CS:4980
Foundations of Embedded Systems

Safety Requirements
Part I

Requirements

- Desirable properties of the executions of the system
  - Informal: either implicit, or stated in natural language
  - Formal: stated explicitly in a mathematically precise way
- Model/design/system meets the requirements if every execution satisfies them all
- Clear separation between
  - requirements, what needs to be implemented, and
  - system, how it is implemented

Requirements

- High assurance / safety-critical systems:

  Typically provided with formal requirements

- Verification problem:

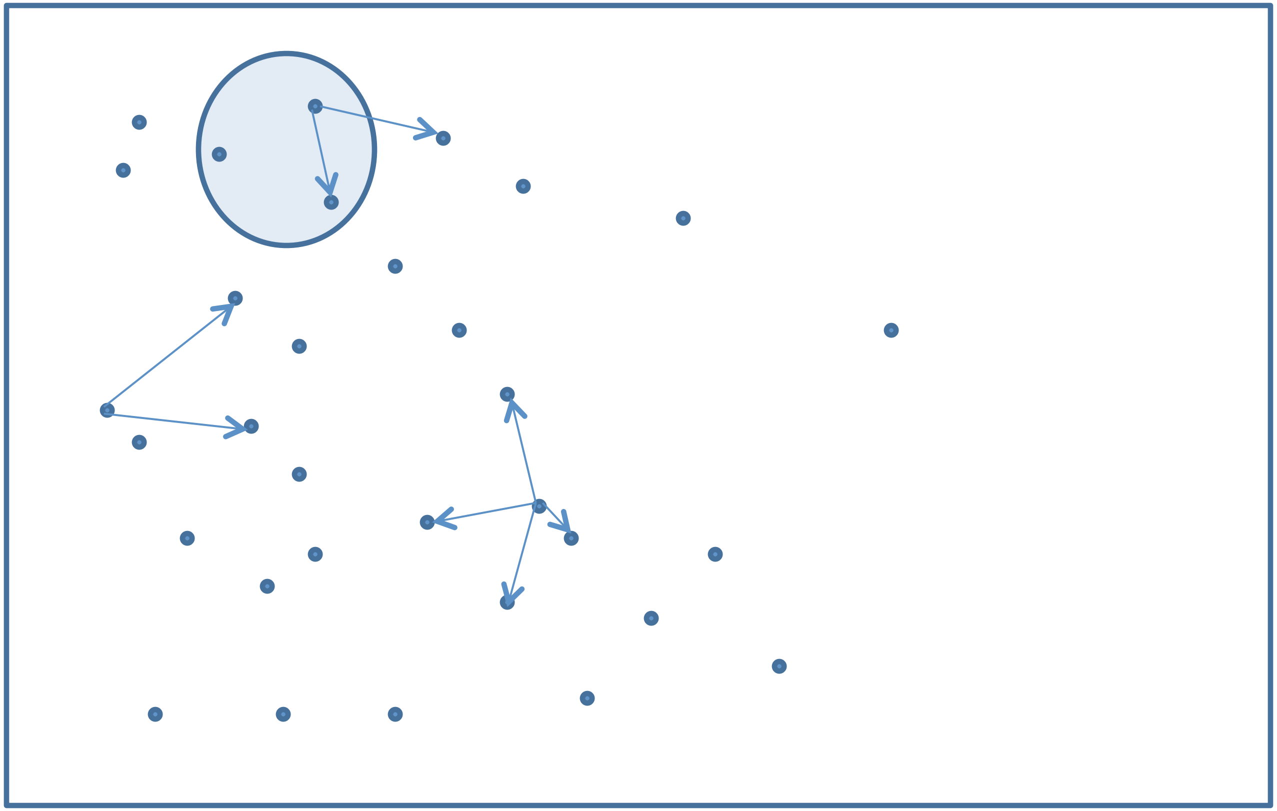
  Given a requirement R and a system/model C, prove or disprove that the C satisfies R

Safety and Liveness Requirements

- A *safety requirement* states that a system always stays within *good* states (i.e., nothing bad ever happens)
  - Leader election: it is never the case that two nodes consider them to be leaders
  - Collision avoidance: Distance between two cars is always greater than some minimum threshold
- A *liveness requirement* states that a system eventually achieves its goal (i.e., something good eventually happens)
  - Leader election: Each node eventually makes a decision
  - Cruise controller: Actual speed eventually equals desired speed
- Formalization and analysis techniques for safety and liveness differ significantly. We will focus on safety

Transition Systems

State space + Initial states + Transitions between states



Definition of Transition System

Syntax: a transition system T has

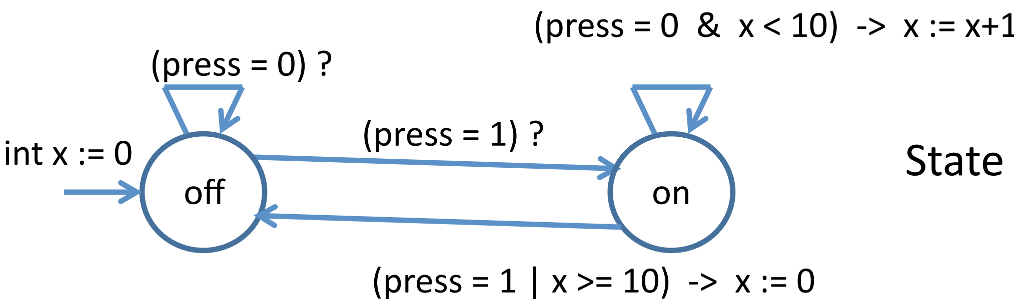
1. A set S of (typed) state variables
2. Initialization $Init$ for state variables
3. A description $Trans$ of how to move from one state to the next

