

CS:5810

Formal Methods in Software Engineering

Sets and Relations

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These Notes

- review the concepts of sets and relations required for working with Alloy
- focus on the kind of set operation and definitions used in specifications
- give some small examples of how we will use sets in specifications

Set

- Collection of distinct objects
- Each set's objects are drawn from a larger *domain* of objects all of which have the same type --- sets are homogeneous
- Examples:

{2,4,5,6,...}

{red, yellow, blue}

{true, false}

{red, true, 2}

set of integers  *domain*

set of colors 

set of boolean values

for us, **not a set!**

Value of a Set

- Is the collection of its members
- Two sets A and B are equal if
 - every member of A is a member of B
 - every member of B is a member of A
- $x \in S$ denotes “ x is a member of S ”

Defining Sets

- We can define a set by *enumeration*
 - PrimaryColors == {red, yellow, blue}
 - Boolean == {true, false}
 - Evens == {..., -4, -2, 0, 2, 4, ...}
- This works fine for finite sets, but
 - what do we mean by “...” ?
 - remember we want to be precise

Defining Sets

- We can define a set by *comprehension*, that is, by describing a property that its elements must share
- Notation:
 - $\{ x : S \mid P(x) \}$
 - Form a new set of elements drawn from set/domain S including exactly the elements that satisfy predicate (i.e., Boolean function) P

- Examples:

$$\{ x : \mathbb{N} \mid x < 10 \}$$

Naturals less than 10

$$\{ x : \mathbb{Z} \mid (\exists y : \mathbb{Z} \mid x = 2y) \}$$

Even integers

$$\{ x : \mathbb{N} \mid \text{false} \}$$

Empty set of natural numbers

Cardinality

- The *Cardinality* (#) of a finite set is the number of its elements
- Examples:
 - # {red, yellow, blue} = 3
 - # {1, 23} = 2
 - # \mathbb{Z} = ?
- Cardinalities are defined for infinite sets too, but we'll be most concerned with the cardinality of finite sets.

Set Operations

- Union:
 - $X \cup Y \equiv \{e \mid e \in X \text{ or } e \in Y\}$
 - $\{\text{red}\} \cup \{\text{blue}\} = \{\text{red}, \text{blue}\}$
- Intersection
 - $X \cap Y \equiv \{e \mid e \in X \text{ and } e \in Y\}$
 - $\{\text{red}, \text{blue}\} \cap \{\text{blue}, \text{yellow}\} = \{\text{blue}\}$
- Difference
 - $X \setminus Y \equiv \{e \mid e \in X \text{ and } e \notin Y\}$
 - $\{\text{red}, \text{yellow}, \text{blue}\} \setminus \{\text{blue}, \text{yellow}\} = \{\text{red}\}$

Subsets

- A *subset* holds elements drawn from another set
 - $X \subseteq Y$ iff $(\forall e \mid e \in X \Rightarrow e \in Y)$
 - $\{1, 7, 17, 24\} \subseteq Z$
- A *proper subset* is a non-equal subset
- Another view of set equality
 - $A = B$ iff $(A \subseteq B \wedge B \subseteq A)$

Power Sets

- The **power set** of set S (denoted $Pow(S)$) is the set of all subsets of S , i.e.,

$$Pow(S) \equiv \{e \mid e \subseteq S\}$$

- Example:
 - $Pow(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\},$
 $\{a,b\}, \{a,c\}, \{b,c\},$
 $\{a,b,c\}\}$

