CS:4980 Topics in Computer Science II

Introduction to Automated Reasoning

DPLL and CDCL

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The University of Iowa

Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, **Emina Torlak** at the University of Washington, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

Plan

- DPLL
    - Abstract DPLL
- CDCL (DP Chapter 2)
    - Abstract CDCL
    - Implication graphs

The Original DPLL Procedure

Modern SAT solvers are based on an extension of the DPLL procedure

DPLL tries to build incrementally a satisfying assignment M for a clause set Δ

M is grown by

- deducing the truth value of a literal from M and Δ, or
- guessing a truth value

If a wrong guess for a literal leads to an inconsistency,
the procedure backtracks and tries the opposite value

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DPLL as a Proof System

To facilitate a deeper look at DPLL, we present it as a proof system: *Abstract DPLL*

The proof system is a re-elaboration of those in [1,2]

[1] Nieuwenhuis et al, “Solving SAT and SAT Modulo Theories: from an Abstract Davis-Putnam-Logemann-Loveland Procedure to DPLL(T).”, Journal of the ACM, 53(6).

[2] Krstić and Goel, “Architecting Solvers for SAT Modulo Theories: Nelson-Oppen with DPLL.”, FroCos 2007.

Abstract DPLL: A Proof System for DPLL

States:

UNSAT $\quad\quad \langle M, \Delta \rangle$

where

- M is a *sequence of literals* and *decision points* $\bullet$ denoting a partial variable assignment
- Δ is a *set of clauses* denoting a CNF formula

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Note: When convenient, we treat M as a set

Provided M contains no complementary literals it determines the assignment

$$v_M(p) = \begin{cases} \text{true} & \text{if } p \in M \\ \text{false} & \text{if } \neg p \in M \\ \text{undef} & \text{otherwise} \end{cases}$$

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Notation: If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level i* of M
- $M^{[i]}$ denotes the subsequence $M_0 \bullet \cdots \bullet M_i$, from decision level 0 to decision level i

Abstract DPLL: A Proof System for DPLL

States:

$$\textsf{UNSAT} \qquad \langle M, \Delta \rangle$$

Initial state:

- $\langle (), \Delta_0 \rangle$, where Δ_0 is to be checked for satisfiability

Abstract DPLL: A Proof System for DPLL

States:

$\textsc{unsat}$ $\quad\quad$ $\langle M, \Delta \rangle$

Initial state:

- $\langle (), \Delta_0 \rangle$, where Δ_0 is to be checked for satisfiability

Expected final states:

- $\textsc{unsat}$ if Δ_0 is unsatisfiable
- $\langle M, \Delta_n \rangle$ otherwise, where Δ_n is equisatisfiable with Δ_0 and satisfied by M

Some clause terminology

Given a *partial assignment*: $v := \{ p_1 \mapsto \text{true}, p_2 \mapsto \text{false}, p_4 \mapsto \text{true} \}$

- clause $\{p_1, p_3, \neg p_4\}$ is *satisfied* by v
- clause $\{\neg p_1, p_2\}$ is *conflicting* with v
- clause $\{\neg p_1, p_3, \neg p_4\}$ is *unit* in v
- clause $\{\neg p_1, p_3, p_5\}$ is *unresolved* by v
- variable p_1 is *assigned* in v
- variable p_3 is *unassigned* in v

Abstract DPLL proof rules: extending the assignment

$$\textsc{Propagate}\ \frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathsf{M} \qquad l, \overline{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$$

Deduce the value of unassigned literal in unit clauses

$$\textsc{Pure}\ \frac{l \text{ literal of } \Delta \qquad \overline{l} \text{ not literal of } \Delta \qquad l, \overline{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$$

Make a pure literal true

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Deduce the value of unassigned literal in unit clauses

The clause $\{l_1, \cdots, l_n, l\}$ is the *antecedent clause* of l, denoted by $\mathrm{Antecedent}(l)$

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Abstract DPLL proof rules: extending the assignment

$$\textsc{Decide}\ \frac{l \in \texttt{Lits}(\Delta) \qquad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

Guess a truth value for an unassigned literal

Notation: $\texttt{Lits}(\Delta) := \{l \mid l \text{ literal of } \Delta\} \cup \{\bar{l} \mid l \text{ literal of } \Delta\}$

l is a *decision literal* of the new M

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Abstract DPLL proof rules: repairing the assignment

$$\textsc{Backtrack}\ \frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \mathrm{M} = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{\mathrm{M} := M_1\, \bar{l}}$$

There is a conflicting clause and a decision point to backtrack to
Backtrack over last decision point and add complement of decision literal

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Note: Premise $\bullet \notin N$ enforces chronological backtracking

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There is a conflicting clause and a decision point to backtrack to
Backtrack over last decision point and add complement of decision literal

$$\textsc{Fail}\ \frac{\{l_1,\ldots,l_n\} \in \Delta \qquad \bar{l}_1,\ldots,\bar{l}_n \in \mathrm{M} \qquad \bullet \notin \mathrm{M}}{\mathrm{UNSAT}}$$

There is a conflicting clause and no decision points to backtrack to
Conclude that clause set is unsatisfiable

