CS:5810 Formal Methods in Software Engineering

Dynamic Models in Alloy

Overview

- Basics of dynamic models
  - Modeling a system’s states and state transitions
  - Modeling operations causing transitions
- Simple example of operations

Static Models

- So far, we've used Alloy to define the allowable values of state components
  - values of sets
  - values of relations
- A model instance is a set of state component values that
  - Satisfies the constraints defined by multiplicities, fact, "realism" conditions, …

Static Model Instances

```
Person = {Matt, Sue}
Man = {Matt}
Woman = {Sue}
Married = {}
spouse = {}
children = {}
siblings = {}
```

```
Person = {Matt, Sue}
Man = {Matt}
Woman = {Sue}
Married = {Matt, Sue}
spouse = {(Matt,Sue), (Sue,Matt)}
children = {}
siblings = {}
```

```
Person = {Matt, Sue, Sean}
Man = {Matt, Sean}
Woman = {Sue}
Married = {Matt, Sue}
spouse = {(Matt,Sue), (Sue,Matt)}
children = {(Matt,Sean), (Sue,Sean)}
siblings = {}
```

Dynamic Models

- Static models let us describe the legal states of a dynamic system
- However, we'd like also to be able to describe possible transitions between states

  E.g.

  – Two unmarried people become each other's spouses once they get married
  – People go from being alive to not being alive when the die
  – A person becomes someone's child after being born

Example

Family Model

```
abstract sig Person {
      children: set Person,
      siblings: set Person
}

sig Man, Woman extends Person {}

sig Married in Person {
      spouse: one Married
}
```

State Transitions

Two people get married

- At time t, `spouse = {}`
- At time t’, `spouse = {(Matt, Sue), (Sue, Matt)}`

⇒ We can add the notion of time in the spouse relation

```
Person = {Matt,Sue}
Man = {Matt}
Woman = {Sue}
Married = {}
spouse = {}
children = {}
siblings = {}
```

Time t

→

```
Person = {Matt, Sue}
Man = {Matt}
Woman = {Sue}
Married = {Matt, Sue}
spouse = {(Matt, Sue), (Sue, Matt)}
children = {}
siblings = {}
```

Time t’

Modeling State Transitions

- Until version 6, Alloy had no predefined notion of time and of state transition
- This is not really a problem since there are several ways to model dynamic aspects of a system in Alloy
- A general and relatively simple way is to:
  1. introduce a `Time` signature expressing time
  2. add a time component to each relation that changes over time

Family Model Signatures

```
abstract sig Person {
	children: set Person,
	siblings: set Person
}

sig Man, Woman extends Person {}

sig Married in Person {
	spouse: one Married
}
```

Family Model Signatures with Time

```
sig Time {}

abstract sig Person {
      children: Person set -> Time,
      siblings: Person set -> Time
}

sig Man, Woman extends Person {}

sig Married in Person {
      spouse: Married one -> Time
}
```

Transitions

Two people get married

- At time t, `Married = {}`
- At time t’, `Married = {Matt, Sue}`

```
Person = {Matt,Sue}
Man = {Matt}
Woman = {Sue}
Married = {}
spouse = {}
children = {}
siblings = {}
```

Time t

→

```
Person = {Matt, Sue}
Man = {Matt}
Woman = {Sue}
Married = {Matt, Sue}
spouse = {(Matt, Sue), (Sue, Matt)}
children = {}
siblings = {}
```

Time t’

Transitions

A person is born

- At time t, `Person = {}`
- At time t’, `Person = {Sue}`

For simplicity, we will not use time-dependent signatures

```
Person = {}
Man = {}
Woman = {}
spouse = {}
children = {}
siblings = {}
```

Time t

→

```
Person = {Sue}
Man = {}
Woman = {Sue}
spouse = {}
children = {}
siblings = {}
```

Time t’

Keeping Signatures Static

```
abstract sig Person {
  children: Person set -> Time,
  siblings: Person set -> Time,
  spouse: Person lone -> Time
}
sig Man, Woman extends Person {}
```

```
sig Married in Person {
      spouse: Married one -> Time
}
```

Keeping Signatures Static

```
abstract sig Person {
  children: Person set -> Time,
  siblings: Person set -> Time,
  spouse: Person lone -> Time,
  alive: set Time
}

sig Man, Woman extends Person {}
```

