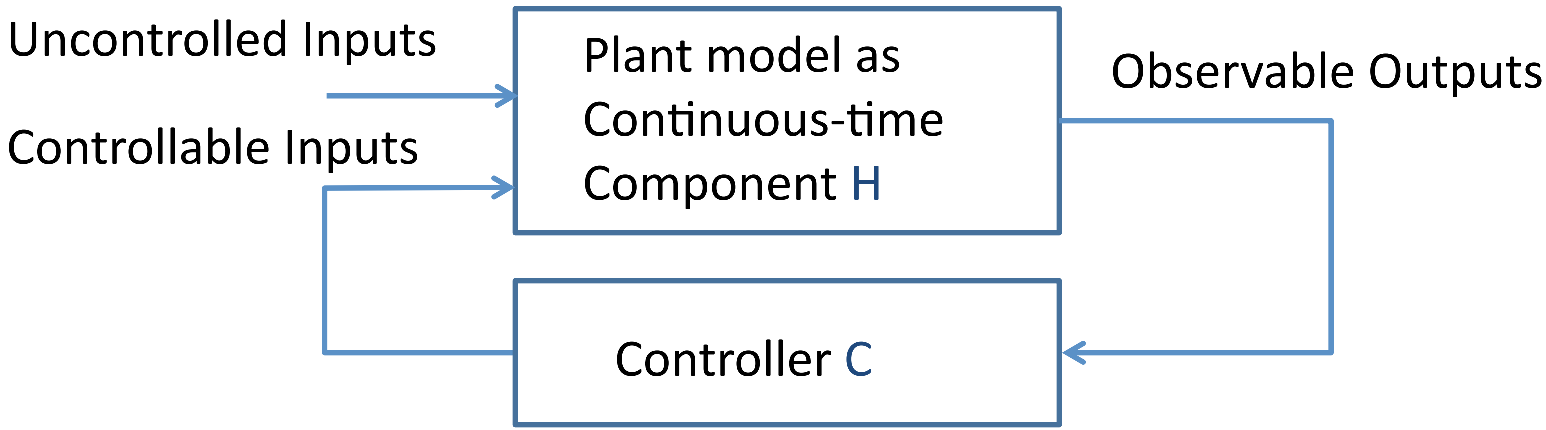
CS:4980
Foundations of Embedded Systems

Dynamical Systems
Part III

Control Design Problem



- We want to design a controller C so that C || H is stable
- Is there a mathematical way to check when a system is stable?
- Is there in fact a way to design C so that C || H is stable ?
- Yes, if the plant model is linear

Linear Component

A *linear* expression over variables $x_1, x_2, \ldots x_n$ is of the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$$

where $a_1, a_2, \ldots$ are rational constants

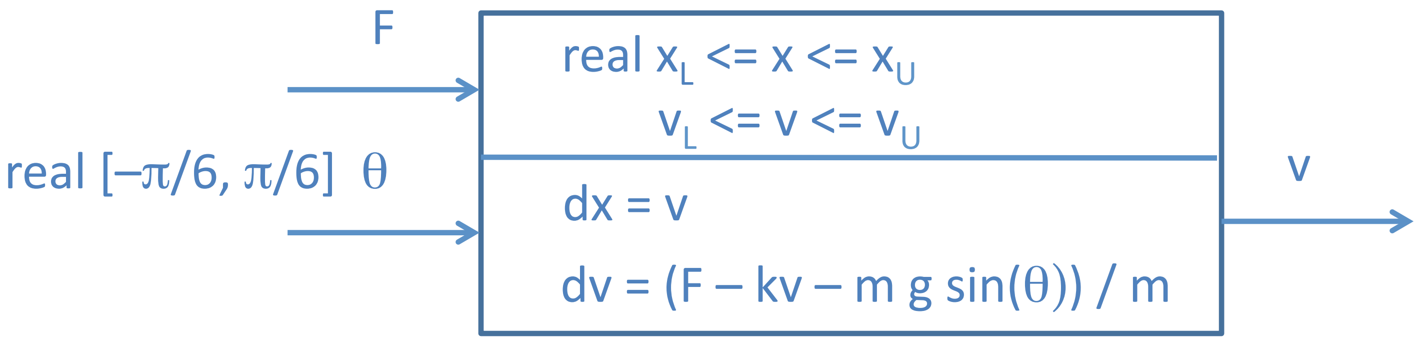
A continuous-time component H with state variables S, input variables I, and output variables O is *linear* if

