CS:4350 Logic in Computer Science

Inference Systems for Propositional Logic

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Spring 2021

THE UNIVERSITY OF IOWA

Credits

Part of these slides are based on Chap. 1 of *Logic in Computer Science* by M. Huth and M. Ryan, Cambridge University Press, 2nd edition, 2004.

Outline

Inference Systems for Propositional Logic

- Semantic consequence/entailment
- Derivability
- Natural deduction
- Soundness and completeness of natural deduction

Logics, formally

A logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{L}$, the language, is
  a class of sentences described by a formal grammar
- $\mathcal{S}$, the semantics, is
  a formal specification for assigning meaning to sentences in $\mathcal{L}$
- $\mathcal{R}$, the inference system, is
  a set of axioms and inference rules to *infer* (i.e., generate) sentences of $\mathcal{L}$ from given sentences of $\mathcal{L}$

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We will study a few of them

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Formal properties of inference systems

A formal system is defined by a set of inference rules that allow us to generate formulas from given formulas

We will focus on these properties of our inference systems:

Soundness Every inferred formula is a semantic consequence of the given ones

Completeness Only semantic consequences are inferable

Termination Only finitely many inferences are needed to prove or disprove semantic consequence

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Semantic consequence (or entailment)

Given

- a set $U = \{A_1, \ldots, A_n\}$ of formulas and
- a formula B

we write

$$\{A_1, \ldots, A_n\} \models B$$

iff every interpretation that satisfies all formulas in U satisfies B too

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$U \models B$ is read as B *is a semantic/logical consequence of* U*, or*
B *logically follows from* U*, or*
U *entails* B

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$U \models A$ formally captures the notion of a fact A following from assumptions U

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Note 2: Do not confuse this use $\models$ with that in $\mathcal{I} \models B$ where $\mathcal{I}$ is an interpretation

Entailment, Examples

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\begin{array}{lll}
\{p\} & \models & p \lor q \\
\{p,\, p \to q\} & \models & q \\
\{p,\, q\} & \models & p \land q \\
\{\} & \models & r \to r \\
\{p,\, \neg r\} & \models & (p \lor q) \land (q \lor \neg r) \\
\{q\} & \models & (p \lor q) \land (q \lor \neg r) \\
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Exercise

Determine which of the following entailments hold

$$
\begin{array}{lcl}
p \wedge q,\ r & \overset{?}{\models} & q \wedge r \\
p,\ \neg\neg(q \wedge r) & \overset{?}{\models} & \neg\neg p \wedge r \\
p,\ p \to q,\ q \to r & \overset{?}{\models} & r \\
p \vee q,\ p \to q,\ q \to r & \overset{?}{\models} & r \\
p \vee q,\ p \to r,\ q \to r & \overset{?}{\models} & r \\
p \to q & \overset{?}{\models} & \neg q \to \neg p \\
p \to q & \overset{?}{\models} & \neg p \to \neg q \\
p \vee (q \wedge r) & \overset{?}{\models} & (p \vee q) \wedge (p \vee r) \\
 & \overset{?}{\models} & p \to (q \to p) \\
p \to q,\ p \to \neg q & \overset{?}{\models} & \neg p
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Properties of entailment

- $U \models A$ for all $A \in U$ (*inclusion*)
- if $U \models A$ then $V \models A$ for all $V \supseteq U$ (*monotonicity*)
- A is valid iff $\emptyset \models A$ (also written as $\models A$)
- A is unsatisfiable iff $A \models \bot$
- $U \models A$ iff $U \cup \{\neg A\}$ is unsatisfiable
- $\{A_1, \ldots, A_n\} \models B$ iff $\{A_1, \ldots, A_{n-1}\} \models A_n \to B$ (*deduction*)
- $\{A_1, \ldots, A_n\} \models B$ iff $\{A_1 \wedge \cdots \wedge A_n\} \models B$ iff $\emptyset \models (A_1 \wedge \cdots \wedge A_n) \to B$
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Given

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we write

$$\{A_1, \ldots, A_n\} \vdash_I B$$

iff it is possible to infer B from U with the rules of I

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iff it is possible to infer B from U with the rules of I

$U \vdash_I A$ is read as U *derives* B *in* I, or
B *derives from* U *in* I, or
B *is derivable from* U *in* I

Inference systems for propositional logic

An *inference system* I is a collection of formal rules for inferring formulas from formulas

Given

- a set $U = \{A_1, \ldots, A_n\}$ of formulas (*premises*) and
- a formula B (*conclusion*)

we write

$$\{A_1, \ldots, A_n\} \vdash_I B$$

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We write just $U \vdash A$ when I is clear from context

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iff it is possible to infer B from U with the rules of I

Intuitively, I is designed so that $U \vdash_I A$ only if $U \models A$

Ideally, I should be such that $U \vdash_I A$ if $U \models A$

All these symbols!

Note:

$A \wedge B \rightarrow C$ is a formula, a sequence of symbols manipulated by an inference system I

$A \wedge B \models C$ is a mathematical abbreviation for the statement:
“every interpretation that satisfies $A \wedge B$, also satisfies C”

$A \wedge B \vdash_I C$ is a mathematical abbreviation for the statement:
“I derives C from $A \wedge B$”

All these symbols!