Abstract DPLL proof rules

$$\textsc{Propagate}\ \frac{\{l_1,\ldots,l_n,l\}\in\Delta \qquad \bar{l}_1,\ldots,\bar{l}_n\in \mathrm{M} \qquad l,\bar{l}\notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

$$\textsc{Decide}\ \frac{l\in \mathtt{Lits}(\Delta) \qquad l,\bar{l}\notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\bullet l}$$

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This proof system captures the main steps of the DPLL procedure

Note: There are no rules to update Δ, the set of clauses
Such rules are present in CDCL, as we will see

Abstract DPLL proof rules

Propagate $\dfrac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \qquad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$

Decide $\dfrac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$

Pure $\dfrac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$

Fail $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \qquad \bullet \notin \mathsf{M}}{\mathsf{UNSAT}}$

Backtrack $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \qquad \mathsf{M} = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{\mathsf{M} := M_1\, \bar{l}}$

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DPLL derivation example

$\Delta_0 := \{\ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}\ \}$ **Note:** we abbreviate p_n as n

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Pure
$4 \bullet 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Decide
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$4\ \neg 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4\ \neg 1\ \neg 2\ \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
	UNSAT	by Fail

Propagate $\dfrac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$

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Fail $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \bullet \notin \mathrm{M}}{\mathrm{UNSAT}}$

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Fail $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \bullet \notin \mathrm{M}}{\mathrm{UNSAT}}$

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$4\ \neg 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by BACKTRACK
$4\ \neg 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4\ \neg 1\ \neg 2\ \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
	UNSAT	by FAIL

PROPAGATE
$$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

DECIDE
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PURE
$$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

FAIL
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \bullet \notin \mathrm{M}}{\mathrm{UNSAT}}$$

BACKTRACK
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \mathrm{M} = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{\mathrm{M} := M_1\, \bar{l}}$$

DPLL derivation example

$\Delta_0 := \{\ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}\ \}$ **Note:** we abbreviate p_n as n

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PURE
$4 \bullet 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by DECIDE
$4 \bullet 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4 \bullet 1\ \neg 2\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4\ \neg 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by BACKTRACK
$4\ \neg 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4\ \neg 1\ \neg 2\ \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
	UNSAT	by FAIL

$$\textsc{Propagate}\ \frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

$$\textsc{Decide}\ \frac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M} \bullet l}$$

$$\textsc{Pure}\ \frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

$$\textsc{Fail}\ \frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \bullet \notin \mathrm{M}}{\mathrm{UNSAT}}$$

$$\textsc{Backtrack}\ \frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \mathrm{M} = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{\mathrm{M} := M_1\, \bar{l}}$$

DPLL derivation example

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$ **Note:** we abbreviate p_n as n

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Pure
$4 \bullet 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Decide
$4 \bullet 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4 \bullet 1\ \neg 2\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4\ \neg 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Backtrack
$4\ \neg 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4\ \neg 1\ \neg 2\ \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
	UNSAT	by Fail

Propagate $$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

Decide $$\frac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M} \bullet l}$$

Pure $$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

Fail $$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \bullet \notin \mathrm{M}}{\mathrm{UNSAT}}$$

Backtrack $$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad \mathrm{M} = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{\mathrm{M} := M_1\, \bar{l}}$$

DPLL derivation example

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$ **Note:** we abbreviate p_n as n

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PURE
$4 \bullet 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by DECIDE
$4 \bullet 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4 \bullet 1\ \neg 2\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4\ \neg 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by BACKTRACK
$4\ \neg 1\ \neg 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4\ \neg 1\ \neg 2\ \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
	UNSAT	by FAIL

PROPAGATE $\dfrac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$

DECIDE $\dfrac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$

PURE $\dfrac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$

FAIL $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$

BACKTRACK $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$

The DPLL proof system

$$\textsc{Propagate}\ \frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathrm{M} \qquad l, \overline{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

$$\textsc{Pure}\ \frac{l \text{ literal of } \Delta \qquad \overline{l} \text{ not literal of } \Delta \qquad l, \overline{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

$$\textsc{Decide}\ \frac{l \text{ or } \overline{l} \text{ occurs in } \Delta \qquad l, \overline{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M} \bullet l}$$

$$\textsc{Backtrack}\ \frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathrm{M} \qquad \mathrm{M} = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{\mathrm{M} := M_1\, \overline{l}}$$

$$\textsc{Fail}\ \frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathrm{M} \qquad \bullet \notin \mathrm{M}}{\mathrm{UNSAT}}$$

DPLL derivation exercise

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Pure**
$4 \bullet \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Decide**
$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4 \bullet \neg 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Backtrack**
$4\ 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
	UNSAT	by **Fail**

Propagate
$$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$$

Decide
$$\frac{l \in \texttt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$$

Pure
$$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$$

Fail
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$$

Backtrack
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$$

DPLL derivation exercise

$\Delta_0 := \{\, \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \,\}$

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Pure**
$4 \bullet \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Decide**
$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4 \bullet \neg 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Backtrack**
$4\ 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
	UNSAT	by **Fail**

Propagate
$$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$$

Decide
$$\frac{l \in \texttt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$$

