CS:4980 Topics in Computer Science II

# Introduction to Automated Reasoning

# Decision Procedures for Satisfiability in Propositional Logic

Cesare Tinelli

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The University of Iowa

# Credits

These slides are based on slides originally developed by **Cesare Tinelli** at the University of Iowa, and by **Clark Barrett**, **Caroline Trippel**, and **Andrew (Haoze) Wu** at Stanford University. Adapted by permission.

# Decision procedures for propositional logic

From now on, instead of wffs, we consider only their clausal form (clause sets)

# Decision procedures for propositional logic

**Observe:**

- Each clause $l_1 \vee \cdots \vee l_n$ can be itself regarded as a set, of literals: $\{ l_1, \ldots, l_n \}$
- A set of clauses is satisfiable iff there is an interpretation of its variables that satisfies at least one literal in each clause

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**Example:**

- The clause set $\Delta := \{ p_1 \vee p_3, \neg p_1 \vee p_2 \vee \neg p_3 \}$ can be represented as $\{ \{ p_1, p_3 \}, \{ \neg p_1, p_2, \neg p_3 \} \}$
- $v := \{ p_1 \mapsto \text{true}, p_2 \mapsto \text{true}, p_3 \mapsto \text{false} \}$ is a satisfying assignment for $\Delta$

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# SAT Solver Overview: features

Automated reasoners for the satisfiability problem in PL are called *SAT solvers*

1. **Backtracking search solvers**
   - Traversing and backtracking on a binary tree
   - Sound, complete and terminating
2. **Stochastic search solvers**
   - Solver guesses a full assignment $v$
   - If the set is falsified by $v$, starts to flip values of variables according to some (greedy) heuristic
   - Sound but neither complete nor terminating
   - Nevertheless, quite effective in certain applications

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We focus on backtracking solvers in this course

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# SAT Solver Overview: performance

The SAT problem is hard (NP-complete). How well do SAT solvers do in practice?

- Modern SAT solvers can solve many real-life CNF formulas with hundreds of thousands or even millions of variables in a reasonable amount of time
- There are also instances of problems two orders of magnitude smaller that the same tools cannot solve
- In general, it is very hard to predict which instance is going to be hard to solve, without actually attempting to solve it

SAT portfolio solvers: use machine-learning techniques to extract features of CNF formulas in order to select the most suitable SAT solver for the job

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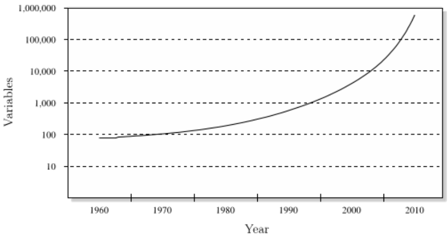


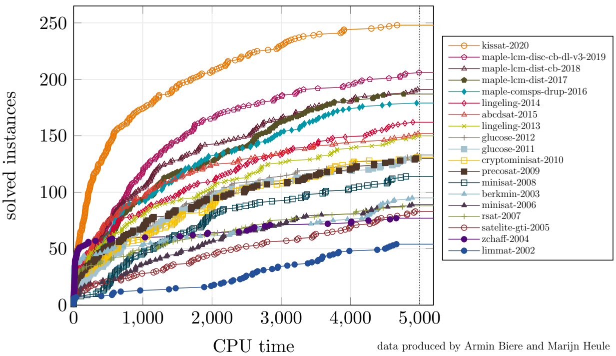


**Left:** Size of industrial clause sets (y-axis) regularly solved by solvers in a few hours each year (x-axis). Instances come from realistic problems like planning or hardware verification

**Right:** Top contenders in SAT solver competitions from 2002 to 2020; each point shows number of solved instances (y-axis) per unit of time (x-axis). Note that no. of instances solved within 20 minutes more than doubled in less than a decade

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Success of SAT solvers can largely be attributed to their ability to:

- Learn from failed assignments
- Prune large parts of the search spaces quickly
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# The DIMACS format

A standard format for clause sets accepted by most modern SAT solvers

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- Comment lines: Start with a lower-case letter *c*
- Problem line: *p cnf <#variables > <#clauses >*
- Clause lines:
  - Each variable is assigned a unique index *i* greater than 0
  - A positive literal is represented by an index
  - A negative literal is represented by the negation of its complement's index
  - A clause is represented as a list of literals separated by white space
  - Value 0 is used to mark the end of a clause

**Example:**

$\{\, p_1 \vee \neg p_3,\ p_2 \vee p_3 \vee \neg p_1 \,\}$

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c example.cnf
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$\{ p_1 \vee \neg p_3,\ p_2 \vee p_3 \vee \neg p_1 \}$ $\Rightarrow$

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# Basic SAT solvers

- 1960: *Davis-Putnam (DP)* algorithm
- 1961: *Davis-Putnam-Logemann-Loveland (DPLL)* algorithm
- 1996: Modern SAT solver based on *Conflict-Driven Clause Learning (CDCL)* derived from DP and DPLL

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# A proof system for clause sets: resolution

There is a refutation sound and complete proof system for clause sets $\Delta$ that consists of just one proof rule!