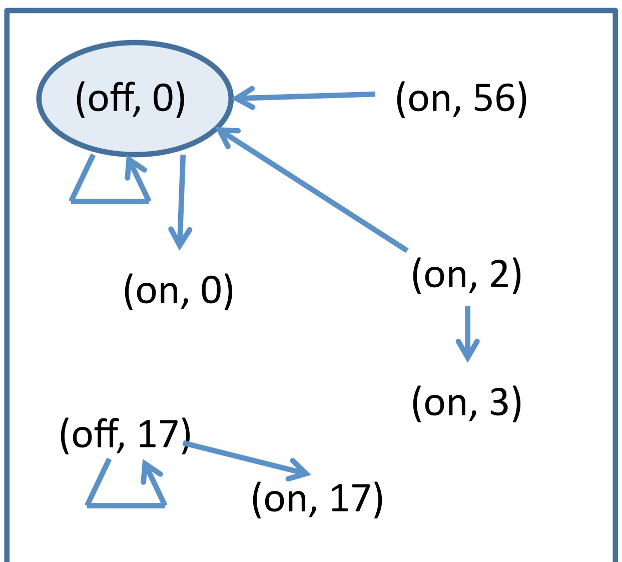
Semantics:

1. Set Q_S of states
2. Set $[Init]$ of initial states, a subset of Q_S
3. Set $[Trans]$ of transitions, a subset of Q_S x Q_S

Synchronous reactive components, programs, and more generally systems, all have an underlying transition system

Switch Transition System





State variables:

{off, on} mode, int x

Initialization:

mode := off ; x := 0

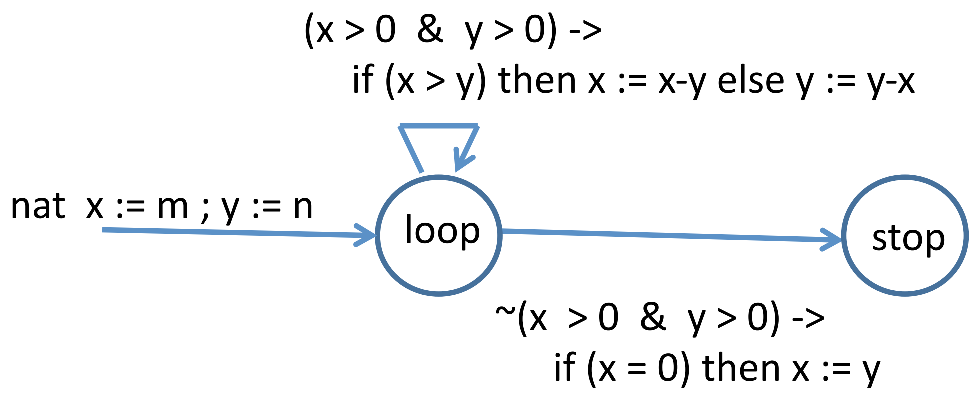
Transitions:

(off, n) -> (off, n) ;
(off, n) -> (on, n) ;
(on, n) -> (on, n+1) if n < 10 ;
(on, n) -> (off, 0)

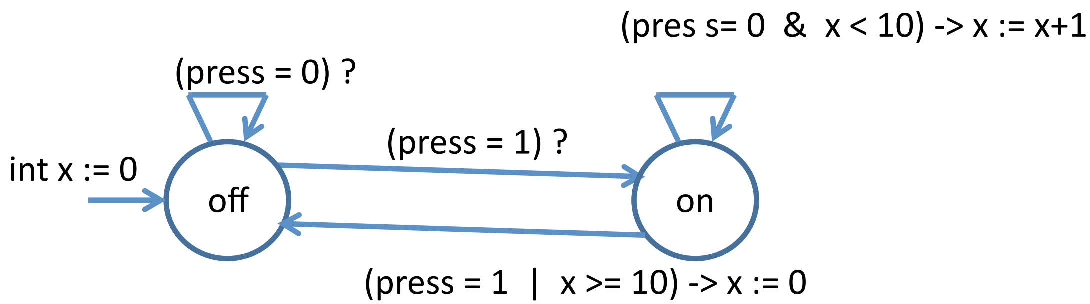
- Input/output variables become local
- Values for input vars chosen nondeterministically