Note: for any S , $\emptyset \subseteq S$ and thus $\emptyset \in Pow(S)$

Exercises

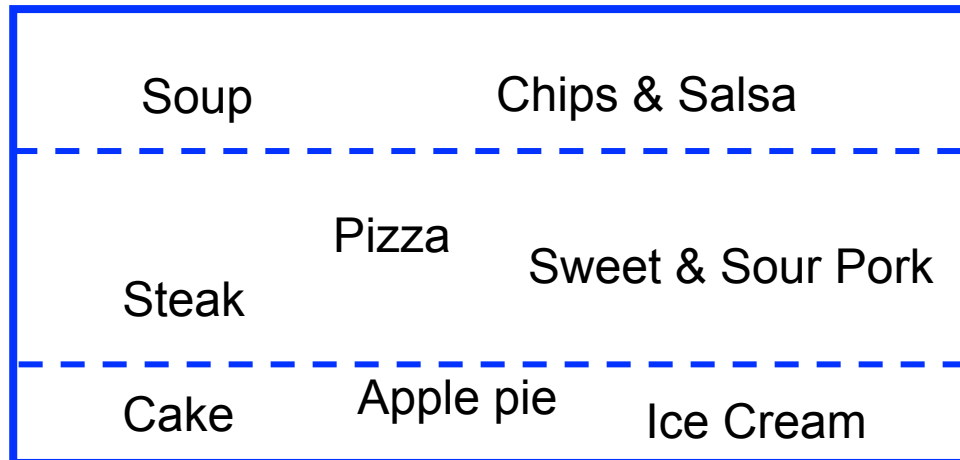
- These slides include questions that you should be able to solve at this point
- They may require you to think some
- You should spend some effort in solving them
 - ... and may in fact appear on exams

Exercises

- Specifying using comprehension notation
 - Odd positive integers
 - The squares of integers, i.e. $\{1,4,9,16,\dots\}$
- Express the following logic properties on sets without using the # operator
 - Set has at least one element
 - Set has no elements
 - Set has exactly one element
 - Set has at least two elements
 - Set has exactly two elements

Set Partitioning

- Sets are *disjoint* if they share no elements
- Often when modeling, we will take some set S and divide its members into disjoint subsets called *blocks* or *parts*
- We call this division a *partition*
- Each member of S belongs to exactly one block of the partition



Example

Model residential scenarios

- Basic domains: *Person, Residence*
- Partitions:
 - Partition *Person* into *Child, Student, Adult*
 - Partition *Residence* into *Home, DormRoom, Apartment*

Exercises

- Express the following properties of pairs of sets
 - Two sets are disjoint
 - Two sets form a partitioning of a third set

Expressing Relationships

- It's useful to be able to refer to **structured values**
 - a group of values that are bound together
 - e.g., struct, record, object fields
- Alloy is a calculus of *relations*
- All of our Alloy models will be built using relations (sets of tuples).
- ... but first some basic definitions

Product

- Given two sets A and B , the **product** of A and B , usually denoted $A \times B$, is the set of all possible pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B \equiv \{(a, b) \mid a \in A, b \in B\}$$

- Example: PrimaryColor \times Boolean:

{ (red,true), (red, false),
(blue,true), (blue, false),
(yellow, true), (yellow, false) }

Relation

- A **binary relation** R between A and B is an element of $Pow(A \times B)$, i.e., $R \subseteq A \times B$
- Examples:
 - Parent : Person \times Person
 - Parent == {(John, Autumn), (John, Sam)}
 - Square : $\mathbb{Z} \times \mathbb{N}$
 - Square == {(1,1), (-1,1), (-2,4)}
 - ClassGrades : Person \times {A, B, C, D, F}
 - ClassGrades == {(Todd,A), (Jane,B)}

Relation

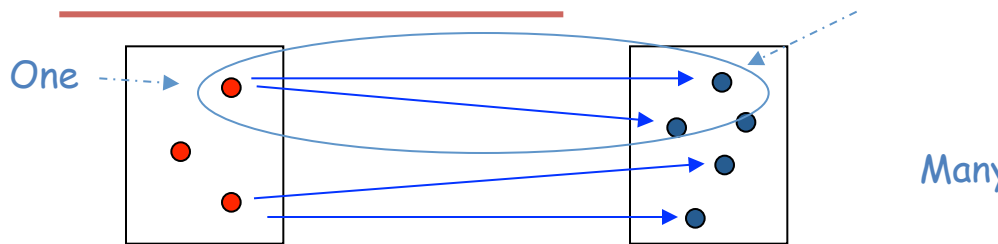
- A **ternary relation** R between A , B and C is an element of $Pow(A \times B \times C)$
- Example:
 - FavoriteBeer : Person x Beer x Price
 - FavoriteBeer == {(John, Miller, \$2), (Ted, Heineken, \$4), (Steve, Miller, \$2)}
- **N-ary relations** with $n > 3$ are defined analogously (n is the **arity** of the relation)