- for every state variable x, the dynamics is given by $dx = f_x(S,I)$, where f_x is a linear expression
- for every output variable y, y is defined by $y = h_y(S,I)$, where h_y is a linear expression

Examples

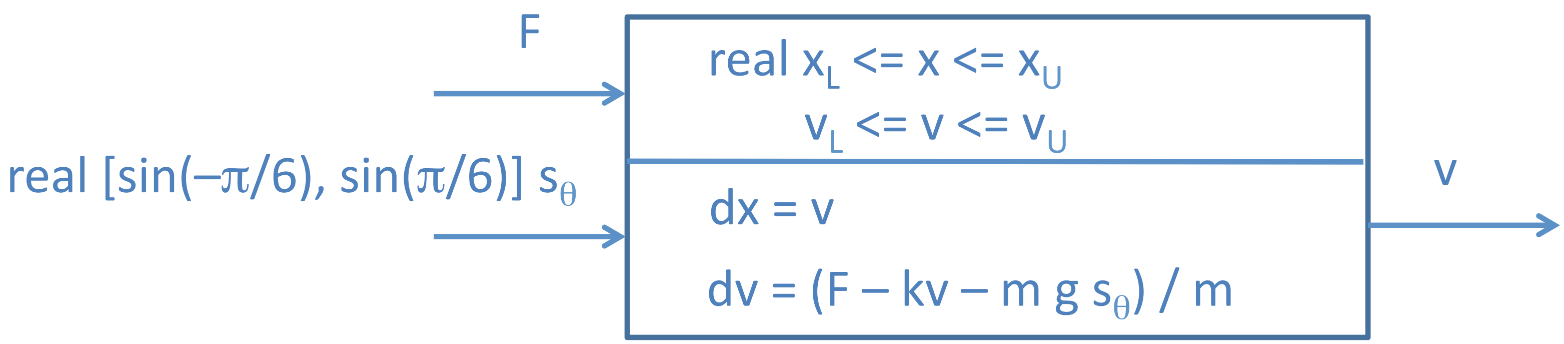
- linear: heatflow, car, helicopter
- nonlinear: pendulum

Continuous-time Component Car2



- Right-hand side of dv equation not linear
- Easy fix: replace disturbance θ by another variable $s_\theta = \sin\theta$

Continuous-time Component Car2



Rewriting to normal form:

$$dx/dt = 0x + 1v + 0F + 0s_\theta$$

$$dv/dt = 0x + (-k/m)v + (1/m)F + (-g)s_\theta$$

$$v = 0x + 1v + 0F + 0s_\theta$$

Matrix-based representation:

$$S = (x \;\; v)^T \quad I = (F \;\; s_\theta)^T \quad O = (v)$$

$$dS/dt = A\,S + B\,I$$

$$O = C\,S + D\,I$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -k/m \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1/m & -g \end{pmatrix}$$

$$C = (0 \;\; 1) \quad D = (0 \;\; 0)$$

(A,B,C,D) Representation of Linear Components

Suppose a linear continuous-time component has

- n state variables $S = \{x_1, x_2, \ldots x_n\}$
- m input variables $I = \{u_1, u_2, \ldots u_m\}$
- k output variables $O = \{y_1, y_2, \ldots y_k\}$