Pure
$$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$$

Fail
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$$

Backtrack
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$$

DPLL derivation exercise

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Pure**
$4 \bullet \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Decide**
$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4 \bullet \neg 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Backtrack**
$4\ 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
	UNSAT	by **Fail**

Propagate $\dfrac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$

Decide $\dfrac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$

Pure $\dfrac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$

Fail $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$

Backtrack $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$

DPLL derivation exercise

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Pure**
$4 \bullet \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Decide**
$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4 \bullet \neg 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Backtrack**
$4\ 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
$4\ 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by **Propagate**
	UNSAT	by **Fail**

Propagate
$$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$$

Decide
$$\frac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$$

Pure
$$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$$

Fail
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$$

Backtrack
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$$

DPLL derivation exercise

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PURE
$4 \bullet \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by DECIDE
$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4 \bullet \neg 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by BACKTRACK
$4\ 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
$4\ 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by PROPAGATE
	UNSAT	by FAIL

PROPAGATE
$$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$$

DECIDE
$$\frac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$$

PURE
$$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$$

FAIL
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$$

BACKTRACK
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$$

DPLL derivation exercise

$\Delta_0 := \{\ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}\ \}$

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Pure
$4 \bullet \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Decide
$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4 \bullet \neg 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Backtrack
$4\ 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4\ 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
	UNSAT	by Fail

Propagate
$$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$$

Decide
$$\frac{l \in \texttt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$$

Pure
$$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$$

Fail
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$$

Backtrack
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$$

DPLL derivation exercise

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$

M	Δ	Justification
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4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Pure
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$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4 \bullet \neg 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Backtrack
$4\ 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
$4\ 3\ 2\ 1$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
	UNSAT	by Fail

Propagate $\dfrac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$

Decide $\dfrac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$

Pure $\dfrac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$

Fail $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$

Backtrack $\dfrac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$

DPLL derivation exercise

$\Delta_0 := \{ \{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\} \}$

M	Δ	Justification
	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	
4	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Pure
$4 \bullet \neg 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Decide
$4 \bullet \neg 3\ 2$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Propagate
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$4\ 3$	$\{1, \neg 2\}, \{\neg 1, \neg 2\}, \{2, 3\}, \{\neg 3, 2\}, \{1, 4\}$	by Backtrack
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	UNSAT	by Fail

Propagate
$$\frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad l, \bar{l} \notin M}{M := M\, l}$$

Decide
$$\frac{l \in \mathtt{Lits}(\Delta) \qquad l, \bar{l} \notin M}{M := M \bullet l}$$

Pure
$$\frac{l \text{ literal of } \Delta \qquad \bar{l} \text{ not literal of } \Delta \qquad l, \bar{l} \notin M}{M := M\, l}$$

Fail
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad \bullet \notin M}{\text{UNSAT}}$$

Backtrack
$$\frac{\{l_1, \ldots, l_n\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in M \qquad M = M_1 \bullet l\, M_2 \qquad \bullet \notin M_2}{M := M_1\, \bar{l}}$$

Transforming DPLL to Resolution

The search procedure of DPLL can be reduced a posteriori to a resolution proof (a sequence of applications of resolution rules)

DPLL Shortcomings

OK for randomly generated CNFs, but not for practical ones. Why?

- No learning: throws away all work performed to conclude that the current assignment is bad
  Revisits bad partial assignments leading to conflicts due to the same root cause
- Chronological backtracking: backtracks only one level, even if it can be concluded that the current partial assignment became doomed at a lower level
- Naïve decisions: picks an arbitrary variable to branch on
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Conflict-Driven Clause Learning (CDCL)

Learning: Δ is augmented with a conflict clause that summarizes the root cause of the conflict

Non-chronological backtracking: can backtrack several levels, based on the cause of the conflict (*conflict-driven backjumping*)

Decision heuristics: chooses the next literal to add to the current assignment based on the current state of the search

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From DPLL to CDCL Solvers

To model conflict-driven backjumping and learning, we add a third component C to states whose value is either $\texttt{no}$ or a clause C (the *conflict clause*)

States:

$$\text{UNSAT} \qquad \langle M, \Delta, C \rangle$$

Initial state:

- $\langle (), \Delta_0, \texttt{no} \rangle$, where Δ_0 is to be checked for satisfiability

Expected final states:

- UNSAT if Δ_0 is unsatisfiable
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$$\textsc{Conflict}\ \frac{\mathsf{C} = \mathtt{no} \qquad \{l_1, \ldots, l_n\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathsf{M}}{\mathsf{C} := \{l_1, \ldots, l_n\}}$$

There is no conflict clause but a clause of Δ is falsified bt M

So we set C to be that clause

From DPLL to CDCL Solvers

Replace **Backtrack** with three rules:

$$\textsc{Explain}\ \frac{\mathsf{C} = \{l\} \cup D \quad \{l_1, \ldots, l_n, \overline{l}\} \in \Delta \quad \overline{l}_1, \ldots, \overline{l}_n, \overline{l} \in \mathsf{M} \quad \overline{l}_1, \ldots, \overline{l}_n \prec_{\mathsf{M}} \overline{l}}{\mathsf{C} := \{l_1, \ldots, l_n\} \cup D}$$