In other words,

- $\rightarrow$ is a symbol of propositional logic, processed by inference systems
- $\models$ denotes a relation from sets of formulas to formulas, based on their meaning in propositional logic
- $\vdash_I$ denotes a relation from sets of formulas to formulas, based on their derivability in I

Implication vs. Entailment

The connective $\rightarrow$ and the relation $\models$ are related as follows:

$$A \rightarrow B \text{ is valid iff } A \models B$$

Example: $p \rightarrow (p \vee q)$ is valid and $p \models (p \vee q)$

	p	q	$p \vee q$	$p \rightarrow (p \vee q)$
1.	0	0	0	1
2.	0	1	1	1
3.	$\underline{1}$	0	1	1
4.	$\underline{1}$	1	1	1

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3.	$\underline{1}$	0	1	1
4.	$\underline{1}$	1	1	1

Soundness and completeness

The relations $\models$ and $\vdash_I$ are related as by these two properties of inference systems I

Soundness I is *sound* if it can derive from any set U of formulas only formulas entailed by U:

$$\text{if } U \vdash_I A \text{ then } U \models A$$

Completeness I is *complete* if it can derive from any set U of formulas all formulas entailed by U:

$$\text{if } U \models A \text{ then } U \vdash_I A$$

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Natural deduction

There are many inference systems for propositional logic

Natural deduction is a family of inference systems with inference rules designed to mimic the way people reason deductively

Note

- “Natural” here is meant in contraposition to “mechanical / automated”
- Other inference systems for PL are more machine-oriented and so arguably not as natural for people
- Natural deduction is actually automatable but less conveniently than other, more machine-oriented inference systems

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$\wedge$ introduction and elimination

$$\frac{A \qquad B}{A \wedge B}\ \wedge\mathrm{i} \qquad\qquad \frac{A \wedge B}{A}\ \wedge\mathrm{e}_1 \qquad\qquad \frac{A \wedge B}{B}\ \wedge\mathrm{e}_2$$

Usage Given: A set U of formulas

- $\wedge\mathrm{i}$: for any two formulas A and B in U, add $A \wedge B$ to U
- $\wedge\mathrm{e}_1$: for any formula of the form $A \wedge B$ in U, add A to U
- $\wedge\mathrm{e}_2$: for any formula of the form $A \wedge B$ in U, add A to U

$\wedge$ introduction and elimination

$$\frac{A \qquad B}{A \wedge B}\,\wedge\mathrm{i} \qquad\qquad \frac{A \wedge B}{A}\,\wedge\mathrm{e}_1 \qquad\qquad \frac{A \wedge B}{B}\,\wedge\mathrm{e}_2$$

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Example derivation

$$\frac{A \qquad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$p \wedge q,\ r \ \vdash\ q \wedge r$$

Example derivation

$$\frac{A \qquad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

Example derivation

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Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

I like cats and (like) dogs, Jill likes birds $\vdash$ I like dogs and Jill likes birds

Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

Proof

1 $p \wedge q$ premise

Example derivation

$$\frac{A \qquad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

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$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

Proof

1	$p \wedge q$	premise
2	r	premise

Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

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Proof

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge\mathrm{e}_2$ applied to 1

Example derivation

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{i} \qquad \frac{A \wedge B}{A}\wedge\mathrm{e}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{e}_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

Proof

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge\mathrm{e}_2$ applied to 1
4	$q \wedge r$	$\wedge\mathrm{i}$ applied to 3, 2

Proof tree

$$\frac{\dfrac{p \wedge q}{q}\wedge\mathrm{e}_2 \qquad r}{q \wedge r}\wedge\mathrm{i}$$

$\neg$ introduction and elimination

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

$\neg$ introduction and elimination

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

$\neg$ introduction and elimination

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise

$\neg$ introduction and elimination

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$

1	p	premise
2	$\neg\neg(q \land r)$	premise
3	$q \land r$	$\neg\neg$e 2

$\neg$ introduction and elimination

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$q \wedge r$	$\neg\neg$e 2
4	r	$\wedge$e 3

$\neg$ introduction and elimination

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$

1	p	premise
2	$\neg\neg(q \land r)$	premise
3	$q \land r$	$\neg\neg\mathrm{e}$ 2
4	r	$\land\mathrm{e}$ 3
5	$\neg\neg p$	$\neg\neg\mathrm{i}$ 1

$\neg$ introduction and elimination

$$\frac{A}{\neg\neg A}\ \neg\neg\mathrm{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\mathrm{e}$$

Example Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$q \wedge r$	$\neg\neg\mathrm{e}$ 2
4	r	$\wedge\mathrm{e}$ 3
5	$\neg\neg p$	$\neg\neg\mathrm{i}$ 1
6	$\neg\neg p \wedge r$	$\wedge\mathrm{i}$ 5, 4

$\rightarrow$ elimination

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e}$$

$\rightarrow$ elimination

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e}$$

Example Prove $p, p \rightarrow q, q \rightarrow r \vdash r$

$\to$ elimination

$$\frac{A \qquad A \to B}{B} \to\mathrm{e}$$

Example Prove $p, p \to q, q \to r \vdash r$

1	p	premise
2	$p \to q$	premise
3	$q \to r$	premise

$\to$ elimination

$$\frac{A \qquad A \to B}{B} \to\mathrm{e}$$

Example Prove $p, p \to q, q \to r \vdash r$

1	p	premise
2	$p \to q$	premise
3	$q \to r$	premise
4	q	$\to\mathrm{e}$ 1,2

$\rightarrow$ elimination

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e}$$

Example Prove $p, p \rightarrow q, q \rightarrow r \vdash r$

1	p	premise
2	$p \rightarrow q$	premise
3	$q \rightarrow r$	premise
4	q	$\rightarrow$e 1,2
5	r	$\rightarrow$e 4,3

$\rightarrow$ elimination

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e} \qquad\qquad \frac{A \rightarrow B \qquad \neg B}{\neg A} \text{MT}$$

$\rightarrow$ elimination

$$\frac{A \qquad A \rightarrow B}{B} \rightarrow\mathrm{e} \qquad\qquad \frac{A \rightarrow B \qquad \neg B}{\neg A}\ \mathrm{MT}$$

- $\rightarrow\mathrm{e}$ is also known as *Modus Ponens*
- MT is known as *Modus Tollens*