Then the dynamics is given by

$$dS/dt = A\,S + B\,I \quad \text{and} \quad O = C\,S + D\,I$$

where

A is an $n \times n$ matrix   C is a $k \times n$ matrix

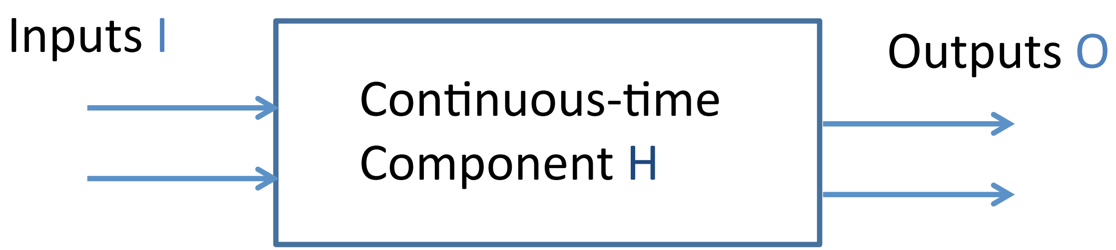
B is an $n \times m$ matrix   D is a $k \times m$ matrix

The rate of change of i-th state variable and the value of j-th output are

$$dx_i/dt = A_{i,1}\,x_1 + A_{i,2}x_2 + \ldots + A_{i,n}\,x_n + B_{i,1}\,u_1 + B_{i,2}u_2 + \ldots + B_{i,m}\,u_m$$

$$y_j = C_{j,1}\,x_1 + C_{j,2}x_2 + \ldots + C_{j,n}\,x_n + D_{j,1}\,u_1 + D_{j,2}u_2 + \ldots + D_{j,m}\,u_m$$

Input-Output Linearity



With a fixed initial state, a continuous-time component H maps input signals **I**(t) to output signals **O**(t)

Theorem: If H is linear, then both of the following hold.

- *Scaling*: If the output response of H to the input signal **I**(t) is **O**(t), then for every constant α, the output response of H to the input signal α **I**(t) is α **O**(t)
- *Additivity*: If the output responses of H to the input signals $\mathbf{I}_1(t)$ and $\mathbf{I}_2(t)$ are $\mathbf{O}_1(t)$ and $\mathbf{O}_2(t)$, then the output response of H to the input signal $(\mathbf{I}_1 + \mathbf{I}_2)(t)$ is $(\mathbf{O}_1 + \mathbf{O}_2)(t)$

Response of Linear Systems

Consider a one-dimensional linear system with no inputs:

$dx/dt = ax$; initial state x_0

Its execution is given by the signal

$\mathbf{x}(t) = x_0 e^{at}$

- Recall that $e^a = 1 + \sum_{n>0} a^n/n!$
- Verify that solution $\mathbf{x}(t)$ satisfies the differential equation
- See textbook on how solution is found

Response of Linear Systems

General Case (no inputs)

- State set S
- Dynamics is given by

  $$dS/dt = A\,S$$

  initial state s_0

- For each input signal $\mathbf{I}(t)$, execution is given by the signal

  $$\mathbf{S}(t) = e^{At}\, s_0$$

  - At = scalar product of A and t
  - $e^M = \mathrm{I} + \sum_{n>0} M^n/n!$
  - I = identity matrix ($\mathrm{I}_{i,j}$ = if $(i = j)$ then 1 else 0)

Response of Linear Systems

General Case (with inputs)

- State set S, input set I
- Dynamics is given by

  $$dS/dt = A\,S + B\,I$$

  initial state s_0

- For each input signal $\mathbf{I}(t)$, execution is given by the signal

  $$\mathbf{S}(t) = e^{At}\,s_0 + \int_0^t \left(e^{A(t-\tau)}\,B\,\mathbf{I}(\tau)\,d\tau\right)$$