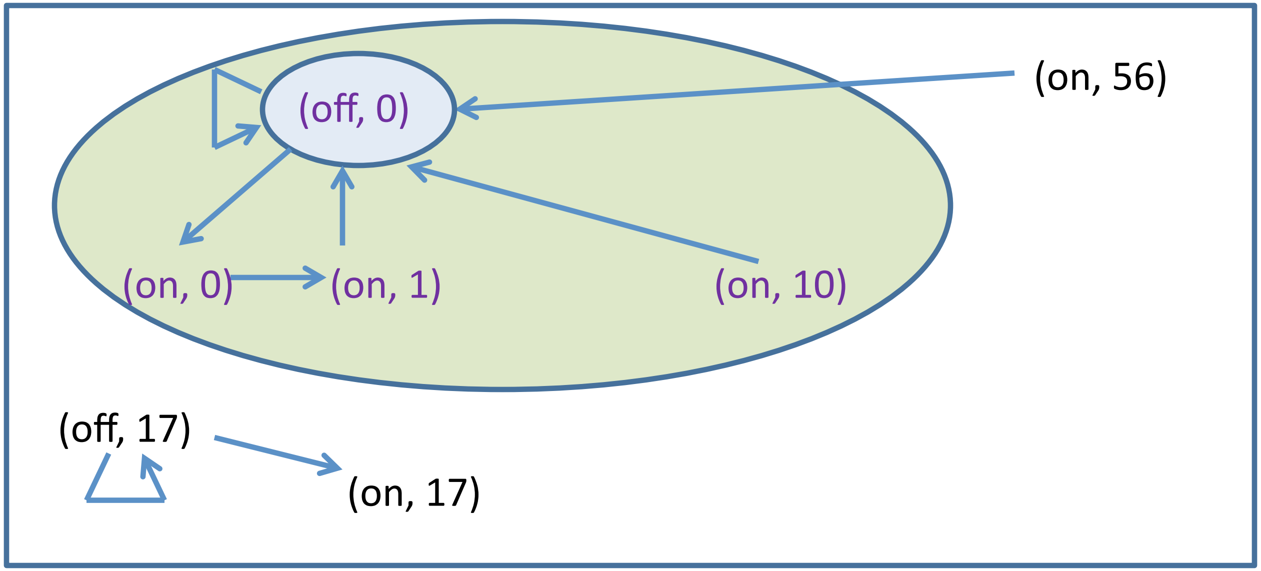
Euclid's GCD Algorithm

Classical program to compute greatest common divisor of (non-negative) input numbers m and n

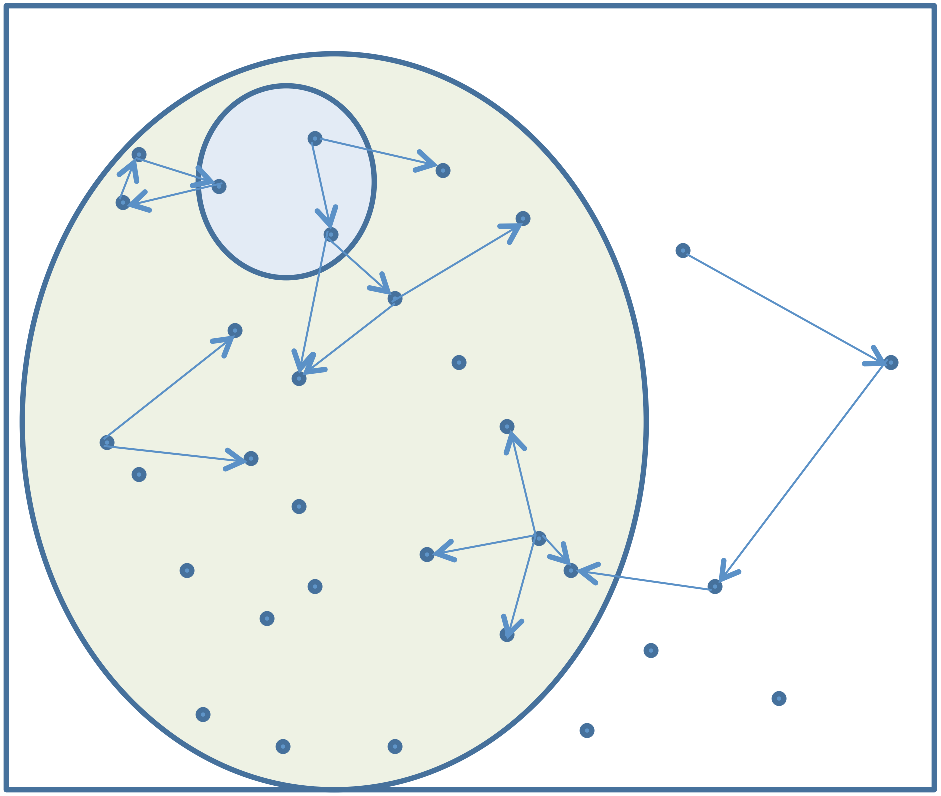


Reachable States



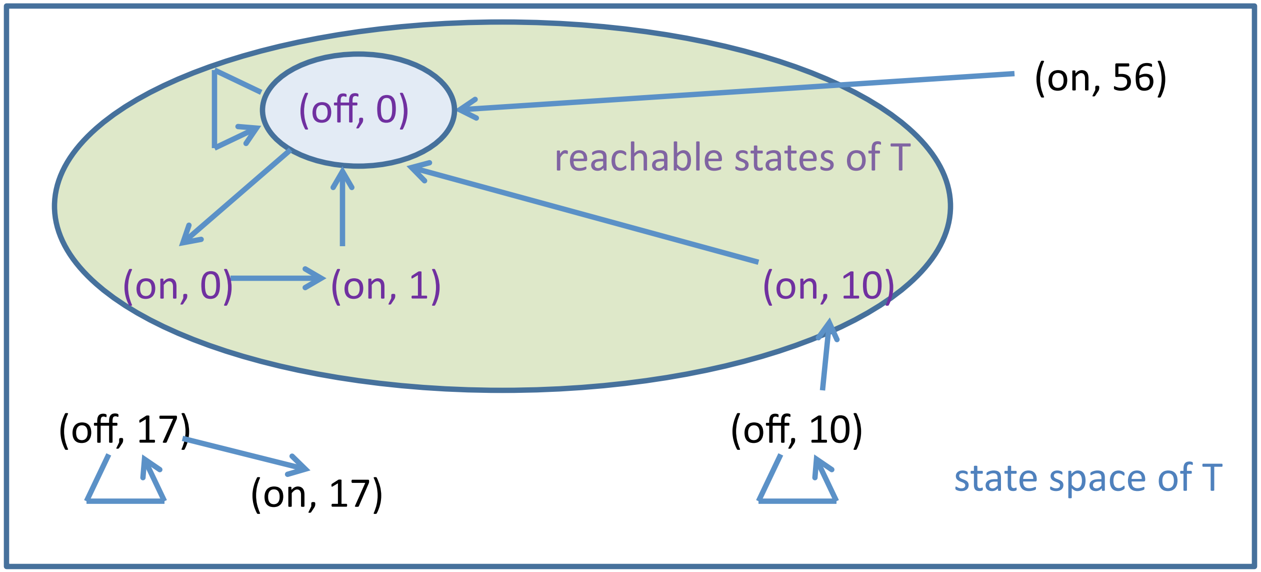


Reachable States of Transition Systems



A state s of a transition system T is *reachable* if there is an execution starting in an initial state of T and ending in s