$\rightarrow$ introduction

$$\frac{A \to B \quad \neg B}{\neg A}\ \text{MT}$$

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B}\ {\to}\mathrm{i}$$

Example Prove $p \to q \vdash \neg q \to \neg p$

$\to$ introduction

$$\frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to\mathrm{i}$$

Example Prove $p \to q \vdash \neg q \to \neg p$

$\rightarrow$ introduction

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ \rightarrow\mathrm{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise

$\rightarrow$ introduction

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \rightarrow B} \rightarrow\mathrm{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption

$\rightarrow$ introduction

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \text{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ \rightarrow\mathrm{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2

$\rightarrow$ introduction

$$\frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT}$$

$$\frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B}\ \to\mathrm{i}$$

Example Prove $p \to q \vdash \neg q \to \neg p$

1	$p \to q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \to \neg p$	$\to$i 2–3

$\rightarrow$ introduction

$$\frac{A \rightarrow B \quad \neg B}{\neg A}\ \mathrm{MT}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ {\rightarrow}\mathrm{i}$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	$\rightarrow$i 2–3

Longer Example

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to\mathrm{i}$$

$$\frac{A \to B \quad \neg B}{\neg A} \mathrm{MT}$$

Longer Example

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to i$$

$$\frac{A \to B \quad \neg B}{\neg A} \text{MT}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption

Longer Example

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to\mathrm{i}$$

$$\frac{A \to B \quad \neg B}{\neg A} \text{ MT}$$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption

Longer Example

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to\mathrm{i}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption

Longer Example

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{A \to B \quad \neg B}{\neg A}\ \text{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to i$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3

Longer Example

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to i$$

$$\frac{A \to B \quad \neg B}{\neg A} \text{ MT}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4

Longer Example

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to i$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5

Longer Example

$$\frac{A \quad A \to B}{B}\ {\to}\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B}\ {\to}\mathrm{i}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg\mathrm{e}$ 5
7	r	$\to\mathrm{e}$ 1,6

Longer Example

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to\mathrm{i}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg\mathrm{e}$ 5
7	r	$\to\mathrm{e}$ 1,6

Longer Example

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to\mathrm{i}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg\mathrm{e}$ 5
7	r	$\to\mathrm{e}$ 1,6
8	$p \to r$	$\to\mathrm{i}$ 3–7

Longer Example

$$\frac{A \qquad A \to B}{B}\ {\to}\mathrm{e} \qquad \frac{A \to B \qquad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\begin{array}{|c|}\hline A \\ \vdots \\ B \\ \hline\end{array}}{A \to B}\ {\to}\mathrm{i}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$e 1,6
8	$p \to r$	$\to$i 3–7

Longer Example

$$\frac{A \quad A \to B}{B}\ {\to}\mathrm{e} \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B}\ {\to}\mathrm{i}$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg\mathrm{e}$ 5
7	r	${\to}\mathrm{e}$ 1,6
8	$p \to r$	${\to}\mathrm{i}$ 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	${\to}\mathrm{i}$ 2–8

Longer Example

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

$$\frac{A \quad A \to B}{B} \to\mathrm{e} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to\mathrm{i}$$

$$\frac{A \to B \quad \neg B}{\neg A} \mathsf{MT}$$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg\mathrm{i}$ 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg\mathrm{e}$ 5
7	r	$\to\mathrm{e}$ 1,6
8	$p \to r$	$\to\mathrm{i}$ 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	$\to\mathrm{i}$ 2–8

Longer Example

$$\frac{A \quad A \to B}{B} \to e \qquad \frac{A \to B \quad \neg B}{\neg A}\ \mathrm{MT} \qquad \frac{\boxed{\begin{matrix} A \\ \vdots \\ B \end{matrix}}}{A \to B} \to i$$

Prove $\vdash (q \to r) \to (\neg q \to \neg p) \to (p \to r)$

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$e 1,6
8	$p \to r$	$\to$i 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	$\to$i 2–8
10	$(q \to r) \to (\neg q \to \neg p) \to (p \to q)$	$\to$e 1–9

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad\qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad\qquad \frac{A \vee B \qquad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \qquad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\,\vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\,\vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\,\vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad\qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad\qquad \frac{A \vee B \quad \boxed{\begin{array}{c}A\\ \vdots\\ C\end{array}} \quad \boxed{\begin{array}{c}B\\ \vdots\\ C\end{array}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\;\vee\mathrm{i}_1 \qquad\qquad \frac{B}{A \vee B}\;\vee\mathrm{i}_w \qquad\qquad \frac{A \vee B \quad \boxed{\begin{array}{c}A\\ \vdots\\ C\end{array}} \quad \boxed{\begin{array}{c}B\\ \vdots\\ C\end{array}}}{C}\;\vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

$$\begin{array}{lll} 1 & p \vee q & \text{premise} \\ \hline 2 & p & \text{assumption} \\ 3 & q \vee p & \vee\mathrm{i}_2\ 2 \\ \hline \end{array}$$

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\,\vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\,\vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c}A\\ \vdots\\ C\end{array}} \quad \boxed{\begin{array}{c}B\\ \vdots\\ C\end{array}}}{C}\,\vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2
4	q	assumption

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c}A\\ \vdots\\ C\end{array}} \quad \boxed{\begin{array}{c}B\\ \vdots\\ C\end{array}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2
4	q	assumption
5	$q \vee p$	$\vee\mathrm{i}_1$ 2

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad\qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad\qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

$$\begin{array}{lll} 1 & p \vee q & \text{premise} \\ \hline 2 & p & \text{assumption} \\ 3 & q \vee p & \vee\mathrm{i}_2\ 2 \\ \hline 4 & q & \text{assumption} \\ 5 & q \vee p & \vee\mathrm{i}_1\ 2 \\ \hline \end{array}$$