$l \prec_M l'$ iff l occurs before l' in M

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Δ contains a clause $C' = \{l_1, \ldots, l_n, \bar{l}\}$ such that

1. l is in the conflict clause and is falsified by M
2. $l_1, \cdots, l_n$ are all falsified by M before l

We derive a new conflict clause by resolution of C and C'

From DPLL to CDCL Solvers

Replace **Backtrack** with three rules:

$$\textsc{Backjump}\ \frac{\mathsf{C} = \{l_1, \ldots, l_n, l\} \qquad \mathtt{lev}(\overline{l_1}), \ldots, \mathtt{lev}(\overline{l_n}) \leq i < \mathtt{lev}(\overline{l})}{\mathsf{M} := \mathsf{M}^{[i]}l \qquad \mathsf{C} := \mathtt{no}}$$

From DPLL to CDCL Solvers

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To compute the level to backjump to:

1. find the literal $\overline{l} \in C$ that was assigned last
2. choose a level i smaller than $\texttt{lev}(\overline{l})$ but not smaller than the levels of the other literals in C

Backtrack to level i and add l to it

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Note: $\mathtt{lev}(l) = n$ iff l occurs in decision level n of M

From DPLL to CDCL Solvers

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Note: The rules maintain the invariant: $\Delta \models C$ and $M \models \neg C$ when $C \neq \mathtt{no}$

From DPLL to CDCL Solvers

Modify **Fail** to

$$\textsc{Fail}\ \frac{C \neq \mathtt{no} \qquad \bullet \notin M}{\mathsf{UNSAT}}$$

C contains a conflict clause but there are no decision points to backjump over

Conclude that Δ is unsatisfiable

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CDCL derivation example

$\Delta := \{\, C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \,\}$

M	Δ	C	rule	
	Δ	no		
1	Δ	no	Propagate	
1 2	Δ	no	Propagate	
1 2 • 3	Δ	no	Decide	
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1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict	
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5	
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4	
1 2 ¬5	Δ	no	Backjump	
1 2 ¬5 • 3	Δ	no	Decide	
1 2 ¬5 • 3 4	Δ	no	Propagate	SAT!

CDCL derivation exan

$$\textsc{Propagate}\ \frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \bar{l}_1, \ldots, \bar{l}_n \in \mathrm{M} \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M}\, l}$$

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1 2 ¬5	Δ	no	Backjump
1 2 ¬5 • 3	Δ	no	Decide
1 2 ¬5 • 3 4	Δ	no	Propagate SAT!

CDCL derivation exan

$$\text{PROPAGATE}\ \frac{\{l_1,\ldots,l_n,l\}\in\Delta \qquad \overline{l}_1,\ldots,\overline{l}_n\in M \qquad l,\overline{l}\notin M}{M := M\,l}$$

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CDCL derivation exam

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1 2 • 3 4 • 5 ¬6 7	Δ	no	Propagate
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4
1 2 ¬5	Δ	no	Backjump
1 2 ¬5 • 3	Δ	no	Decide
1 2 ¬5 • 3 4	Δ	no	Propagate — SAT!

$C = \{\neg 7\} \cup \{\neg 2, \neg 5, 6\} \quad \{\neg 1, \neg 5, 7\} \in \Delta \quad 1, 5 \prec_M 7 \quad \Longrightarrow \quad C = \{\neg 1, \neg 5\} \cup \{\neg 2, \neg 5, 6\} = \{\neg 1, \neg 2, \neg 5, 6\}$

CDCL derivation exan

$$\textsc{Explain}\ \frac{C = \{l\} \cup D \quad \{l_1, \ldots, l_n, \bar{l}\} \in \Delta \quad \bar{l}_1, \ldots, \bar{l}_n, \bar{l} \in M \quad \bar{l}_1, \ldots, \bar{l}_n \prec_M \bar{l}}{C := \{l_1, \ldots, l_n\} \cup D}$$

$\Delta := \{\, C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \,\}$

M	Δ	C	rule
	Δ	no	
1	Δ	no	Propagate
$1\ 2$	Δ	no	Propagate
$1\ 2 \bullet 3$	Δ	no	Decide
$1\ 2 \bullet 3\ 4$	Δ	no	Propagate
$1\ 2 \bullet 3\ 4 \bullet 5$	Δ	no	Decide
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6$	Δ	no	Propagate
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	no	Propagate
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4
$1\ 2\ \neg 5$	Δ	no	Backjump
$1\ 2\ \neg 5 \bullet 3$	Δ	no	Decide

$$C = \{6\} \cup \{\neg 1, \neg 2, \neg 5\} \quad \{\neg 5, \neg 6\} \in \Delta \quad 5 \prec_M \neg 6 \quad \Longrightarrow \quad C = \{\neg 1, \neg 2, \neg 5\} \cup \{\neg 5\} = \{\neg 1, \neg 2, \neg 5\}$$

CDCL derivation exan

$$\textsc{Backjump}\ \frac{C = \{l_1, \ldots, l_n, l\} \qquad \mathtt{lev}(\bar{l}_1), \ldots, \mathtt{lev}(\bar{l}_n) \leq i < \mathtt{lev}(\bar{l})}{M := M^{[i]}l \qquad C := \mathtt{no}}$$