Invariants



- A *property* of a transition system T is a Boolean-valued expression P over state variables
- Property P is an *invariant* of T if every reachable state satisfies P
- Some invariants for T above: x <= 10, x <= 50, mode = off => x = 0
- Some non-invariants for T above: x < 10, mode = off

Invariants

- We express safety requirements for a transition system T as properties P of T's state variables
  - If P is invariant then T is safe
  - If P is not invariant, then some *bad* state, satisfying ¬P is reachable

    (the execution leading to such a state is a counterexample)

- Leader election:

  $(r_n = N) \Rightarrow (id_n = \max I)$ I : set of identifiers of all nodes

- Euclid's GCD Program:

  $(mode = stop) \Rightarrow (x = gcd(m, n))$

Formal Verification

Grand challenge: automate verification as much as possible!

Analysis Techniques

- Dynamic Analysis (runtime)
  - Execute the system, possibly multiple times with different inputs
  - Check if every execution meets the desired requirement
- Static Analysis (design time)
  - Analyze the source code or the model for possible bugs
- Trade-offs
  - Dynamic analysis is incomplete, but accurate (checks real system, and bugs discovered are real bugs
  - Static analysis can be complete and can catch design bugs early
  - Many static analysis techniques are not scalable (solution: analyze approximate versions, can lead to false warnings)

Invariant Verification

Simulation

- Simulate the model, possibly multiple times with different inputs
- Easy to implement, scalable, but no correctness guarantees

Deductive verification

- Construct a proof that system satisfies the invariant
- Usually requires manual effort (but partial automation often possible)

Model checking

- Automatically explores all reachable states to check invariants
- Not scalable, but current tools can analyze many real-world designs (relies on many interesting theoretical advances)

Note: Newer techniques are blurring the differences between deductive verification and model checking

Proving Invariants

- Given a transition system $T = (S, Init, Trans)$, and a property P, prove that all reachable states of T satisfy P

- Inductive definition of reachable states
  - All initial states are reachable using 0 transitions
  - If a state s is reachable in k transitions and $s \to t$ is a transition, then the state t is reachable in $k+1$ transitions
  - Reachable = Reachable in n transitions, for some n

- Prove: for all n, states reachable in n transitions satisfy P
  - *Base case*: Show that all initial states satisfy P
  - *Inductive case*:
    1. Assume that a state s satisfies P
    2. Show that if $s \to t$ is a transition then t must satisfy P

Recall: Inductive Proofs in Arithmetic

- To show that a statement P holds for all natural numbers n,
  - Base case: Prove that P holds for n=0
  - Assume that P holds for an arbitrary natural k
  - Using the assumption, prove that P holds for k+1

- Example statement: For all n,

$$(0 + 1 + 2 + \ldots + n) = n(n+1)/2$$

Inductive Invariant

- A property P is an *inductive invariant* of transition system T if
  1. Every initial state of T satisfies P
  2. If a state satisfies P and s -> t is a transition of T, then t must satisfy P

- If P is an inductive invariant of T, then all reachable states of T must satisfy P, and thus, it is an invariant of T

Proving Inductive Invariant Example (1)

- Consider transition system T given by
  - State variable int x, initialized to 0
  - Transition description given by if (x < m) then x := x+1 for some m >= 0
- Is the property P : 0 <= x <= m an inductive invariant of T?
- Base case: Consider initial state x := 0. Check that it satisfies P
- Inductive case:
  - Consider an arbitrary state s, suppose s(x) = a
  - Assume that s satisfies P, that is, assume 0 <= a <= m
  - Consider the state t obtained by executing a transition from s
  - If a < m then t(x) = a+1, else t(x) = a
  - In either case, 0 <= t(x) <= m
  - So t satisfies the property P, and the proof is complete

Proving Inductive Invariant Example (2)

- Consider transition system T given by
  - State variables int x, y; initially: x := 0 ; y := m for some m > 0
  - Transition description given by if (x < m) then { x := x+1 ; y := y-1 }
- Is the property P : 0 <= y <= m an inductive invariant of T?
- Base case: Consider initial state (x := 0, y := m). Check that it satisfies P
- Inductive case:
  - Consider an arbitrary state s with x = a and y = b
  - Assume that s satisfies P, that is, assume 0 <= b <= m
  - Consider the state t obtained by executing a transition from s
  - If a < m then t(y) = b-1, else t(y) = b
  - Can we conclude that 0 <= t(y) <= m?
  - No! When b = 0, t(y) is negative.
  - The proof fails. In fact, P is not an inductive invariant of T!

Why did the proof fail?

- Consider the state s with $x = 0$ and $y = 0$
  - State s satisfies P: $0 <= y <= m$
  - Executing a transition from s leads to state t with $x = 1$ and $y = -1$
  - State t does not satisfy P
- However, the state s in above argument is not reachable!
- Cause of failure: The property P did not capture correlation between the state components x and y
- Solution: *Inductive Strengthening*
  - Consider property Q : $(0 <= y <= m)$ & $(x + y = m)$
  - Property Q implies property P
  - While P is not an inductive invariant, Q is!
  - It follows that all reachable states must satisfy P

Proving Inductive Invariant Example (3)