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{matrix} A \\ \vdots \\ C \end{matrix}} \quad \boxed{\begin{matrix} B \\ \vdots \\ C \end{matrix}}}{C}\ \vee\mathrm{e}$$

Example 1 Prove $p \vee q \vdash q \vee p$

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee\mathrm{i}_2$ 2
4	q	assumption
5	$q \vee p$	$\vee\mathrm{i}_1$ 2
6	$q \vee p$	$\vee\mathrm{e}$ 1, 2–3, 4–5

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \to r,\ q \to r \vdash r$

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad\qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad\qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \to r,\ q \to r \vdash r$

1	$p \vee q$	premise
2	$p \to r$	premise
3	$q \to r$	premise

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\,\vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\,\vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\,\vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \to r,\ q \to r \vdash r$

1	$p \vee q$			premise
2	$p \to r$			premise
3	$q \to r$			premise
4	p	assumption	q	assumption
5	r	$\to$e 4, 2	r	$\to$e 4, 3

$\vee$ introduction and elimination

$$\frac{A}{A \vee B}\ \vee\mathrm{i}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{i}_w \qquad \frac{A \vee B \quad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \quad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee\mathrm{e}$$

Example 2 Prove $p \vee q,\ p \to r,\ q \to r \vdash r$

1	$p \vee q$			premise
2	$p \to r$			premise
3	$q \to r$			premise
4	p	assumption	q	assumption
5	r	$\to$e 4, 2	r	$\to$e 4, 3
6	r			$\vee$e 1, 4–5

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

$\bot$ elimination and $\neg$ elimination

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \lor q \vdash p \to q$

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \rightarrow q$

1	$\neg p \vee q$	premise

$\bot$ elimination and $\neg$ elimination

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \rightarrow q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption

$\bot$ elimination and $\neg$ elimination

$$\frac{\bot}{A}\ \bot\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\bot}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \rightarrow q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \rightarrow q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\perp$	$\neg$e 3,2

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp e \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg e$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\perp$	$\neg e$ 3,2
5	q	$\perp e$ 4

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\perp$	$\neg$e 3,2
5	q	$\perp$e 4

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\perp$	$\neg$e 3,2
5	q	$\perp$e 4
6	$p \to q$	$\to$i 3–5

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption		
4	$\perp$	$\neg$e 3,2		
5	q	$\perp$e 4		
6	$p \to q$	$\to$i 3–5		

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \lor q \vdash p \to q$

1	$\neg p \lor q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\perp$	$\neg$e 3,2		
5	q	$\perp$e 4		
6	$p \to q$	$\to$i 3–5		

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\perp$	$\neg$e 3,2	q	copy 2
5	q	$\perp$e 4		
6	$p \to q$	$\to$i 3–5		

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\perp$	$\neg$e 3,2	q	copy 2
5	q	$\perp$e 4		
6	$p \to q$	$\to$i 3–5		

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\perp$	$\neg$e 3,2	q	copy 2
5	q	$\perp$e 4	$p \to q$	$\to$i 3–4
6	$p \to q$	$\to$i 3–5		

$\perp$ elimination and $\neg$ elimination

$$\frac{\perp}{A}\ \perp\mathrm{e} \qquad\qquad \frac{A \qquad \neg A}{\perp}\ \neg\mathrm{e}$$

$$\frac{\boxed{\begin{array}{c}A\\ \vdots\\ B\end{array}}}{A \to B}\ \to\mathrm{i}$$

$$\frac{A \vee B \qquad \boxed{\begin{array}{c}A\\ \vdots\\ C\end{array}} \qquad \boxed{\begin{array}{c}B\\ \vdots\\ C\end{array}}}{C}\ \vee\mathrm{e}$$

Example Prove $\neg p \vee q \vdash p \to q$

1	$\neg p \vee q$			premise
2	$\neg p$	assumption	q	assumption
3	p	assumption	p	assumption
4	$\perp$	$\neg$e 3,2	q	copy 2
5	q	$\perp$e 4	$p \to q$	$\to$i 3–4
6	$p \to q$	$\to$i 3–5		
7	$p \to q$			$\vee$e 1, 2–6

$\neg$ introduction and proof by contradiction

$$\dfrac{\begin{array}{|c|}\hline A \\ \vdots \\ \bot \\ \hline\end{array}}{\neg A}\;\neg\mathrm{i} \qquad\qquad \dfrac{\begin{array}{|c|}\hline \neg A \\ \vdots \\ \bot \\ \hline\end{array}}{A}\;\mathsf{PBC} \qquad\qquad \dfrac{}{A \vee \neg A}\;\mathsf{LEM}$$

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathsf{PBC} \qquad \frac{}{A \vee \neg A}\ \mathsf{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \rightarrow q,\ p \rightarrow \neg q \vdash \neg p$

1	$p \rightarrow q$	premise
2	$p \rightarrow \neg q$	premise
3	p	assumption
4	q	$\rightarrow$e 1, 3
5	$\neg q$	$\rightarrow$e 2, 3

$\neg$ introduction and proof by contradiction

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3
6	$\bot$	$\neg$e 4, 5

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad\qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \vdash \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3
6	$\bot$	$\neg$e 4, 5

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 1 Prove $p \to q,\ p \to \neg q \ \vdash\ \neg p$

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to$e 1, 3
5	$\neg q$	$\to$e 2, 3
6	$\bot$	$\neg$e 4, 5
7	$\neg p$	$\neg$i 2–4

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

$\neg$ introduction and proof by contradiction

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad\qquad \dfrac{}{A \lor \neg A}\ \text{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad\qquad \frac{}{A \lor \neg A}\ \text{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2
4	$\neg\neg p$	$\neg$i 2–3

