$ of literals s.t.

$$s \in \llbracket B \rrbracket \text{ and } B[x] \models \exists x' H[x, x']$$
Extracting a Conjunctive Implicant

\[ H[x, x'] := e^+(F[x] \land T[x, x'] \land B'[x']) \]

where

\[
e^+(F) := \begin{cases} 
    e^+(F_1) & \text{if } F = F_1 \lor \cdots \lor F_n \text{ and } \models_L F_1[s, s'] \\
    e^+(F_1) \land \cdots \land e^+(F_n) & \text{if } F = F_1 \land \cdots \land F_n \\
    e^-(F_1) & \text{if } F = \neg F_1 \\
    F & \text{if } F \text{ is an atom}
\end{cases}
\]

\[
e^-(F) := \begin{cases} 
    e^-(F_1) \land \cdots \land e^-(F_n) & \text{if } F = F_1 \lor \cdots \lor F_n \\
    e^-(F_1) & \text{if } F = F_1 \land \cdots \land F_n \text{ and } \models_L \neg F_1[s, s'] \\
    e^+(F_1) & \text{if } (3) \text{if } F = \neg F_1 \\
    \neg F & \text{if } (4) \text{if } F \text{ is an atom}
\end{cases}
\]
Computing One-state Cube

Use a under-approximating version of QE to compute $B[x]$ from $H[x, x']$

Currently done for linear integer arithmetic

Based on Cooper’s QE procedure for LIA

Idea applies similarly to other logics with QE (e.g., real arithmetic)
Experimental Evaluation

Implementation in Kind 2 model checker with \( \mathbb{L} = \text{LIA} \)

Kind 2 is written in OCaml and uses several SMT solvers as reasoning engines

Used Z3 in this case (as it has does QE)

Step 2 of extrapolate can be configured to use either

- our approximate QE for LIA or
- precise QE provided by Z3
Experimental Evaluation

883 benchmark problems, each containing a transition system specified in Lustre and a single property.

About half are valid, i.e., their property is invariant.

Timeout: 300s of wall clock time.

Hardware: AMD Opteron 24-core 2.1GHz with 32GB RAM.
Precise vs. Approximate QE in Kind 2
Precise QE on Implicants
Kind 2 vs. Kind 1 with Invariants

The diagram shows a scatter plot comparing the runtime of Kind 1 invariant generation to the runtime of Kind 2 assertion QE + invariant generation. The x-axis represents the runtime of Kind 2 assertion QE + invariant generation (s), while the y-axis represents the runtime of Kind 1 invariant generation (s). The data points are marked with plus signs (+) for valid data and crosses (×) for invalid data. The line indicates where the two runtimes would be equal.
Kind 2 vs. Z3’s PDR
Conclusions

General version of IC3 procedure applying beyond propositional logic

A QE-based method for generalizing induction counterexamples for frame refinement

Explicit use of the counterexamples to guide approximate QE

Developed simple under-approximate QE method for LIA IC3 procedure and QE mentor implemented within a new, multi-engine version of Kind model checker

Implementation competitive with other IC3-based system for same logic
Future Work

• Develop and integrate approximate QE methods for logics besides LIA

• Developing methods akin to ternary simulation in the propositional case to generalize approximate QE further

• In general, find new methods to weaken refinement lemmas to include more reachable states so as to enable or accelerate convergence in logics of interest