- Consider transition system T given by
  - State variables int x, y; initially: $x := 0$; $y := m$ for some $m > 0$
  - Transition description given by if $(x < m)$ then { $x := x+1$; $y := y-1$ }
- Property Q : $(0 <= y <= m)$ & $(x + y = m)$
- Base case: Consider initial state $(x := 0, y := m)$. Check that it satisfies Q
- Inductive case:
  - Consider an arbitrary state s with $x = a$ and $y = b$
  - Assume that s satisfies Q, that is, assume $0 <= b <= m$ and $a+b = m$
  - Consider the state t obtained by executing a transition from s
  - If $a < m$ then $t(x) = a+1$ and $t(y) = b-1$, else $t(x) = a$ and $t(y) = b$
  - But if $a < m$, since $b = m-a$, then $b > 0$, and thus $b-1 >= 0$
  - In either case, the condition $(0 <= t(y) <= m)$ & $(t(x)+t(y) = m)$ holds
- Conclusion: Property Q is an inductive invariant

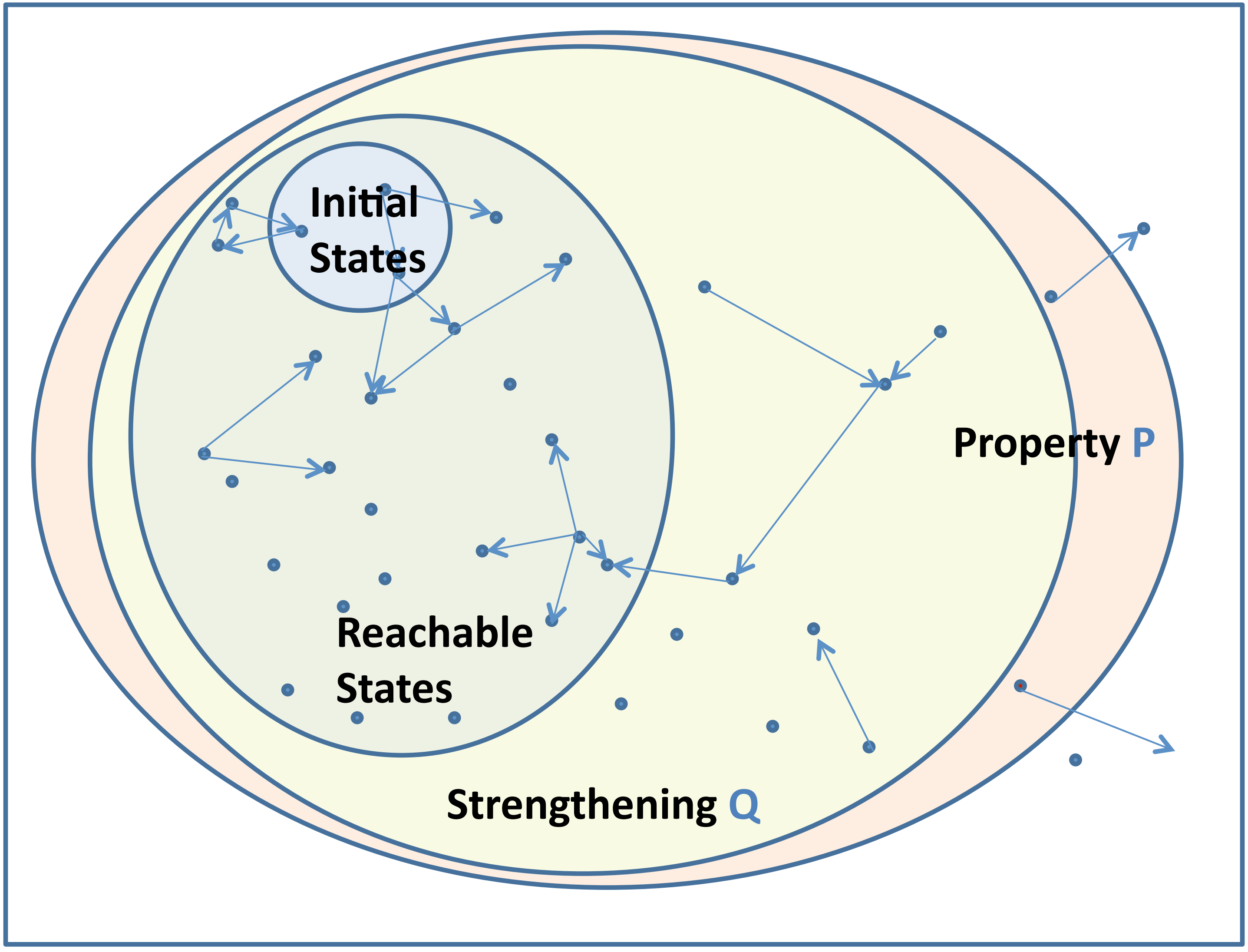
Proof Rule for Proving Invariants

- To establish that a property P is an invariant of transition system T
- Find an *inductive strengthening* of P: a property Q such that
  1. Q implies P (i.e., every state satisfying Q also satisfies P)
  2. Q is an inductive invariant:
     - all initial states satisfies Q
     - For any states s, t such as s satisfies Q and s -> t is a transition, t satisfies Q
- This is a sound and complete strategy for establishing invariants

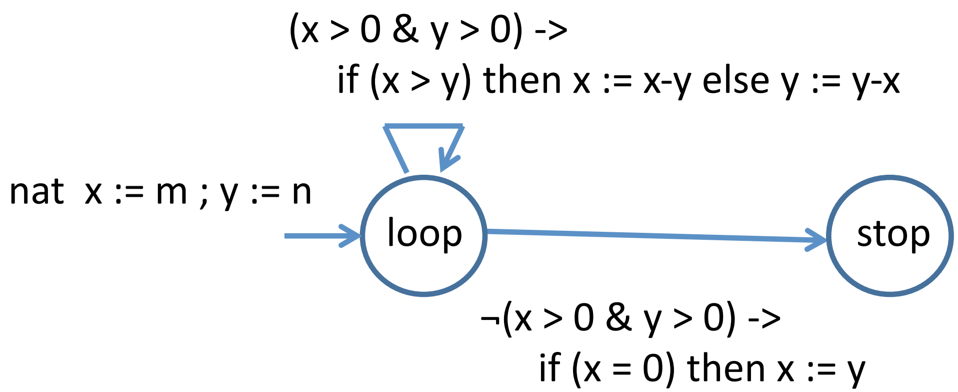
  Sound: If P has an inductive strengthening Q then P is indeed invariant