$\Delta := \{\, C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \,\}$

M	Δ	C	rule
	Δ	no	
1	Δ	no	Propagate
$1\ 2$	Δ	no	Propagate
$1\ 2 \bullet 3$	Δ	no	Decide
$1\ 2 \bullet 3\ 4$	Δ	no	Propagate
$1\ 2 \bullet 3\ 4 \bullet 5$	Δ	no	Decide
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6$	Δ	no	Propagate
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	no	Propagate
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4
$1\ 2\ \neg 5$	Δ	no	Backjump
$1\ 2\ \neg 5 \bullet 3$	Δ	no	Decide

$\mathtt{lev}(1) = \mathtt{lev}(2) = 0 \quad \mathtt{lev}(5) = 2 \quad \Longrightarrow$ backtrack to $M^{[0]}$

(could have backtracked to $M^{[1]}$ as well)

CDCL derivation exan

$$\textsc{Decide}\ \frac{l \in \texttt{Lits}(\Delta) \qquad l, \bar{l} \notin \mathrm{M}}{\mathrm{M} := \mathrm{M} \bullet l}$$

$\Delta := \{\, C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \,\}$

M	Δ	C	rule
	Δ	no	
1	Δ	no	PROPAGATE
$1\ 2$	Δ	no	PROPAGATE
$1\ 2 \bullet 3$	Δ	no	DECIDE
$1\ 2 \bullet 3\ 4$	Δ	no	PROPAGATE
$1\ 2 \bullet 3\ 4 \bullet 5$	Δ	no	DECIDE
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6$	Δ	no	PROPAGATE
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$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	CONFLICT
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	EXPLAIN w. C_5
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5\}$	EXPLAIN w. C_4
$1\ 2\ \neg 5$	Δ	no	BACKJUMP
$1\ 2\ \neg 5 \bullet 3$	Δ	no	DECIDE
$1\ 2\ \neg 5 \bullet 3\ 4$	Δ	no	PROPAGATE SAT!

CDCL derivation exan

$$\textsc{Propagate}\ \frac{\{l_1,\ldots,l_n,l\} \in \Delta \qquad \bar{l}_1,\ldots,\bar{l}_n \in M \qquad l,\bar{l} \notin M}{M := M\, l}$$

$\Delta := \{\, C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \,\}$

M	Δ	C	rule
	Δ	no	
1	Δ	no	Propagate
1 2	Δ	no	Propagate
1 2 • 3	Δ	no	Decide
1 2 • 3 4	Δ	no	Propagate
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1 2 • 3 4 • 5 ¬6	Δ	no	Propagate
1 2 • 3 4 • 5 ¬6 7	Δ	no	Propagate
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4
1 2 ¬5	Δ	no	Backjump
1 2 ¬5 • 3	Δ	no	Decide
1 2 ¬5 • 3 4	Δ	no	Propagate SAT!

Conflict Analysis

CDCL systems learn new clause during search with the goal of

blocking partial assignments leading to the current conflict

A common strategy is to learn an *asserting clause*, a conflict clause that will become unit after backtracking

One way to illustrate different conflict analysis strategies is through implication graphs

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One way to illustrate different conflict analysis strategies is through implication graphs

Conflict Analysis: Implication Graph

An *implication graph* is a labeled directed acyclic graph $G(V, E)$, where:

V collects the literals in the current partial assignment M

- each $l \in V$ is labeled with its decision level in M

$E = \{(l, l') \mid l, l' \in V, \bar{l} \in \mathrm{Antecedent}(l')\}$

- each edge (l, l') is labeled with $C = \mathrm{Antecedent}(l')$

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- each edge (l, l') is labeled with $C = \mathrm{Antecedent}(l')$

G is a *conflict graph* if it also contains

- a single conflict node $\perp$
- $\perp$'s incoming edges are $\{ (l, \perp) \mid \bar{l} \in C \}$ for some falsified clause C
- those edges are labeled with C

Revisiting CDCL example with an implication graph

$\Delta := \{ C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \}$

M	Δ	C	rule
	Δ	no	
1	Δ	no	Propagate
1 2	Δ	no	Propagate
1 2 • 3	Δ	no	Decide
1 2 • 3 4	Δ	no	Propagate
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1 2 ¬5	Δ	no	Backjump

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1 2 ¬5	Δ	no	Backjump

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1 2 ¬5	Δ	no	Backjump

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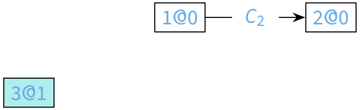
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4
1 2 ¬5	Δ	no	Backjump



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$1\ 2 \bullet 3$	Δ	no	DECIDE
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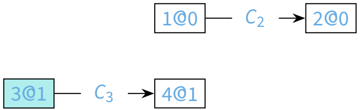
$1\ 2\ \neg 5$	Δ	no	BACKJUMP



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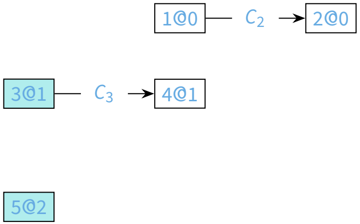
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	no	Propagate
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4
$1\ 2\ \neg 5$	Δ	no	Backjump