Revising Constraints

```
abstract sig Person {
  children: Person set -> Time,
  siblings: Person set -> Time,
  spouse: Person lone -> Time,
  alive: set Time,
  parents: Person set -> Time
}

sig Man, Woman extends Person {}

fun parents[] : Person ->Person { ~children }
fact parentsDef {
  all t: Time | parents.t = ~(children.t)
}
```

Revising Constraints

```
-- Time-dependent parents relation

fact parentsDef {
  all t: Time | parents.t = ~(children.t)
}

-- Two persons are blood relatives (at time t) iff
-- they have a common ancestor (at time t)
pred BloodRelatives [p, q: Person, t: Time]
{
  some p.*(parents.t) & q.*(parents.t)
}
```

Revising *Static* Constraints

```
-- People cannot be their own ancestors (at any time)
all t: Time | no p: Person |
  p in p.^(parents.t)

-- No one can have more than one father or mother (at any time)
all t: Time | all p: Person |
  lone (p.parents.t & Man)
  and
  lone (p.parents.t & Woman)

...
```

Revising *Static* Constraints

```
-- (At all times) your siblings are those people other than you
-- who have the same parents you have
all t: Time | all p: Person |
  p.siblings.t = { q: Person - p | some q.parents.t and
                                   p.parents.t = q.parents.t }


-- (At all times) the spouse relation is symmetric
all t: Time |
  spouse.t = ~(spouse.t)
```

Revising *Static* Constraints

```
-- (At all times) a spouse can't be a sibling
all t: Time | no p: Person |
  some p.spouse.t and p.spouse.t in p.siblings.t


-- (At all times) people can't be married to a blood relative
  all t: Time | no p: Person |
    let s = p.spouse.t |
      some s and BloodRelatives[p, s, t]
```

Revising *Static* Constraints

```
-- (At all times) a person can't have children with a blood relative
all t: Time | all p, q: Person |
  (some (p.children.t & q.children.t) and p != q)
  implies
  not BloodRelatives[p, q, t]
```

A Better Approach: Mutable Relations

Alloy 6 incorporates an implicit, built-in notion of (discrete) time

- The meaning of an Alloy model is actually an infinite sequence of instances, or a trace
- Each instance in a trace corresponds to a state of a dynamic system
- Signatures/relations can change from state to state
- A set of temporal operators allows us to express properties over time as properties over traces

Mutable Fields: Relations that Change Over Time

```
abstract sig Color {}

one sig      Green, Yellow, Red extends Color {}

one sig TrafficLight
{
  var col: Color   -- value here can change over time
}

fun c : Color { TrafficLight.col }
```

Mutable Fields: Relations that Change Over Time

```
-- enum abbreviates a partition of a signature into singletons
enum Color { Green, Yellow, Red }

one sig TrafficLight
{
  var col: Color   -- value here can change over time
}

fun c : Color { TrafficLight.col }
```

From Instances to Traces

- Models with mutable signatures and/or fields represent *dynamic systems*, systems that change over time
- Instead of standing for a set of instances, a dynamic model stands for a set of *traces*
- A trace is an infinite sequence of instances
  - An instance now describes just one possible *state* of a system
  - A trace describes a particular sequence of *state transitions* for the system

From Instances to Traces

An Alloy model captures the behavior of a system over time by means of constraints containing *temporal operators*

Temporal operators implicitly talk about (properties of) traces

Temporal Operators in Alloy 6

Formula	Intuitive meaning
`always p`	p holds from current state/instance forward in a trace
`historically p`	p holds from current state backward
`after p`	p holds in next state (after the current one)
`before p`	p holds in previous state (before the current one)
`eventually p`	p holds in current state or a later one
`once p`	p holds in current state or an earlier one
`p until q`	q holds eventually and p holds continuously until then
`p since q`	p has held continuously since last time q held
`e’`	denotes the value of e in next state