  Complete: If P is an invariant, then it has an inductive strengthening Q

Inductive Strengthening



Correctness of GCD



- Property P : gcd(x, y) = gcd(m, n)
- Verify that P is an inductive invariant (Exercise)
- Captures the core logic of the program: even though x and y are updated at every step, their gcd stays unchanged
- When switching to stop, if x is 0, then gcd(x, y) is y; if y = 0, then gcd(x, y) = x, and thus x = gcd(m, n) upon switching to stop
- Note: (mode = stop) => (x = gcd(m, n)) is invariant, but not inductive

Transition System for Leader Election

- Initial state:
  - For each node n, int id_n := n ; int r_n := 1
- Update during single transition:
  - Round counters: if $r_n < N$ then $r_n := r_n + 1$
  - Identifiers: $id_n := \max \{id_n, \max \{id_m \mid m \text{ is connected to } n\}\}$

Invariants for Leader Election

- Initial state: for each node n, int id_n := n; int r_n := 1
- Update during single transition:
  - if $r_n < N$ then $r_n := r_n + 1$
  - $id_n := \max \{id_n, \max \{id_m \mid m \text{ is connected to } n\}\}$
- Consider: $id_n >= n$ (that is, for node n, id is at least n)
  - Obviously an invariant; is it an inductive invariant?
- Let ID be the set of identifiers of all nodes and consider the property : "id_n belongs to ID", for a specific node n
  - Not an inductive invariant!
  - During a transition s -> t, value of id_n in state t may equal value of id_m in state s, but property says nothing about $s(id_m)$
- What about: "for each node n, id_n belongs to ID" ? (Exercise)

Correctness of Leader Election

- We expect id_n to be maximum of all identifiers after N rounds
- Formal property:
  - For each node n, $(r_n = N) \Rightarrow (id_n = \max ID)$
- Not inductive
- Goal: Find inductive strengthening that captures co-relation among all variables at intermediate steps
- Informal: After k rounds, each r_n equals k, and id_n is max of identifiers of nodes that are $\leq k$ hops away from node n
- Formal property:

$$P_1 : \text{For all nodes m and n, } r_m = r_n$$

$$\& \; P_2 : \text{For each node n, } id_n = \max \{ m \mid distance(m, n) < r_n \}$$

- Prove that P_1 & P_2 is an inductive invariant!

Proof: Base Case

- Initial state s: for each node n, $s(id_n) = n$ and $s(r_n) = 1$
- Goal: Show that the following both hold in this initial state s

  P_1 : For each m and n, $r_m = r_n$

  P_2 : For each n, $id_n = \max \{ m \mid distance(m, n) < r_n \}$

P_1) $s(r_m) = s(r_n) = 1$; so P_1 holds

P_2) Consider a node n, we want to show

$$s(id_n) = \max \{ m \mid distance(m, n) < 1 \}$$

The only node m with $distance(m, n) < 1$ is n itself, and $s(id_n) = n$, so P_2 holds

Proof: Inductive Case

- Consider an arbitrary state s, and assume both P_1 and P_2 hold
- Let $s(r_n) = k$, for each node n
- For $k < N$, consider a successor state t of s
- Goal: Show that both P_1 and P_2 hold in state t
- Consider two nodes m and n
- $t(r_m) = s(r_m) + 1 = k+1$, and similarly, $t(r_n) = k+1$, so P_1 holds in t
- To show P_2, consider a node n, we want to show

$$t(id_n) = \max \{ m \mid distance(m, n) < k+1 \}$$

- Assumption 1 (from inductive hypothesis): for each node m

  $s(id_m) = \max \{ l \mid distance(l, m) < k\}$
- Assumption 2 (from the transition relation):

  $t(id_n) = \max \{ s(id_n), \max \{s(id_m) \mid m \text{ is connected to } n \} \}$

Proof: Inductive Case (Continued)

- Let l be the node with highest identifier with $distance(l, n) < k+1$
- Goal: show that $t(id_n) = l$
- Let $distance(l, n) = d$. We know $d < k+1$, so either $d < k$ or $d = k$

  Case ($d < k$)

  - By Assumption 1, $s(id_n)$ cannot be less than l, so must be l
  - By Assumption 2, $t(id_n)$ cannot be less, and thus, must be l

  Case ($d = k$)

  - By basic properties of graphs, there must be a node m such that $distance(l,m) = k-1$ and m is connected to n
  - By Assumption 1, $s(id_m)$ cannot be less than l, so must be l
  - By Assumption 2, $t(id_n)$ cannot be less, and thus, must be l
- The proof is complete!

Summary of Invariants

- General way to formulate and prove safety properties of programs/models/systems
- Inductive invariant:
  - Holds in initial states
  - Is preserved by every transition
- To be inductive, property needs to capture relevant relationships among all state variables
- Benefit of finding inductive invariants:
  - Correctness reasoning becomes local (one needs to think about what happens in one step)
  - Tools available to check if a given property is inductive invariant
- Area of active research: can a tool discover them automatically?

Automated Invariant Verification

Can such a verifier exist?

If so, what is the computational complexity of the verification problem?