Matrix Exponential

- Matrix exponential $e^A = I + A + A^2/2 + A^3/3! + A^4/4! + \ldots$
- Each term in the sum is an $n \times n$ matrix
- How do we compute e^A?
- If $A^k = 0$ for some k, the sum is finite and can be computed directly
- If A is a diagonal matrix $\mathbf{D}(a_1, a_2, \ldots, a_n)$ (A_{ij} = if $(i = j)$ then a_i else 0), then $e^A = \mathbf{D}(e^{a1}, e^{a2}, \ldots, e^{an})$
- In general, the sum of the first k terms will give an approximation (whose quality is proportional to k)
- Otherwise, we can use analytical methods based on eigenvalues and similarity transformations

Eigenvalues and Eigenvectors

- Let A be an $n \times n$ matrix, λ a scalar value and x an n-dimensional non-zero vector.

  If the equation $A\,x = \lambda\,x$ holds, then x is an *eigenvector* of A, and λ is the corresponding *eigenvalue*

- How to compute eigenvalues and eigenvectors?

- We solve the *characteristic equation* of A:

$$\det(A - \lambda I) = 0$$

- Recall: the *determinant* $\det(M)$ of a 2×2 matrix M is $M_{1,1}M_{2,2} - M_{1,2}M_{2,1}$

Eigenvalues and Eigenvectors

The eigenvalues of an $n \times n$ matrix A are the roots of the characteristic polynomial $p = \det(A - \lambda I)$

Note:

- The multiplicity of an eigenvalue (as a root of p) can be > 1
- An eigenvalue can be a complex number
- If A is a diagonal matrix then the diagonal entries are the eigenvalues
- For a given eigenvalue λ, we can compute the corresponding eigenvector(s) by solving the linear system $A x = \lambda x$, with unknown vector x
- If all eigenvalues of are A distinct, then the set of corresponding eigenvectors is linearly independent

Similarity Transformation

- Consider dynamical system H with dynamics given by:

  $dS/dt = A\,S$; initial state s_0

- Let P be an invertible $n \times n$ matrix
- Consider system H' with state vector $S' = P^{-1} S$ and dynamics

  $$d/dt\, S' = d/dt\,(P^{-1} S) = P^{-1}\, dS/dt = P^{-1} A\, S = P^{-1} A\, P\, S' = J\, S'$$

- The matrix $J = P^{-1} A\, P$ is said to be *similar* to A
- The initial state of transformed system H' is $S'(0) = P^{-1} s_0$
- The response of the transformed system H' is given by

  $\mathbf{S'}(t) = e^{Jt}\, P^{-1} s_0$

- The response of the original system is $\mathbf{S}(t) = P\, e^{Jt}\, P^{-1} s_0$
- When is all this useful?
- When we can choose P so that J is diagonal!

Similarity Transformation using Eigenvectors

- Consider system H with dynamics given by:

$$dS/dt = A\,S \text{ ; initial state } s_0$$

- Calculate eigenvalues $\lambda_1, \ldots, \lambda_n$ and suppose they are all distinct
- Calculate corresponding eigenvectors $x_1, \ldots, x_n$ (which must be linearly independent)
- Consider the $n \times n$ matrix $P = (x_1 \ \ x_2 \ \ \ldots \ \ x_n)$
- Find its inverse P^{-1} (which must exist in this case)
- **Claim:** The matrix $J = P^{-1} A\, P$ is the diagonal matrix $\mathbf{D}(\lambda_1, \ldots, \lambda_n)$
- Execution of the system is given by

$$\mathbf{S}(t) = P\,\mathbf{D}(e^{\lambda 1\, t}, \ldots, e^{\lambda n\, t})\, P^{-1} s_0$$