$\neg$ introduction and proof by contradiction

$$\dfrac{\boxed{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \dfrac{\boxed{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}}{A}\ \text{PBC} \qquad\qquad \dfrac{}{A \vee \neg A}\ \text{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to\mathrm{e}$ 1, 2
4	$\neg\neg p$	$\neg\mathrm{i}$ 2–3
5	p	$\neg\neg\mathrm{e}$ 4

$\neg$ introduction and proof by contradiction

$$\dfrac{\boxed{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \dfrac{\boxed{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}}{A}\ \mathrm{PBC} \qquad\qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 2 Prove $\neg p \to \bot \vdash p$

1	$\neg p \to \bot$	premise
2	$\neg p$	assumption
3	$\bot$	$\to$e 1, 2
4	$\neg\neg p$	$\neg$i 2–3
5	p	$\neg\neg$e 4

PBC can be simulated

$\neg$ introduction and proof by contradiction

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \perp \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \perp \end{array}}}{A}\ \text{PBC} \qquad \dfrac{}{A \vee \neg A}\ \text{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1 $\neg(p \vee \neg p)$ assumption

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption

$\neg$ introduction and proof by contradiction

$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee \mathrm{i}_1$ 2

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \text{PBC} \qquad \frac{}{A \vee \neg A}\ \text{LEM}$$

Example 3 Prove $\vdash p \vee \neg p$

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1

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$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad\qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

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5	$\neg p$	$\neg\mathrm{i}$ 2–4

$\neg$ introduction and proof by contradiction

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6	$p \vee \neg p$	$\vee\mathrm{i}_2$ 5

$\neg$ introduction and proof by contradiction

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5	$\neg p$	$\neg\mathrm{i}$ 2–4
6	$p \vee \neg p$	$\vee\mathrm{i}_2$ 5
7	$\bot$	$\neg\mathrm{e}$ 6, 1

$\neg$ introduction and proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad\qquad \frac{}{A \lor \neg A}\ \mathrm{LEM}$$

Example 3 Prove $\vdash p \lor \neg p$

1	$\neg(p \lor \neg p)$	assumption
2	p	assumption
3	$p \lor \neg p$	$\lor\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1
5	$\neg p$	$\neg\mathrm{i}$ 2–4
6	$p \lor \neg p$	$\lor\mathrm{i}_2$ 5
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$$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg\mathrm{i} \qquad \dfrac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

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8	$p \vee \neg p$	PBC 7

$\neg$ introduction and proof by contradiction

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Example 3 Prove $\vdash p \vee \neg p$

LEM can be simulated too

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee\mathrm{i}_1$ 2
4	$\bot$	$\neg\mathrm{e}$ 3, 1
5	$\neg p$	$\neg\mathrm{i}$ 2–4
6	$p \vee \neg p$	$\vee\mathrm{i}_2$ 5
7	$\bot$	$\neg\mathrm{e}$ 6, 1
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$\neg$ introduction and proof by contradiction

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PBC and LEM are *derived* rules

$\neg$ introduction and proof by contradiction

$$\dfrac{\boxed{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}}{\neg A}\ \neg\mathrm{i} \qquad\qquad \dfrac{\boxed{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}}{A}\ \mathrm{PBC} \qquad\qquad \dfrac{}{A \vee \neg A}\ \mathrm{LEM}$$

PBC and LEM are *derived* rules

MT and $\neg\neg\mathrm{i}$ are derived rules too

Soundness of natural deduction

We will prove a crucial property of natural deduction:

any formula A derived from a set U of premises is a logical consequence of U

Theorem 1 (Soundness)
For all formulas $A_1, \ldots, A_n$ and A such that $A_1, \ldots, A_n \vdash A$, we have that

$$A_1, \ldots, A_n \models A\,.$$

For the proof of the theorm, we will rely on this lemma:

Lemma 2
For all formulas $A_1, \ldots, A_n, A$ and B,

1. $A_1, \ldots, A_n, A \models B$ *iff* $A_1, \ldots, A_n \models A \to B$
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Soundness proof

The proof of Theorem 1 is by induction on proof length

The *length* of a natural deduction proof is the number of lines in it

Proof of Theorem 1.

Let P be the a proof of $A_1, \ldots, A_n \vdash A$, seen as a sequence of formulas.

Assume, without loss of generality, that A is the last formula in the sequence.

By induction on the length l of P.

($l = 1$)

Then $A = A_i$ for some $i \in \{1, \ldots, n\}$. Trivially, $A_1, \ldots, A_n \models A_i$.

(continued)

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(continued)

Soundness proof (continued)

($l > 1$)

Assume by induction that the theorem holds for all proofs of length $l' < l$.

The proof depends on the final rule used to derive A.

($\wedge\mathrm{e}_1$) If A was derived by $\wedge\mathrm{e}_1$, then P looks like:

$$\begin{array}{ll} A_1 & \text{premise} \\ \vdots & \\ A \wedge B & \ldots \\ \vdots & \\ A & \wedge\mathrm{e}_1 \end{array}$$

for some formula B.

Note that the subsequence of P from A_1 to $A \wedge B$ is a proof of $A \wedge B$ of length $< l$.

Then, by inductive hypothesis, $A_1, \ldots, A_n \models A \wedge B$. Hence, $A_1, \ldots, A_n \models A$.