A Brief Detour into Computational Complexity

- Goal: Classify computational problems in terms of (roughly) how many basic operations it takes to solve the problem, as function of input size
- Example 1: Finding maximum of a list of numbers
  - Time complexity is linear: $O(n)$
- Example 2: Sorting a list of numbers
  - Algorithm (e.g. selection-sort) with doubly-nested loop: $O(n^2)$
  - More efficient algorithm (e.g. quicksort) possible: $O(n \log n)$
- Example 3: Expression evaluation: Given an expression e (with not/or/and as operations) over Boolean vars, and an assignment a of 0/1 values to vars, determine whether e evaluates to 1 or 0. Linear-time $O(n)$
- Example 4: Boolean satisfiability: Given an expression e, determine if there exists an assignment a to vars that makes the expression 1
  - Naïve algorithm: Evaluate e on every possible assignment a
  - Exponentially many choices for a : Algorithm is $O(2^k)$, k = no. of vars

The Class P

- *Polynomial-time* algorithm means an algorithm with time complexity such as $O(n)$, $O(n \log n)$, $O(n^2)$, $O(n^3)$, or $O(n^c)$, for constant c
- A problem is in P if there is a polynomial-time algorithm to solve it
- Examples:
  - Finding maximum
  - Sorting
  - Expression evaluation
  - Finding shortest path in a graph
- P is the class of *tractable* (i.e. efficiently solvable) problems
  - Problem can be solved exactly
  - Solution will scale reasonably well as input size grows
  - In principle, $O(n)$ is better than $O(n^2)$

NP-Complete Problems

- SAT: Given an expression e over Boolean variables, check if there exists an assignment of 0/1 values to vars for which e evaluates to 1
  - No proof that SAT is in P (no known polynomial-time algorithm)
  - No proof that SAT is not in P
- Cook (1972): SAT is NP-complete
- Hundreds of problems equivalent to SAT
  - Hamiltonian Path: Is there a path in a graph from source to destination that visits each vertex exactly once
  - Max Clique: Given a graph, find largest subset of vertices such that there is an edge between every pair of vertices in this set
- Grand Challenge Open Problem : Is P = NP?
  - If you find a polynomial-time algorithm for SAT, then P = NP, and many other problems will have polynomial-time algorithms
  - If you prove SAT is not in P, then P != NP, and many other problems then provably don't have efficient algorithms

NP-Completeness Continued

- Known algorithms for SAT are exponential-time in the worst-case, but
  - Highly efficient implementations, SAT solvers, exist
  - Can handle millions of variables
  - Many practical problems solved by encoding into SAT
- Key feature of NP problems such as SAT: suffices to find one satisfying assignment
- This does not hold for all intractable problems
  - Validity: Given a Boolean expression e, is it the case that e evaluates to 1 no matter what values we give to its variables
- Many complexity classes beyond NP: coNP, PSPACE, Exptime, …
  - Problems may require exponential-time (or more) to solve
  - Not all exponential-time problems are equal…

(Un)Decidability

- Some problems cannot be solved by a computer at all!
- Fundamental Theorem of CS: Alan Turing (1936):
  - The *Halting problem* for Turing machines is undecidable

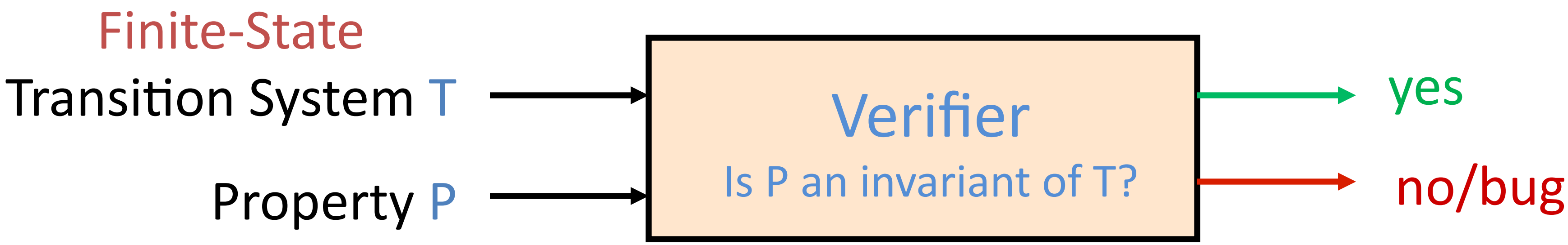
    There is no program that takes as its input an arbitrary program C and an arbitrary input x, and determines if C terminates on x
- Intuition: If a program could analyze other programs exactly, then it can analyze itself, and this suffices to set up a logical contradiction!
- A surprisingly undecidable problem: Does a given a polynomial (e.g. $x^3 + 2xy^2 - 15xy + 156$) have integer roots?
- Decidable Problems: There exists a program (or Turing machine) that solves the problem correctly (gives the right answer and stops)
  - Includes problems in P as well as intractable classes such as NP, Exptime, etc.

Back To Invariant Verification Problem

Theorem: The invariant verification problem is undecidable.

Proof idea: undecidable problems for Turing machines can be recast as invariant verification problems for transition systems with integer state variables