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{\,p\,\}) \cup (C_2 \setminus \{\,\neg p\,\}) \quad C \notin \Delta}{\Delta \cup \{C\}}$$

Clause $C$ is a *($p$-)resolvent* of $C_1$ and $C_2$, and $p$ is the *pivot*

**Example:** $\Delta := \{\,\{\,p_1, p_3\,\}, \{\,p_2, \neg p_3\,\}\,\}$ has a $p_3$-resolvent: $\{\,p_1, p_2\,\}$

**Note:** if $C$ is a resolvent of $C_1, C_2 \in \Delta$ then $\{C_1, C_2\} \models C$ and so $\Delta \models \Delta \cup \{C\}$

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# Proofs by resolution

**Example:** Prove that the following clause set is unsatisfiable

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- The last clause set is unsatisfiable since it contains the empty clause $\{\,\}$
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# Proofs by resolution

**Example:** Prove that the following clause set is unsatisfiable

$$
\begin{array}{l}
\{\{p_1, p_2\}, \{p_1, \neg p_2\}, \{\neg p_1, p_3\}, \{\neg p_1, \neg p_3\}\} \\
\hline
\{\{p_1, p_2\}, \{p_1, \neg p_2\}, \{\neg p_1, p_3\}, \{\neg p_1, \neg p_3\}, \{p_1\}\} \\
\hline
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# A resolution-based satisfiability proof system

- In addition to the SAT and UNSAT states, we consider states of the form

$$\langle \Delta, \Phi \rangle$$

  with $\Delta$ and $\Phi$ clause sets

- Initial states have the form

$$\langle \Delta_0, \{\} \rangle$$

  where $\Delta_0$ is the clause set to be checked for satisfiability

# A resolution-based satisfiability proof system

We modify the resolution rule **Resolve** as highlighted below and add three more rules

$$\textsc{Resolve}\ \frac{C_1, C_2 \in \Delta \quad p \in C_1 \quad \neg p \in C_2 \quad C = (C_1 \setminus \{p\}) \cup (C_2 \setminus \{\neg p\}) \quad C \notin \Delta \cup \Phi}{\Delta := \Delta \cup \{C\}}$$

$$\textsc{Clash}\ \frac{C \in \Delta \quad p, \neg p \in C}{\Delta := \Delta \setminus \{C\} \quad \Phi := \Phi \cup \{C\}}$$

$$\textsc{Unsat}\ \frac{\{\} \in \Delta}{\text{UNSAT}} \qquad \textsc{Sat}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

This proof system is sound, complete and terminating

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This proof system is sound, complete and terminating

# A resolution-based decision procedure

Given a clause set $\Delta$, apply **Clash** or **Resolve** until either

1. an empty clause is derived (return UNSAT)
2. neither applies (return SAT)

This procedure is terminating and decides the SAT problem

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# Unit resolution

**Notation** If $l$ is a literal and $p$ is its variable, $\overline{l} = \begin{cases} \neg p & \text{if } l = p \\ p & \text{if } l = \neg p \end{cases}$

The *unit resolution rule* is a special case of resolution where one of the resolving clauses is a *unit clause*, i.e., a clause with only one literal

$$\textsc{Unit Resolve}\ \frac{C_1, C_2 \in \Delta \quad C_1 = \{\, l \,\} \quad C_2 = \{\, \overline{l} \,\} \cup D}{\Delta \cup \{\, D \,\}}$$

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A proof system with unit resolution alone is not refutation-complete (consider an unsat $\Delta$ with no unit clauses)

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Modern SAT solvers use unit resolution plus backtracking search for deciding SAT

# Davis-Putnam (DP) procedure

A decision procedure for the SAT problem

First procedure to implement something more sophisticated than truth tables

DP leverages 4 satisfiability-preserving transformations:

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DP leverages 4 satisfiability-preserving transformations:

- Unit propagation
- Pure literal elimination
- Tautology elimination
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The first two transformations reduce the total number of literals in the clause set

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The third transformation reduces the number of clauses

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First procedure to implement something more sophisticated than truth tables

DP leverages 4 satisfiability-preserving transformations:

- Unit propagation
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Repeatedly applying these tranformations, eventually leads to
an empty clause (indicating unsatisfiability) or
an empty clause set (indicating satisfiability)