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\vdots & \\
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$$

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($\wedge$i) Then A has the form $B_1 \wedge B_2$ and P looks like:

$$\begin{array}{ll} A_1 & \text{premise} \\ \vdots & \\ B_1 & \ldots \\ \vdots & \\ B_2 & \ldots \\ \vdots & \\ B_1 \wedge B_2 & \wedge\mathrm{i} \end{array} \qquad \text{or} \qquad \begin{array}{ll} A_1 & \text{premise} \\ \vdots & \\ B_2 & \ldots \\ \vdots & \\ B_1 & \ldots \\ \vdots & \\ B_1 \wedge B_2 & \wedge\mathrm{i} \end{array}$$

Soundness proof (continued)

($\wedge$i) Then A has the form $B_1 \wedge B_2$ and P looks like:

$$
\begin{array}{ll}
A_1 & \text{premise} \\
\vdots & \\
B_1 & \ldots \\
\vdots & \\
B_2 & \ldots \\
\vdots & \\
B_1 \wedge B_2 & \wedge\mathrm{i}
\end{array}
\quad \text{or} \quad
\begin{array}{ll}
A_1 & \text{premise} \\
\vdots & \\
B_2 & \ldots \\
\vdots & \\
B_1 & \ldots \\
\vdots & \\
B_1 \wedge B_2 & \wedge\mathrm{i}
\end{array}
$$

This implies that P contains a (shorter) proof of B_1 and of B_2.

Soundness proof (continued)

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$$
\begin{array}{ll}
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\vdots & \\
B_1 & \ldots \\
\vdots & \\
B_2 & \ldots \\
\vdots & \\
B_1 \wedge B_2 & \wedge\mathrm{i}
\end{array}
\qquad \text{or} \qquad
\begin{array}{ll}
A_1 & \text{premise} \\
\vdots & \\
B_2 & \ldots \\
\vdots & \\
B_1 & \ldots \\
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B_1 \wedge B_2 & \wedge\mathrm{i}
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This implies that P contains a (shorter) proof of B_1 and of B_2.

Then, by inductive hypothesis, $A_1, \ldots, A_n \models B_i$ for $i = 1, 2$.

Soundness proof (continued)

($\wedge$i) Then A has the form $B_1 \wedge B_2$ and P looks like:

$$
\begin{array}{ll}
A_1 & \text{premise} \\
\vdots & \\
B_1 & \ldots \\
\vdots & \\
B_2 & \ldots \\
\vdots & \\
B_1 \wedge B_2 & \wedge\mathrm{i}
\end{array}
\qquad \text{or} \qquad
\begin{array}{ll}
A_1 & \text{premise} \\
\vdots & \\
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\vdots & \\
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This implies that P contains a (shorter) proof of B_1 and of B_2.

Then, by inductive hypothesis, $A_1, \ldots, A_n \models B_i$ for $i = 1, 2$.

Hence, $A_1, \ldots, A_n \models B_1 \wedge B_2$.

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P looks like:

1	A_1	premise
2	$\vdots$	
3	B_1	assumption
4	$\vdots$	
5	B_2	$\dots$
6	$B_1 \rightarrow B_2$	$\rightarrow$i

Soundness proof (continued)

($\rightarrow$i) Then A has the form $B_1 \rightarrow B_2$ and

P looks like:

1	A_1	premise
2	$\vdots$	
3	B_1	assumption
4	$\vdots$	
5	B_2	...
6	$B_1 \rightarrow B_2$	$\rightarrow$i

but then

Soundness proof (continued)

($\rightarrow$i) Then A has the form $B_1 \rightarrow B_2$ and

P looks like:

1	A_1	premise
2	$\vdots$	
3	B_1	assumption
4	$\vdots$	
5	B_2	$\ldots$
6	$B_1 \rightarrow B_2$	$\rightarrow$i

but then

1	A_1	premise
2	$\vdots$	
3	B_1	premise
4	$\vdots$	
5	B_2	$\ldots$

is a proof of B_2 from $A_1, \ldots, A_n, B_1$ that is shorter than P.

Soundness proof (continued)

$(\to\text{i})$ Then A has the form $B_1 \to B_2$ and

P looks like:

1	A_1	premise
2	$\vdots$	
3	B_1	assumption
4	$\vdots$	
5	B_2	$\ldots$
6	$B_1 \to B_2$	$\to\text{i}$

but then

1	A_1	premise
2	$\vdots$	
3	B_1	premise
4	$\vdots$	
5	B_2	$\ldots$

is a proof of B_2 from $A_1, \ldots, A_n, B_1$ that is shorter than P.

Then, by inductive hypothesis, $A_1, \ldots, A_n, B_1 \models B_2$.

Soundness proof (continued)

($\to$i) Then A has the form $B_1 \to B_2$ and

P looks like:

1	A_1	premise
2	$\vdots$	
3	B_1	assumption
4	$\vdots$	
5	B_2	$\ldots$
6	$B_1 \to B_2$	$\to$i

but then

1	A_1	premise
2	$\vdots$	
3	B_1	premise
4	$\vdots$	
5	B_2	$\ldots$

is a proof of B_2 from $A_1, \ldots, A_n, B_1$ that is shorter than P.

Then, by inductive hypothesis, $A_1, \ldots, A_n, B_1 \models B_2$.

It follows from Lemma 2(1) that $A_1, \ldots, A_n \models B_1 \to B_2$.

Soundness proof (continued)

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P looks like:

1	A_1	premise
2	$\vdots$	
3	B	assumption
4	$\vdots$	
5	$\perp$	$\ldots$
6	$\neg B$	$\neg$i

Soundness proof (continued)

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P looks like:

1	A_1	premise
2	$\vdots$	
3	B	assumption
4	$\vdots$	
5	$\perp$	$\ldots$
6	$\neg B$	$\neg$i

but then

Soundness proof (continued)

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P looks like:

1	A_1	premise
2	$\vdots$	
3	B	assumption
4	$\vdots$	
5	$\bot$	$\ldots$
6	$\neg B$	$\neg$i

but then

1	A_1	premise
2	$\vdots$	
3	B	premise
4	$\vdots$	
5	$\bot$	$\ldots$

is a proof of $\bot$ from $A_1, \ldots, A_n, B$ that is shorter than P.