Example: Response of Linear Systems

Consider 2-dimensional system with dynamics given by

$$ds_1 = 4\,s_1 + 6\,s_2 \qquad \text{initial state } (s_1, s_2) = (1, 1)^T$$

$$ds_2 = s_1 + 3\,s_2$$

1. Compute eigenvalues λ_1 and λ_2 of $A = ((4\ \ 1)^T\ \ (6\ \ 3)^T)$
   - $\lambda_1 = 6$ and $\lambda_2 = 1$
2. Compute eigenvectors x_1 and x_2
   - $x_1 = (3\ \ 1)^T$ and $x_2 = (2\ \ -1)^T$
3. Choose the similarity transformation matrix $P = (x_1\ \ x_2) = ((3\ \ 1)^T\ \ (2\ \ -1)^T)$
4. Compute the inverse P^{-1} of P
   - $P^{-1} = ((-1\ \ -1)^T\ \ (-2\ \ 3)^T) / (-3-2) = ((1/5\ \ 1/5)^T\ \ (2/5\ \ -3/5)^T)$
5. Verify that $J = P^{-1}\,A\,P$ is diagonal matrix $\mathbf{D}(\lambda_1, \lambda_2) = ((6\ \ 0)^T\ \ (0\ \ 1)^T)$
   - $J = P^{-1}\,A\,P = ((6/5\ \ 1/5)^T\ \ (12/5\ \ -3/5)^T)\,((3\ \ 1)^T\ \ (2\ \ -1)^T) = ((6\ \ 0)^T\ \ (0\ \ 1)^T)$
6. Desired solution is $\mathbf{S}(t) = P\,\mathbf{D}(e^{\lambda 1\,t}, e^{\lambda 2\,t})\,P^{-1}\,(1, 1)^T$
   - $\mathbf{S}(t) = ((3\ \ 1)^T\ \ (2\ \ -1)^T)\,((e^{6t}\ \ 0)^T\ \ (0\ \ e^{t})^T)\,((1/5\ \ 1/5)^T\ \ (2/5\ \ -3/5)^T)\,(1, 1)^T = \ldots$

Back to Equilibria and Stability

- Consider a closed linear system H with dynamics given by:

  $$dS/dt = A\,S$$

- Recall: a state s is an equilibrium state of H if $A\,s = 0$
- How to compute equilibria: solve system of linear equations
- **Claim 1:** State 0 is an equilibrium
- **Claim 2:** If A is invertible, then 0 is the sole equilibrium
- If state s is a non-zero equilibrium of H, consider the transformed system H' with state $S' = S - s$
  - The equilibria 0 of H' and s of H have the same properties

Back to Equilibria and Stability

Henceforth, we will focus on closed linear systems H and their equilibrium state 0

Definition:

1. H is *stable* if state 0 is stable
2. H is *asymptotically stable* if state 0 is asymptotically stable

Stability: One-Dimensional System

- Consider a one-dimensional linear system H with dynamics given by: $dx/dt = a\,x$ with some initial state s_0

- Recall: H is asymptotically stable iff
  1. (Stable) For every $\varepsilon > 0$, there is a $\delta > 0$ such that for all initial states s with $||s|| < \delta$ and for all times t, $||e^{at}\,s|| < \varepsilon$
  2. (Asymptotically) There is a $\delta > 0$ such that for all initial states s with $||s|| < \delta$, $||e^{at}\,s||$ goes to 0 as t goes to ∞

- **Case a < 0**: $e^{at}\,s$ converges exponentially to 0 as t goes to ∞, regardless of s. Asymptotically stable
- **Case a = 0**: dynamics is $dx/dt = 0$. The state stays equal to the initial state s. Stable but not asymptotically stable
- **Case a > 0**: $e^{at}\,s$ grows exponentially as t increases, and thus, state diverges away from 0. Unstable!

Stability: Diagonal State Dynamics

- Consider n-dimensional linear system H with dynamics given by $dS/dt = A\,S$, where A is the diagonal matrix $\mathbf{D}(a_1, \ldots, a_n)$
- Each dimension evolves independently: the i-th component of $\mathbf{S}(t)$ is $e^{a_i t}\, s_{0i}$ where s_0 is the initial state vector
- **All coefficients $a_i < 0$**: State converges to the equilibrium 0 no matter the initial state. Asymptotically stable
- **All coefficients $a_i <= 0$**: Stable but not asymptotically stable if some coefficient $a_j = 0$ (j-th state component stays unchanged)
- **Some coefficient $a_i > 0$**: Some state component grows unboundedly away from equilibrium 0. Unstable!