Example Traces

Time steps	1	2	3	4	5	6	7	8	9	…												
`p (state prop.)`	•	•	•	•	•		•	•	•	•	•				•	•	•	•	•	•	•	…
`q (state prop.)`						•								•	•							…
`always` `p`															•	•	•	•	•	•	•	…
`historically` `p`	•	•	•	•	•																	…
`after` `p`	•	•	•	•		•	•	•	•	•				•	•	•	•	•	•	•	•	…
`before` `p`		•	•	•	•	•		•	•	•	•	•				•	•	•	•	•	•	…
`eventually` `q`	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•							…
`once` `q`						•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	…
`p` **`until`** `q`	•	•	•	•	•	•								•	•							…
`p` **`since`** `q`						•	•	•	•	•	•			•	•	•	•	•	•	•	•	…

• = true

Temporal Operator Precedence

High

```
! _    not _
always _          eventually _    after _
historically _    once _          before _
_ until _    _ since _
_ && _    _ and _
_ => _    _ implies _    _ else _
_ <=> _    _ iff _
_ || _    _ or _
let _ | _    no _ | _    some _ | _    lone _ | _    one _ | _
```

Low

The Family Model with Mutable Fields

```
enum Liveness { Alive, Dead, Unborn }

abstract sig Person {
  var children: set Person,
  var parents: set Person,
  var siblings: set Person,
  var spouse: lone Person,
  var liveness: Liveness
}
sig Man, Woman extends Person {}
```

Revising the Model

```
enum Liveness { Alive, Dead, Unborn }

abstract sig Person {
  var children: set Person,
  var spouse: lone Person,
  var liveness: Liveness
}
sig Man, Woman extends Person {}

fun parents : Person -> Person { ~children }
fun siblings [p: Person]: Person { {q: Person | … } }
```

Revising Constraints

```
pred BloodRelatives [p, q: Person] {
  some p.*parents & q.*parents
}
pred isAlive [p: Person] { p.liveness = Alive }
pred isDead [p: Person] { p.liveness = Dead }
pred isUnborn [p: Person] { p.liveness = Unborn }

-- a newborn is someone who has just been born
pred newBorn[p: Person] {
  isAlive[p] and before isUnborn[p]
}

pred isMarried [p: Person] { some p.spouse }
```

Revising *Static* Constraints

```
-- People cannot be their own ancestors
always no p: Person | p in p.^parents

-- No one can have more than one father or mother
always all p: Person |
  lone (p.parents & Man) and lone (p.parents & Woman)

-- The spouse relation is symmetric
always spouse = ~spouse
```

Revising *Static* Constraints

```
-- A spouse can't be a sibling

always no p: Person |
  some p.spouse and p.spouse in p.siblings

-- People can't be married to a blood relative

always no p: Person | let s = p.spouse |
  some s and BloodRelatives[p, s]

-- A person can't have children with a blood relative
always all disj p, q: Person |
  some (p.children & q.children) implies
    not BloodRelatives[p, q]
```

Adding *Temporal* Constraints

```
-- Dead people stay dead
always all p: Person |
  isDead[p] implies after isDead[p]

-- Dead people were once alive
always all p: Person |
  isDead[p] implies once isAlive[p]

-- No one lives forever
always all p: Person |
  isAlive[p] implies eventually isDead[p]
```

Adding *Temporal* Constraints

```
-- Living people don't become unborn
always all p: Person |
  isAlive[p] implies always not isUnborn[p]

-- Living people stay alive until they die
always all p: Person |
  isAlive[p] implies (isAlive[p] until isDead[p])

-- Newborns have a father and a mother
always all p: Person | newBorn[p] implies
  some m: Man | some w: Woman | p.parents = m + w
```

Adding *Temporal* Constraints

```
-- Children were born from previously alive parents
always all p, q: Person |
  p in q.children implies
    once (newBorn[p] and once isAlive[q])

-- People with parents have had those parents since birth
always all p, q: Person |
  p in q.children implies
    (p in q.children since newBorn[p])
```

Exercises

- Load `family-6-elec.als` in Alloy
- Execute it
- Analyze the model
- Look at the generated instance
- Does it look correct?
- What, if anything, would you change about it?