Finite-State Invariant Verification Problem

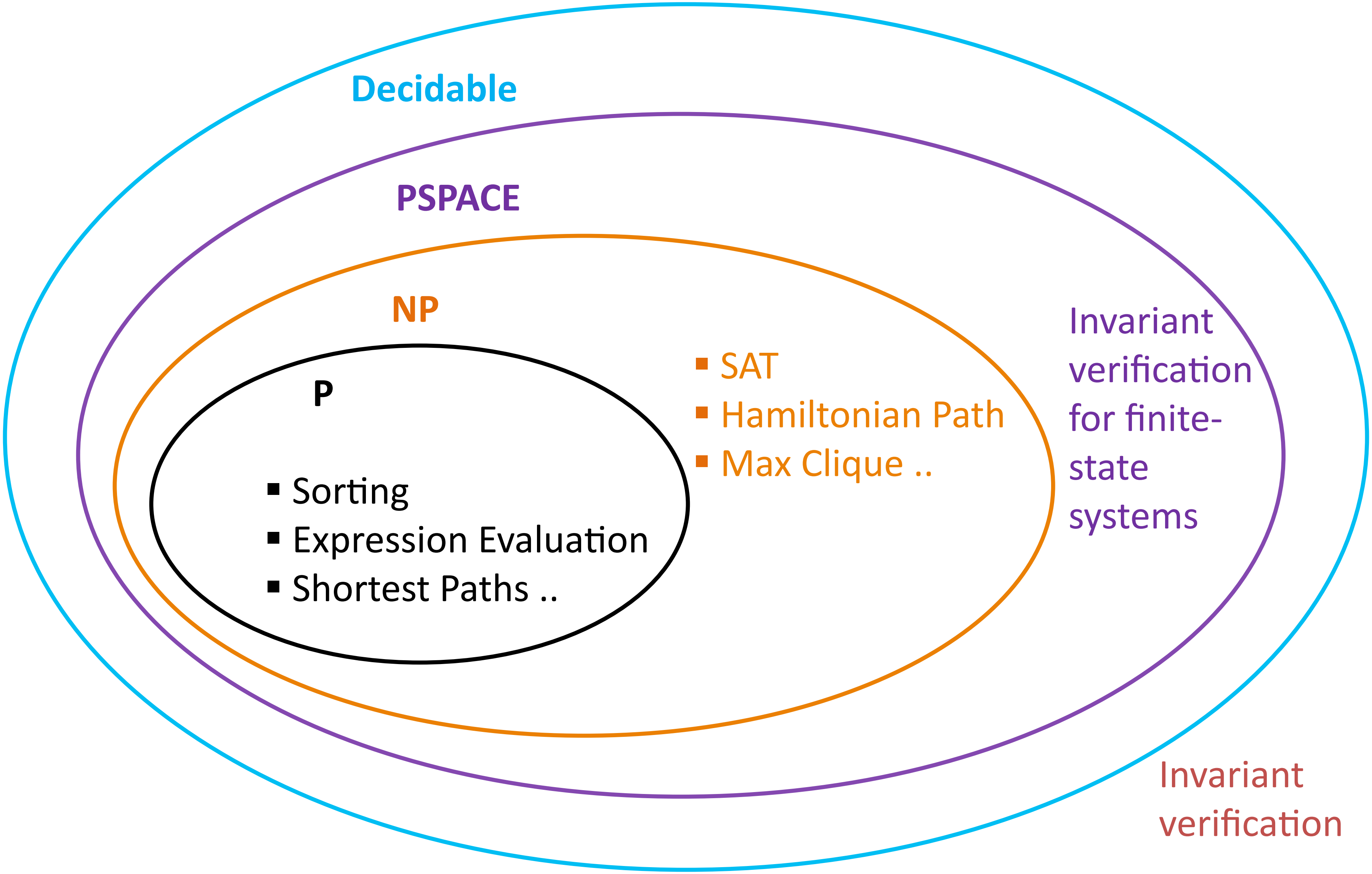


Theorem: The invariant verification problem for finite-state systems is decidable

Proof sketch: If T has k Boolean state vars, then total number of states is 2^k.

Verifier can systematically search through all possible states.

Complexity is exponential. More precisely, it is PSPACE, a class of problems harder than NP-complete problems such as SAT.


Decidable
PSPACE
NP
P
Sorting
Expression Evaluation
Shortest Paths ..
SAT
Hamiltonian Path
Max Clique ..
Invariant verification for finite-state systems
Invariant verification

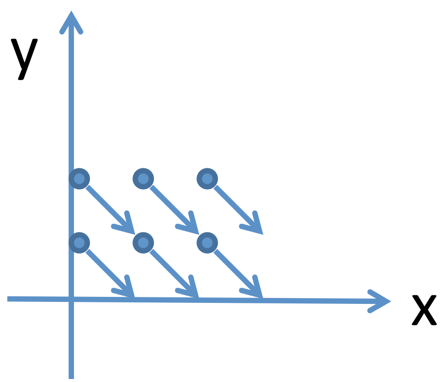
Solving Invariant Verification

- Establishing that the system is safe is important, but there is no generally efficient procedure to solve the verification problem
- Solution 1: Use Simulation-based analysis
  - Simulate the model multiple times, and check that each state encountered on each execution satisfies desired safety property
  - Useful, practical in real-world, but gives only partial guarantee (and is known to miss hard-to-find bugs)
- Solution 2: Write a formal proof using inductive invariants
  - Only partial tool support possible, so requires considerable effort
  - Recent successes: verified microprocessor, web browser, JVM
- Solution 3: Exhaustive search through state-space
  - Fully automated, but has scalability limitations (may not work!)
  - Complementary to simulation, increasingly used in industry
  - Two approaches: On-the-fly enumerative search, Symbolic search

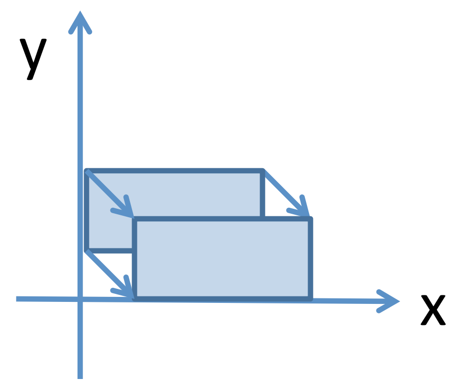
Computing Reachable States

- Search algorithm can start with initial states, and explore transitions out of initial states systematically

- Example: state vars are integers x, y; we know that initially 0 <= x <= 2 and 1 < y <= 2, and a single transition increments x and decrements y



Enumerative:
Consider individual states



Symbolic:
Consider set of states

Credits

Notes based on Chapter 3 of

Principles of Cyber-Physical Systems
by Rajeev Alur
MIT Press, 2015