# DP procedure: unit propagation

Also called the *1-literal rule*

**Premise:** The clause set $\Delta$ contains a unit clause $C = \{l\}$

**Conclusion:**

- Remove all occurrences of $\bar{l}$ from clauses in $\Delta$
- Remove all clauses containing $l$ (including $C$)

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**Justification:** The only way to satisfy $C$ is to make $l$ true; thus, *(i)* $\bar{l}$ cannot be used to satisfy any clause, and *(ii)* any clause containing $l$ is satisfied and can be ignored

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**Example:**

$$\Delta_0 := \{ \{ p_1 \}, \{ p_1, p_4 \}, \{ p_2, p_3, \neg p_1 \} \}$$

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**Premise**: A literal $l$ occurs in $\Delta$ but $\bar{l}$ does not

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# DP procedure: tautology elimination

Also called the *clashing clause rule*

**Premise**: a clause $C \in \Delta$ contains both $p$ and $\neg p$

**Conclusion**: remove $C$ from $\Delta$

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# DP procedure: resolution

Also called the *rule for eliminating atomic formulas*

**Premise:** A variable $p$ occurs in a clause of $\Delta$ and $\neg p$ occurs in another clause

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**Premise**: A variable $p$ occurs in a clause of $\Delta$ and $\neg p$ occurs in another clause

**Conclusion**:

- Let $P$ be the set of clauses in $\Delta$ where $p$ occurs positively and let $N$ be the set of clauses in $\Delta$ where $p$ occurs negatively
- Replace the clauses in $P$ and $N$ with those obtained by resolution on $p$ using all pairs of clauses from $P \times N$

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$\Delta := \{\, \{\, p_1, p_2, p_3 \,\}, \{\, p_2, \neg p_3, \neg p_6 \,\}, \{\, \neg p_2, p_5 \,\} \,\}$

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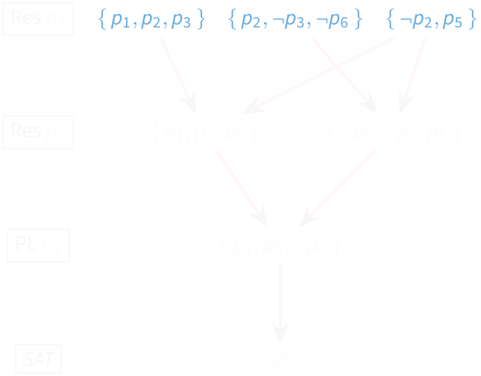

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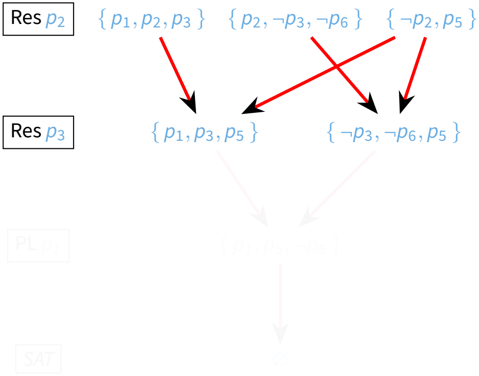

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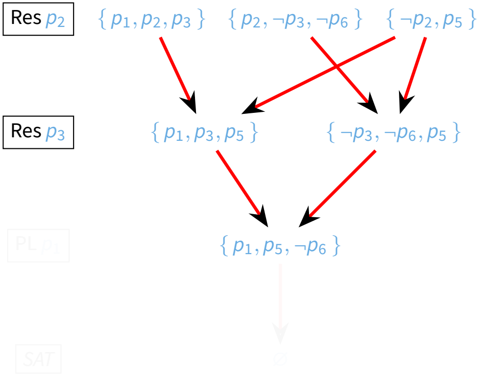

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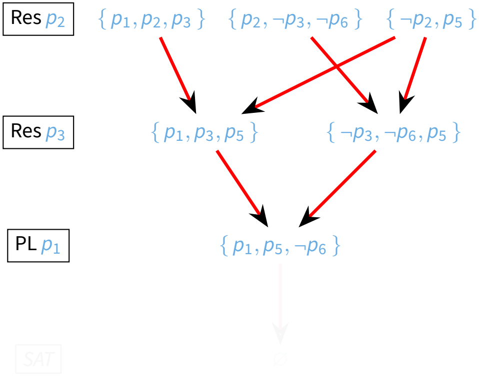