Dynamics as State Transitions

Recall

- The evolution of a dynamic system can be modeled as a set of traces
- Each trace is a sequence of transitions from one state to another

A transition can be thought of as caused by the application of a state transformer

A state transformer is an operator that modifies the current state

Possible Trace

```
Person = {Matt, Sue, Sean}
Man = {Matt, Sean}
Woman = {Sue}
spouse = {}
children = {}
liveness = {(Matt,U),
(Sue,U),(Sean,U)}
```

```
Person = {Matt, Sue, Sean}
Man = {Matt, Sean}
Woman = {Sue}
spouse = {}
children = {}
liveness = {(Matt, U),
(Sue,A),(Sean,U)}
```

```
Person = {Matt, Sue, Sean}
Man = {Matt, Sean}
Woman = {Sue}
spouse = {(Matt,Sue),(Sue,Matt)}
children = {}
liveness = {(Matt,A),(Sue,A),(Sean,U)}
```

```
Person = {Matt, Sue, Sean}
Man = {Matt, Sean}
Woman = {Sue}
spouse = {(Matt,Sue),(Sue,Matt)}
children = {(Matt,Sean),(Sue,Sean)}
liveness = {(Matt,A),(Sue,A),(Sean,A)}
```

Transitions

A person is born from parents

State transformer that modifies the `children` and `liveness` relations

```
Person = {Matt, Sue, Sean}
Man = {Matt, Sean}
Woman = {Sue}
spouse = {(Matt,Sue), (Sue,Matt)}
children = {}
liveness = {(Matt,Alive), (Sue,Alive),
            (Sean,Unborn)}
```

↓

```
Person = {Matt, Sue, Sean}
Man = {Matt, Sean}
Woman = {Sue}
spouse = {(Matt,Sue), (Sue,Matt)}
children = {(Matt,Sean), (Sue,Sean)}
liveness = {(Matt,Alive), (Sue,Alive),
            (Sean,Alive)}
```

Expressing State Transitions in Alloy

A state transformer is modeled as a predicate over two states:

1. the state right before the transition (current state) and
2. the state right after it (next state)

We use the temporal operators of Alloy to express constraints on the current and the next state

(Single) primed field names refer to values in the next state

Expressing State Transformers

Pre-condition constraints

– Describe the states to which the transformer applies

Post-condition constraints

– Describes the effects of the transformer in generating the next state

Frame-condition constraints

– Describes what does not change between current state and next state of a transition

Distinguishing the pre-, post- and frame-conditions in comments provides useful documentation

Example: Marriage

```
pred getMarried [p, q: Person] {
-- preconditions
    -- p and q are both alive
    isAlive[p] and isAlive[q]
    -- neither is married
    no (p + q).spouse
    -- they are not blood relatives
    not BloodRelatives[p, q]
-- post-conditions
    -- p and q are each other’s spouses
    p.spouse' = q
    q.spouse' = p
-- frame conditions
          ??
}
```

```
enum Liveness { Alive, Dead, Unborn }
abstract sig Person {
  var children: set Person,
  var spouse: lone Person,
  var liveness: Liveness }
sig Man,Woman extends Person {}
pred isAlive [p: Person] {
  p.liveness = Alive
}
fun parents : Person -> Person { ~children }
fun siblings [p: Person]: Person {
  {q: Person | … }
}
```

`spouse'` is the next version of `spouse`

Frame Condition

How is each relation impacted by marriage?

- 5 relations :
  - `children`, ~~`parents`~~, ~~`siblings`~~
  - `spouse`
  - `liveness`
- The `parents` and `siblings` relations are defined in terms of the `children` relation
- Thus, the frame condition has only to consider `children`, `spouse` and `liveness`

Frame Condition Predicates

```
pred noChildrenChange [Ps: set Person] {
  all p: Ps |
    p.children' = p.children
}

pred noSpouseChange [Ps: set Person] {
  all p: Ps |
    p.spouse' = p.spouse
}

pred noLivenessChange [Ps: set Person] {
  all p: Ps |
    p.liveness' = p.liveness
}
```

Marriage Operator

```
pred getMarried [p, q: Person]
{
-- preconditions
   isAlive[p] and isAlive[q]
   no (p + q).spouse
   not BloodRelatives[p, q]
-- post-conditions
   p.spouse' = q and q.spouse' = p
-- frame conditions
   noSpouseChange[Person – (p + q)]
   noChildrenChange[Person]
   noLivenessChange[Person]
}
```