Revisiting CDCL example with an implication graph

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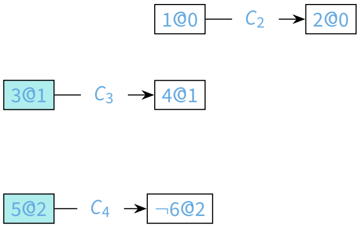
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4
$1\ 2\ \neg 5$	Δ	no	Backjump



Revisiting CDCL example with an implication graph

$\Delta := \{ C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \}$

M	Δ	C	rule
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1	Δ	no	PROPAGATE
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$1\ 2 \bullet 3$	Δ	no	DECIDE
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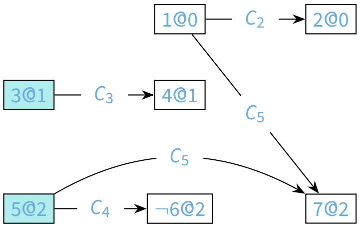
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	CONFLICT
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	EXPLAIN w. C_5
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5\}$	EXPLAIN w. C_4
$1\ 2\ \neg 5$	Δ	no	BACKJUMP



Revisiting CDCL example with an implication graph

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	Δ	no	
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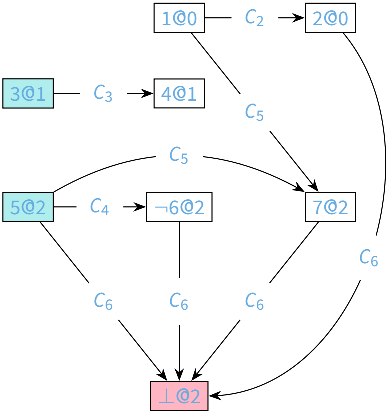
1 2 ¬5	Δ	no	Backjump



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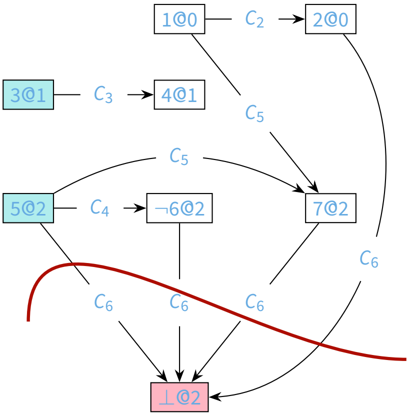
1 2 ¬5	Δ	no	Backjump



Any *separating cut* that breaks all paths from root nodes to the conflict node, with roots on the *reasons side* and conflict node on the *conflict side*, determines a conflict clause

Revisiting CDCL example with an implication graph

$\Delta := \{\, C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \,\}$

M	Δ	C	rule
	Δ	no	
1	Δ	no	PROPAGATE
1 2	Δ	no	PROPAGATE
1 2 • 3	Δ	no	DECIDE
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1 2 • 3 4 • 5	Δ	no	DECIDE
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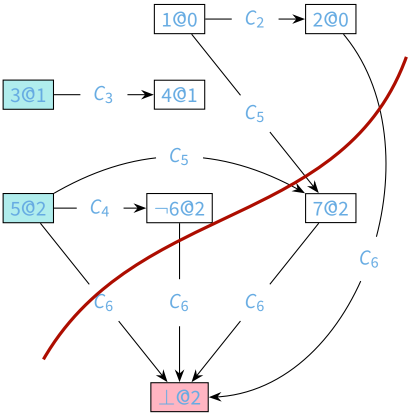
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	EXPLAIN w. C_5
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5\}$	EXPLAIN w. C_4
1 2 ¬5	Δ	no	BACKJUMP



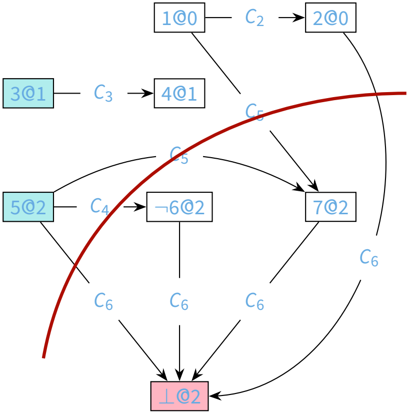
EXPLAIN can be viewed as picking a literal l in the conflict clause C, and replacing C with the l-resolvent of C and $\mathrm{Antecedent}(\bar{l})$

In this case, $l = \neg 7$ and $\mathrm{Antecedent}(\bar{l}) = C_5$

Revisiting CDCL example with an implication graph

$\Delta := \{\, C_1 : \{1\}, C_2 : \{\neg 1, 2\}, C_3 : \{\neg 3, 4\}, C_4 : \{\neg 5, \neg 6\}, C_5 : \{\neg 1, \neg 5, 7\}, C_6 : \{\neg 2, \neg 5, 6, \neg 7\} \,\}$

M	Δ	C	rule
	Δ	no	
1	Δ	no	Propagate
1 2	Δ	no	Propagate
1 2 • 3	Δ	no	Decide
1 2 • 3 4	Δ	no	Propagate
1 2 • 3 4 • 5	Δ	no	Decide
1 2 • 3 4 • 5 ¬6	Δ	no	Propagate
1 2 • 3 4 • 5 ¬6 7	Δ	no	Propagate
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	Conflict
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	Explain w. C_5
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5\}$	Explain w. C_4