Similarity Transformations and Stability

- Consider system H with dynamics given by: $dS/dt = A\,S$
- Let P be an invertible $n \times n$ matrix, and consider $J = P^{-1} A\, P$
- Consider system H' with state $S' = P^{-1} S$ (and note that $S = P\, S'$)

- Response signal of transformed system H': $\mathbf{S'}(t) = e^{Jt} P^{-1} s_0$
- Response signal of original system H: $\mathbf{S}(t) = P\, e^{Jt} P^{-1} s_0$

- Note: response of H' is a linear transformation of response of H
  - If a signal is bounded, so is its linear transformation
  - If a signal converges to 0, so does its linear transformation

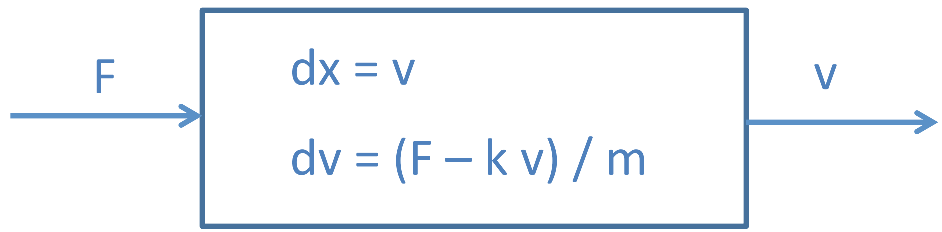
- **Claim:** H is stable iff H' is stable
- **Claim:** H is asymptotically stable iff H' is asymptotically stable

Eigenvalues and Stability

- Consider H with dynamics is given by: $dS/dt = A\,S$
- Suppose all eigenvalues $\lambda_1, \ldots, \lambda_n$ are real and distinct
- Then the set of eigenvectors, $x_1, \ldots, x_n$ is guaranteed to be linearly independent
- Choose $n \times n$ matrix $P = (x_1\ \ x_2\ \ \ldots\ \ x_n)$ for similarity transformation
- The matrix $J = P^{-1} A\,P$ is the diagonal matrix $\mathbf{D}(\lambda_1, \ldots, \lambda_n)$
- If all eigenvalues are negative, then the transformed system H' is asymptotically stable, and so is H
- If all eigenvalues are non-positive, then H' is stable, and so is H

Theorem: A linear system H with dynamics $dS/dt = A\,S$ is asymptotically stable iff each eigenvalue of A has a negative real part

Continuous-time Component Car



- Let $S = (x\ \ v)^T$
- The matrix A is $(\ (0\ \ 0)^T\ \ (1\ \ -k/m)^T\ )$
- Eigenvalues: 0 and $-k/m$
- Stable but not asymptotically stable
- If we consider only the dimension v, then asymptotically stable

Exercise: Set $\mathbf{F}(t) = 0$ for all t, and analyze what happens if we perturb the system from the equilibrium $(0\ \ 0)^T$

Lyapunov Stability vs BIBO Stability

- Consider linear component H with dynamics given by

  $$dS/dt = A\,S + B\,I \qquad O = C\,S + D\,I$$

- BIBO stability: Starting from initial state 0, if the input is a bounded signal, output must be a bounded signal
- **Theorem:** For linear components, asymptotic stability implies BIBO stability
- **Note:** Asymptotic stability depends only on the properties of matrix A
- Proof of the theorem relies of analysis of dynamical systems using transfer functions

Credits

Notes based on Chapter 6 of

Principles of Cyber-Physical Systems
by Rajeev Alur
MIT Press, 2015