Instance of Marriage

```
…
pred someMarriage {
  some m: Man | some w: Woman | getMarried[m, w]
}

-- there is a marriage initially
run { someMarriage }
-- there is a marriage initially or later
run { eventually someMarriage }
-- there is a marriage eventually but not initially
run { not someMarriage and eventually someMarriage }
```

Birth from Parents Operator

```
pred isBornFromParents [p: Person, m: Man, w: Woman] {
-- Pre-conditions
   isUnborn[p]
   isAlive[w]
   once isAlive[m]

-- Post-condition
   liveness’ = liveness + p
-- Post-condition and frame condition
   children' = children + (m -> p) + (w -> p)

-- Frame conditions
   noLivenessChange[Person - p]
   noSpouseChange[Person]
   noChildrenChange[Person – (m + w)]  // redundant
}
```

Instance of Birth

```
pred someBirth {
  some b: Person, m: Man, w: Woman |
    isBornFromParents[b, m, w]
}

run { eventually someBirth }

run { some b: Person, m: Man, w: Woman |
      eventually (getMarried[m, w] and
                  eventually isBornFromParents[b, m, w])
```

Death Operator

```
pred dies [p: Person] {
-- Pre-condition
   isAlive[p]

-- Post-condition
   after isDead[p]

-- Post-condition and frame condition
   let q = p.spouse |
     spouse' = spouse - ((p -> q) + (q -> p))

-- Frame conditions
   noChildrenChange[Person]
   noLivenessChange[Person - p]
}
```

Instance of Death

```
pred someDeath {
  some p: Person | dies[p]
}

run {
  eventually someDeath
}

run {
  some p: Person |
    isAlive[p] and after (isAlive[p] and eventually dies[p])
}
```

Specifying Transition Systems

- A transition system can be defined as a set of traces (aka executions):

  sequences of states generated by the operators

- In our family example, for every execution:
  - The initial state satisfies some initialization condition
  - All pairs of consecutive states are related by
    - a birth operation, or
    - a death operation, or
    - a marriage operation

Initial State Specification

`init` specifies constraints on the initial state

```
pred init {
  no children
  no spouse
  #LivingPeople > 2
  #Person > #LivingPeople
}
```

```
fun LivingPeople [] : Person {
  liveness.Alive
}
```

Transition Relation Specification

`trans` specifies that each transition is a consequence of the application of one of the operators to some individuals

```
pred trans []  {
  (some m: Man, w: Woman | getMarried [m, w])
  or
  (some p: Person, m: Man, w: Woman |
    isBornFromParents [p, m, w])
  or
  (some p: Person | dies [p])
  or
  other   ???
}
```

The Need for a No-op

- For convenience, Alloy considers only infinite traces
- So, we need a do-nothing operator for systems that can have finite executions

```
pred other [] {
  -- the relevant relations stay the same
  children' = children
  spouse' = spouse
  liveness' = liveness
}
```

System Specification

A `System` predicate specifies that each execution

- starts in a state satisfying the initial state condition and
- moves from one state to the next as specified
  by one of the operator predicates

```
pred System {
  init and always trans
}

run { System }
```

System Invariants

- Many of the facts that we stated in our static model now become expected *system invariants*
- These are properties that
  - should hold in initial states
  - should be preserved by system transitions
- We can check that a property is invariant for a given system `System` (within a given scope) by
  - encoding the property as a formula `F` and
  - checking the assertion `System implies always F`

Expected Invariants: Examples

```
-- People cannot be their own ancestors
assert a1 { System implies
  always no p: Person | p in p.^parents
}
check a1 for 6

-- No one can have more than one father or mother
assert a2 { System implies
  always all p: Person |
    lone (p.parents & Man) and
    lone (p.parents & Woman)
}
check a2 for 8
```

Exercises

- Load `family-7-elec.als` in Alloy
- Execute it
- Look at the generated instance
- Does it look correct?
- What if anything would you change about it?
- Check each of the given assertions
- Are they all valid?
- If not, how would you change the model to fix that?

Exercises

- Load `dynamic/trash-1-elec.als` in Alloy 5
- Complete the model as instructed there
- Execute it
- Check each of the assertions you have written
- Are they all valid?
- If not, how would you change the model to fix that?