Explain can be viewed as picking a literal l in the conflict clause C, and replacing C with the l-resolvent of C and $\mathrm{Antecedent}(\bar{l})$

In this case, $l = 6$ and $\mathrm{Antecedent}(\bar{l}) = C_4$

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$$\textsc{Backjump}\ \frac{C = \{l_1, \ldots, l_n, l\} \qquad \mathtt{lev}(\bar{l}_1), \ldots, \mathtt{lev}(\bar{l}_n) \leq i < \mathtt{lev}(\bar{l})}{M := M^{[i]} l \qquad C := \mathtt{no}}$$

$1\ 2$	Δ	no	PROPAGATE
$1\ 2 \bullet 3$	Δ	no	DECIDE
$1\ 2 \bullet 3\ 4$	Δ	no	PROPAGATE
$1\ 2 \bullet 3\ 4 \bullet 5$	Δ	no	DECIDE
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6$	Δ	no	PROPAGATE
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	no	PROPAGATE
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	CONFLICT
$1\ 2 \bullet 3\ 4 \bullet 5\ \neg 6\ 7$	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	EXPLAIN w. C_5
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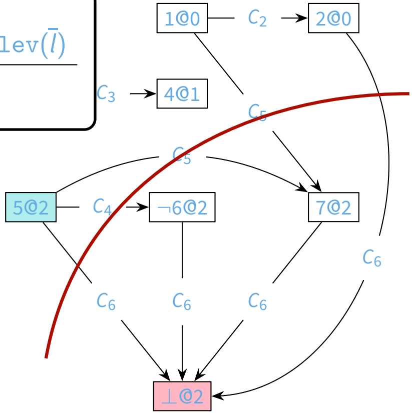
$1\ 2\ \neg 5$	Δ	no	BACKJUMP



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A *Unique Implication Point (UIP)* is any node other than $\bot$ that is on all paths from the current decision node to $\bot$

A *first UIP* is a UIP that is closest to the conflict node

In this case, 5@2 is the only UIP and thus also the first UIP

1 2 • 3 4	Δ	no	PROPAGATE
1 2 • 3 4 • 5	Δ	no	DECIDE
1 2 • 3 4 • 5 ¬6	Δ	no	PROPAGATE
1 2 • 3 4 • 5 ¬6 7	Δ	no	PROPAGATE
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 2, \neg 5, 6, \neg 7\}$	CONFLICT
1 2 • 3 4 • 5 ¬6 7	Δ	$\{\neg 1, \neg 2, \neg 5, 6\}$	EXPLAIN w. C_5
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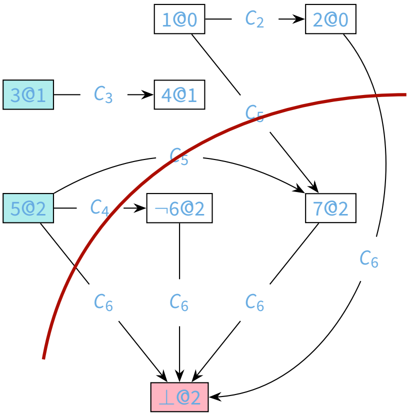
1 2 ¬5	Δ	no	BACKJUMP



From DPLL to full CDCL Solvers

Also add

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$$\textsc{Learn}\ \frac{D \text{ is a clause} \qquad \Delta \models D \qquad D \notin \Delta}{\Delta := \Delta \cup \{D\}}$$

Can be applied to any clause entailed by Δ

In particular, to any conflict clause C when present (because then $\Delta \models C$)

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Can be applied to any clause entailed by Δ

In particular, to any conflict clause C when present (because then $\Delta \models C$)

The learned clause D is called a *lemma*

From DPLL to full CDCL Solvers

Also add

$$\textsc{Forget}\ \frac{C = \texttt{no} \qquad \Delta = \Delta' \cup \{C\} \qquad \Delta' \models C}{\Delta := \Delta'}$$

Learning can quickly add millions of clauses to Δ

So it is useful to be able to delete redundant clauses that might not be useful anymore

From DPLL to full CDCL Solvers

Also add

$$\textsc{Restart}\ \frac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad C := \mathtt{no}}$$

If we are stuck in a hopeless area of the search space it may be better to just restart

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Note: Restart is not from scratch since propagations at level 0 are maintained, together with all the learned lemmas not eliminated by **Forget**

Learning the First UIP

Empirical studies show it is a good strategy to

- learn a conflict clause C that contains a first UIP for the current decision level
- backjump to the second lowest decision level among C's literals

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That l is a first UIP and the resulting C is an asserting clause

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Learn and **Backjump** could in fact be combined into

$$\textbf{L\&B}\ \frac{\mathsf{C} = \{l_1, \ldots, l_n, l\} \qquad \texttt{lev}(\bar{l}_1), \ldots, \texttt{lev}(\bar{l}_n) \leq i < \texttt{lev}(\bar{l})}{\mathsf{M} := \mathsf{M}^{[i]} \qquad \Delta := \Delta \cup \{C\} \qquad C := \texttt{no}}$$

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Note: We do not need to append l to $M^{[i]}$ as **Propagate** will be able to do that

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Note: The first UIP for a decision level is not necessarily the decision literal d for that level. However, applying **L&B** guarantees that $\Delta \cup M \models \bar{d}$

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- backjump to the second lowest decision level among C's literals

Possible explanations for the empirical results:

- The strategy has a low computational cost, compared with strategies that choose UIPs further away from the conflict
- It still backtracks to the lowest decision level possible

Non-chronological vs. chronological backtracking

Backjumping is not necessarily better than chronological backtracking

See “Chronological Backtracking” by Nadel and Ryvchin, SAT 2018.