Soundness proof (continued)

($\neg$i) Then A has the form $\neg B$ and

P looks like:

1	A_1	premise
2	$\vdots$	
3	B	assumption
4	$\vdots$	
5	$\perp$	$\ldots$
6	$\neg B$	$\neg$i

but then

1	A_1	premise
2	$\vdots$	
3	B	premise
4	$\vdots$	
5	$\perp$	$\ldots$

is a proof of $\perp$ from $A_1, \ldots, A_n, B$ that is shorter than P.

Then, by inductive hypothesis, $A_1, \ldots, A_n, B \models \perp$.

Soundness proof (continued)

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P looks like:

1	A_1	premise
2	$\vdots$	
3	B	assumption
4	$\vdots$	
5	$\bot$	$\ldots$
6	$\neg B$	$\neg$i

but then

1	A_1	premise
2	$\vdots$	
3	B	premise
4	$\vdots$	
5	$\bot$	$\ldots$

is a proof of $\bot$ from $A_1, \ldots, A_n, B$ that is shorter than P.

Then, by inductive hypothesis, $A_1, \ldots, A_n, B \models \bot$.

It follows from Lemma 2 that $A_1, \ldots, A_n \models \neg B$.

Soundness proof (continued)

($\wedge i_2$) Analogous to $\wedge i_2$ case.

($\vee i_1$) Exercise.

($\vee i_1$) Exercise.

($\vee e$) Exercise.

($\rightarrow e$) Exercise.

($\neg e$) Exercise.

($\bot e$) Exercise.

($\neg\neg e$) Exercise.

$\square$

Completeness of natural deduction

We will now prove another important property of natural deduction:

any logical consequence A of a set U of formulas has a proof with premises U

Assumption: We remove $\top$ from the language and simulate it with $p \vee \neg p$

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For all formulas $A_1, \ldots, A_n$ and A such that $A_1, \ldots, A_n \models A$, we have that

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Completeness of natural deduction

We will now prove another important property of natural deduction:

any logical consequence A of a set U of formulas has a proof with premises U

Assumption: We remove $\top$ from the language and simulate it with $p \vee \neg p$

Theorem 3 (Completeness)

For all formulas $A_1, \ldots, A_n$ and A such that $A_1, \ldots, A_n \models A$, we have that

$$A_1, \ldots, A_n \vdash A .$$

To prove this theorem, we will rely on several intermediate results

Completeness of natural deduction

Lemma 4

For all formulas $A_1, \ldots, A_n$ *and* A *the following holds:*

1. $A_1, A_2, \ldots, A_n \models A$ *implies* $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.
2. $\vdash A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$ *implies* $A_1, A_2, \ldots, A_n \vdash A$.

Completeness of natural deduction

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Proof.

By induction on n in both cases (see Huth & Ryan). □

Completeness of natural deduction

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Theorem 5 (Completeness for validity)

All valid formulas B are provable in natural deduction: if $\models B$ *then* $\vdash B$.

Completeness of natural deduction

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Proof of Theorem 3 ($A_1, \ldots, A_n \models A$ implies $A_1, \ldots, A_n \vdash A$).

Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

Completeness of natural deduction

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For all formulas $A_1, \ldots, A_n$ and A the following holds:

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Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

By Lemma 4(1), $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

Completeness of natural deduction

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Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

By Lemma 4(1), $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

By Theorem 5, $\vdash A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

Completeness of natural deduction

Lemma 4

For all formulas $A_1, \ldots, A_n$ *and* A *the following holds:*

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2. $\vdash A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$ *implies* $A_1, A_2, \ldots, A_n \vdash A$.

Theorem 5 (Completeness for validity)

All valid formulas B *are provable in natural deduction: if* $\models B$ *then* $\vdash B$.

Proof of Theorem 3 ($A_1, \ldots, A_n \models A$ implies $A_1, \ldots, A_n \vdash A$).

Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

By Lemma 4(1), $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

By Theorem 5, $\vdash A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

By Lemma 4(2), $A_1, A_2, \ldots, A_n \vdash A$.

Completeness of natural deduction

Lemma 4

For all formulas $A_1, \ldots, A_n$ *and* A *the following holds:*

1. $A_1, A_2, \ldots, A_n \models A$ *implies* $\models A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.
2. $\vdash A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$ *implies* $A_1, A_2, \ldots, A_n \vdash A$.

Theorem 5 (Completeness for validity)

All valid formulas B *are provable in natural deduction: if* $\models B$ *then* $\vdash B$.

So we are left with proving Theorem 5

Towards a proof of Theorem 5

Lemma 6

Let A be a formula over variables $p_1, \ldots, p_n$ with $n \geq 0$ and let $\mathcal{I}$ be an interpretation. Let $\hat{p}_i = p$ if $\mathcal{I} \models p$ and $\hat{p}_i = \neg p$ otherwise. Then,

$$\hat{p}_1, \ldots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \quad \text{and} \quad \hat{p}_1, \ldots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A\,.$$

Proof of Lemma 6. By structural induction on A.

(Base case)
If A is just a variable, say p_1, then it is immediate that $p_1 \vdash p_1$ and $\neg p_1 \vdash \neg p_1$.
If A is $\perp$ then $n = 0$ and $\mathcal{I} \not\models A$. We can prove $\neg\perp$ from no premises by $\neg$i.

(Inductive Step) If A is not a variable or $\perp$, assume the result holds for all proper subformulas of A.
We reason by cases on the form of A.

(cont.)

Towards a proof of Theorem 5

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If A is $\bot$ then $n = 0$ and $\mathcal{I} \not\models A$. We can prove $\neg\bot$ from no premises by $\neg$i.