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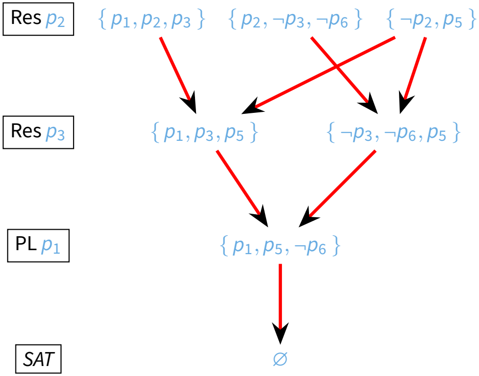

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$\Delta := \{ \{ p_1, p_2 \}, \{ p_1, \neg p_2 \}, \{ \neg p_1, p_3 \}, \{ \neg p_1, \neg p_3 \} \}$

$\{ p_1, p_2 \} \; \{ p_1, \neg p_2 \} \; \{ \neg p_1, p_3 \} \; \{ \neg p_1, \neg p_3 \}$

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$\boxed{\mathit{UNSAT}}$ $\{ \}$

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$\{\, p_1 \,\}\quad \{\, \neg p_1, p_3 \,\}\{\, \neg p_1, \neg p_3 \,\}$

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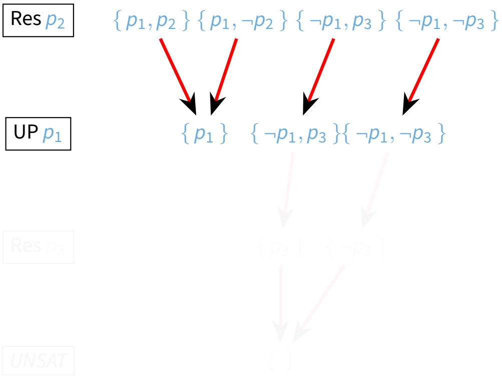

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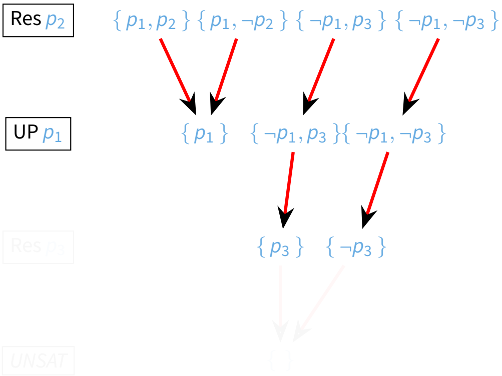

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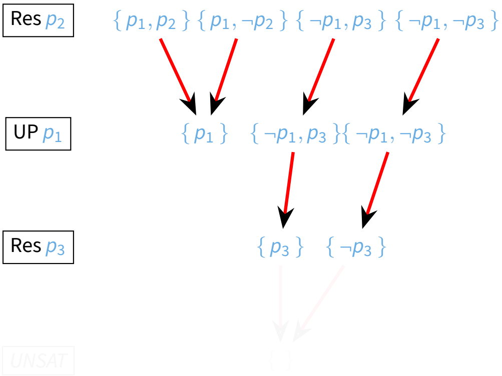
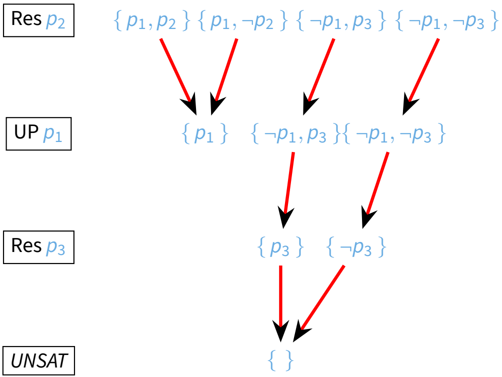


# DP Example 2

$\Delta := \{ \{ p_1, p_2 \}, \{ p_1, \neg p_2 \}, \{ \neg p_1, p_3 \}, \{ \neg p_1, \neg p_3 \} \}$



# From DP to DPLL

The resolution transformation does not increase the number of variables
However, it may increase the size of the clause set

**Question**: If a variable appears positively in 3 clauses and negatively in 3 clauses, how many clauses after applying resolution? 9

In the worst case, the resolution transformation can cause a quadratic expansion each time it is applied

For large enough formulas, this can quickly exhaust the available memory

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The DPLL procedure improves on DP by replacing resolution with *splitting*:

- Let $\Delta$ be the input clause set
- Arbitrarily choose a literal $l$ occurring in $\Delta$
- Recursively check the satisfiability of $\Delta \cup \{ \{ l \} \}$
  - If result is SAT, return SAT
  - Otherwise, recursively check the satisfiability of $\Delta \cup \{ \{ \neg l \} \}$ and return that result

We will discuss DPLL in more detail next time

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