Modeling Modern SAT Solvers

At the core, current CDCL SAT solvers are implementations of the proof system with rules

Propagate, **Pure**, **Decide**,

Conflict, **Explain**, **Backjump**, **Fail**

Learn, **Forget**, **Restart**

Basic CDCL := { **Propagate**, **Pure**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail** }

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The Basic CDCL System – Correctness

Irreducible state: state for which no Basic CDCL rules apply

Execution: a (single-branch) derivation tree starting with $M = \emptyset$ and $C = \texttt{no}$

Exhausted execution: execution ending in an irreducible state

Theorem 1 (Refutation Soundness)

For every exhausted execution starting with $\Delta = \Delta_0$ and ending with UNSAT*, the clause set Δ_0 is unsatisfiable*

Theorem 2 (Solution Soundness)

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Every execution in Basic CDCL is finite

Note: This is not so immediate, because of **EXPLAIN** and **BACKJUMP**

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Lemma 3

All clause sets along an execution are equivalent (i.e., satisfied by the same interpretations)

Theorem 4 (Refutation Soundness)

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The CDCL System – Strategies

To ensure termination for the full system,

1. apply at least one Basic CDCL rule between each two **Learn** applications
2. apply **Restart** less and less often

The CDCL System – Strategies

A common basic strategy applies the rules with the following priorities, using a bound n initially set to 0, until an irreducible state is reached:

1. If $n > 0$ conflicts have been found so far, increase n and apply **Restart**
2. If M falsifies a clause and has no decision points, apply **Fail** and stop
3. If M falsifies a clause, apply **Conflict**
   3.1 Apply **Explain** repeatedly
   3.2 Apply **Learn** to the current conflict clause
   3.3 Apply **Backjump**
4. Apply **Propagate** to completion
5. Apply **Decide**

Steps 3.1-3.3 achieve a form of *conflict analysis* and involve some heuristic choices:

1. When to stop applying **Explain** to a conflict?
2. Which level to **Backjump** to?

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The CDCL proof system

$$\textsc{Propagate}\ \frac{\{l_1, \ldots, l_n, l\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in M \qquad l, \overline{l} \notin M}{M := M\, l}$$

$$\textsc{Pure}\ \frac{l \text{ literal of } \Delta \qquad \overline{l} \text{ not literal of } \Delta \qquad l, \overline{l} \notin M}{M := M\, l}$$

$$\textsc{Decide}\ \frac{l \text{ or } \overline{l} \text{ occurs in } \Delta \qquad l, \overline{l} \notin M}{M := M \bullet l}$$

The CDCL proof system (continued)

$$\textsc{Conflict}\ \frac{\mathsf{C} = \texttt{no} \qquad \{l_1, \ldots, l_n\} \in \Delta \qquad \overline{l}_1, \ldots, \overline{l}_n \in \mathsf{M}}{\mathsf{C} := \{l_1, \ldots, l_n\}}$$

$$\textsc{Explain}\ \frac{\mathsf{C} = \{l\} \cup D \quad \{l_1, \ldots, l_n, \overline{l}\} \in \Delta \quad \overline{l}_1, \ldots, \overline{l}_n, \overline{l} \in \mathsf{M} \quad \overline{l}_1, \ldots, \overline{l}_n \prec_{\mathsf{M}} \overline{l}}{\mathsf{C} := \{l_1, \ldots, l_n\} \cup D}$$

$$\textsc{Backjump}\ \frac{\mathsf{C} = \{l_1, \ldots, l_n, l\} \qquad \texttt{lev}(\overline{l}_1), \ldots, \texttt{lev}(\overline{l}_n) \leq i < \texttt{lev}(\overline{l})}{\mathsf{M} := \mathsf{M}^{[i]} l \qquad \mathsf{C} := \texttt{no}}$$

$$\textsc{Fail}\ \frac{\mathsf{C} \neq \texttt{no} \qquad \bullet \notin \mathsf{M}}{\mathsf{UNSAT}}$$

The CDCL proof system (continued)

$$\textsc{Learn}\ \frac{D \text{ is a clause} \qquad \Delta \models D \qquad D \notin \Delta}{\Delta := \Delta \cup \{D\}}$$

$$\textsc{Forget}\ \frac{\mathtt{C} = \mathtt{no} \qquad \Delta = \Delta' \cup \{C\} \qquad \Delta' \models C}{\Delta := \Delta'}$$

$$\textsc{Restart}\ \frac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad C := \mathtt{no}}$$