(Inductive Step) If A is not a variable or $\bot$, assume the result holds for all proper subformulas of A.
We reason by cases on the form of A.

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Towards a proof of Theorem 5

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Towards a proof of Theorem 5

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Towards a proof of Theorem 5

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(Inductive Step) If A is not a variable or $\perp$, assume the result holds for all proper subformulas of A.
We reason by cases on the form of A.

(cont.)

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = \neg B$) (that is, suppose A has the form $\neg B$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B$. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg B$.
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B$. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B$.
  Take a proof of B from $\hat{p}_1, \ldots, \hat{p}_n$ and apply $\neg\neg\mathrm{i}$ to B.
  The resulting proof is a proof of $\neg A$.

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$)
(continued)

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- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B$. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B$.
  Take a proof of B from $\hat{p}_1, \ldots, \hat{p}_n$ and apply $\neg\neg i$ to B.
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Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

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Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \wedge B_2$)

-

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \wedge B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \models B_2$.

  By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B_1$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash B_2$.

  A proof of A from $\hat{p}_1, \ldots, \hat{p}_n$ is obtained by chaining a proof of B_1 and a proof of B_2 and applying $\wedge\mathrm{i}$ to B_1 and B_2.

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \wedge B_2$)

- If $\mathcal{I} \not\models A$ then $\mathcal{I} \not\models B_k$ for some $k \in \{1, 2\}$. Say $k = 1$ (the other case is similar).
  By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B_1$.
  A proof of $\neg B_1$ can be extended to a proof of $\neg A$ as follows:

1	$\vdots$	
2	$\neg B_1$	
3	$B_1 \wedge B_2$	assumption
4	B_1	$\wedge e_1$ 3
5	$\perp$	$\perp$i 4, 2
6	$\neg(B_1 \wedge B_2)$	$\perp$i 3, 5

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \vee B_2$)

-

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \vee B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \models B_k$ for some $k \in \{1, 2\}$.

  A proof of A from $\hat{p}_1, \ldots, \hat{p}_n$ is obtained from a proof of B_k by applying $\vee\mathrm{i}_k$ to B_k to get $B_1 \vee B_2$.

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$)
(continued)

($A = B_1 \vee B_2$)

- If $\mathcal{I} \not\models A$ then $\mathcal{I} \not\models B_1$ and $\mathcal{I} \not\models B_2$.
  A proof of $\neg A$ from $\hat{p}_1, \ldots, \hat{p}_n$ is obtained by chaining a proof of $\neg B_1$ and a proof of $\neg B_2$ and continuing as follows:

1	$\vdots$			
2	$B_1 \vee B_2$			assumption
3	B_1	assumption	B_2	assumption
4	$\bot$	$\bot$i (with $\neg B_1$)	$\bot$	$\bot$i (with $\neg B_2$)
5	$\bot$			$\vee$e 2, 3 – –4
6	$\neg(B_1 \vee B_2)$			$\bot$i 2 – –5

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \rightarrow B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B_1$ or $\mathcal{I} \models B_2$.
  (exercise)
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \not\models B_2$.
  (exercise)

$\square$

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$)
(continued)

($A = B_1 \to B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B_1$ or $\mathcal{I} \models B_2$.
  (exercise)
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \not\models B_2$.
  (exercise)

$\square$

Towards a proof of Theorem 5

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A$ if $\mathcal{I} \models A$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$ if $\mathcal{I} \not\models A$) (continued)

($A = B_1 \to B_2$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B_1$ or $\mathcal{I} \models B_2$.
  (exercise)
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \not\models B_2$.
  (exercise)

$\square$

Towards a proof of Theorem 5

Lemma 7

Let $L_2, \ldots, L_n, A$ *be formulas and let* p *one of* $A's$ *variables.*
If $p, L_2, \ldots, L_n \vdash A$ *and* $\neg p, L_2, \ldots, L_n \vdash A$ *then* $L_2, \ldots, L_n \vdash A$*.*

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise	and	1	$\neg p$	premise
2	L_2	premise		2	L_2	premise
3	$\vdots$			3	$\vdots$	
4	A	$\ldots$		4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee e$

$\square$

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

and

1	$\neg p$	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee e$

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

and

1	$\neg p$	premise
2	L_2	premise
3	$\vdots$	
4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee e$

$\square$

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

1	p	premise	and	1	$\neg p$	premise
2	L_2	premise		2	L_2	premise
3	$\vdots$			3	$\vdots$	
4	A	$\ldots$		4	A	$\ldots$

The following is a proof of A from $L_2, \ldots, L_n$:

1	$p \vee \neg p$			LEM
2	p	assumption	$\neg p$	assumption
3	L_2	premise	L_2	premise
4	$\vdots$		$\vdots$	
5	A	$\ldots$	A	$\ldots$
6	A			$\vee$e

$\square$

Proof of Theorem 5 ($\models A$ implies $\vdash A$).

Let $p_1, \dots, p_n$ be all of A's variables and consider the set

$$S = \{ p_1, \neg p_1 \} \times \cdots \times \{ p_n, \neg p_n \} ,$$

of all tuples $(\hat{p}_1, \dots, \hat{p}_n)$ where each $\hat{p}_i$ is either p_i or $\neg p_i$.
We prove by induction on $i = 1, \dots, n+1$ that

$$\hat{p}_i, \dots, \hat{p}_n \vdash A \quad \text{for every } (\hat{p}_1, \dots, \hat{p}_n) \in S . \tag{1}$$

The theorem then follows from Property (1) for $i = n + 1$.

($i = 1$) Property (1) holds by Lemma 6 since every $(\hat{p}_1, \dots, \hat{p}_n) \in S$ corresponds to an interpretation of A and all interpretations satisfy A (by def. of validity).

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