Binary Relations

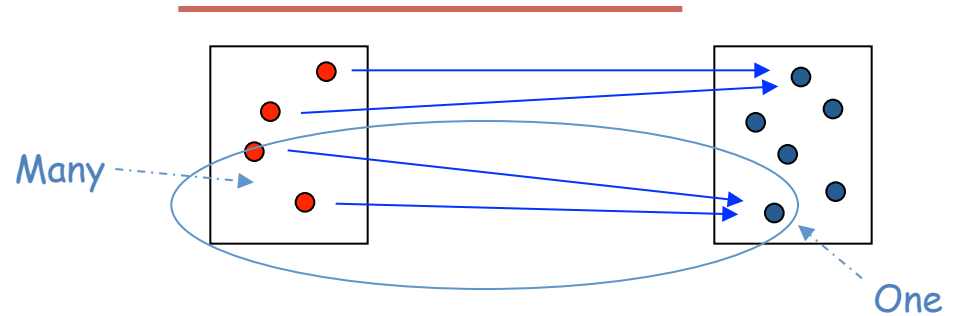
- The set of first elements is the *definition domain* of the relation
 - $\text{domain}(\text{Parent}) = \{\text{John}\}$ NOT Person!
- The set of second elements is the *image* of the relation
 - $\text{image}(\text{Square}) = \{1,4\}$ NOT **N**!
- How about $\{(1,\text{blue}), (2,\text{blue}), (1,\text{red})\}$
 - domain? image?

Common Relation Structures

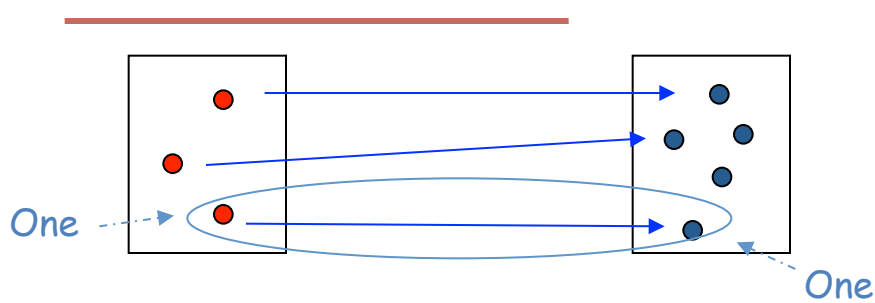
One-to-Many



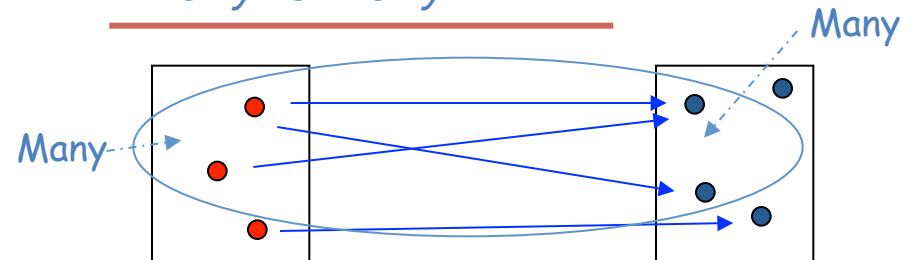
Many-to-One



One-to-One



Many-to-Many



Functions

- A *function* is a relation F of arity $n+1$ containing no two distinct tuples with the same first n elements,

– i.e., for $n = 1$,

$$\forall (a_1, b_1) \in F, \forall (a_2, b_2) \in F, (a_1 = a_2 \Rightarrow b_1 = b_2)$$

- Examples:

– $\{(2, \text{red}), (3, \text{blue}), (5, \text{red})\}$

– $\{(4, 2), (6, 3), (8, 4)\}$

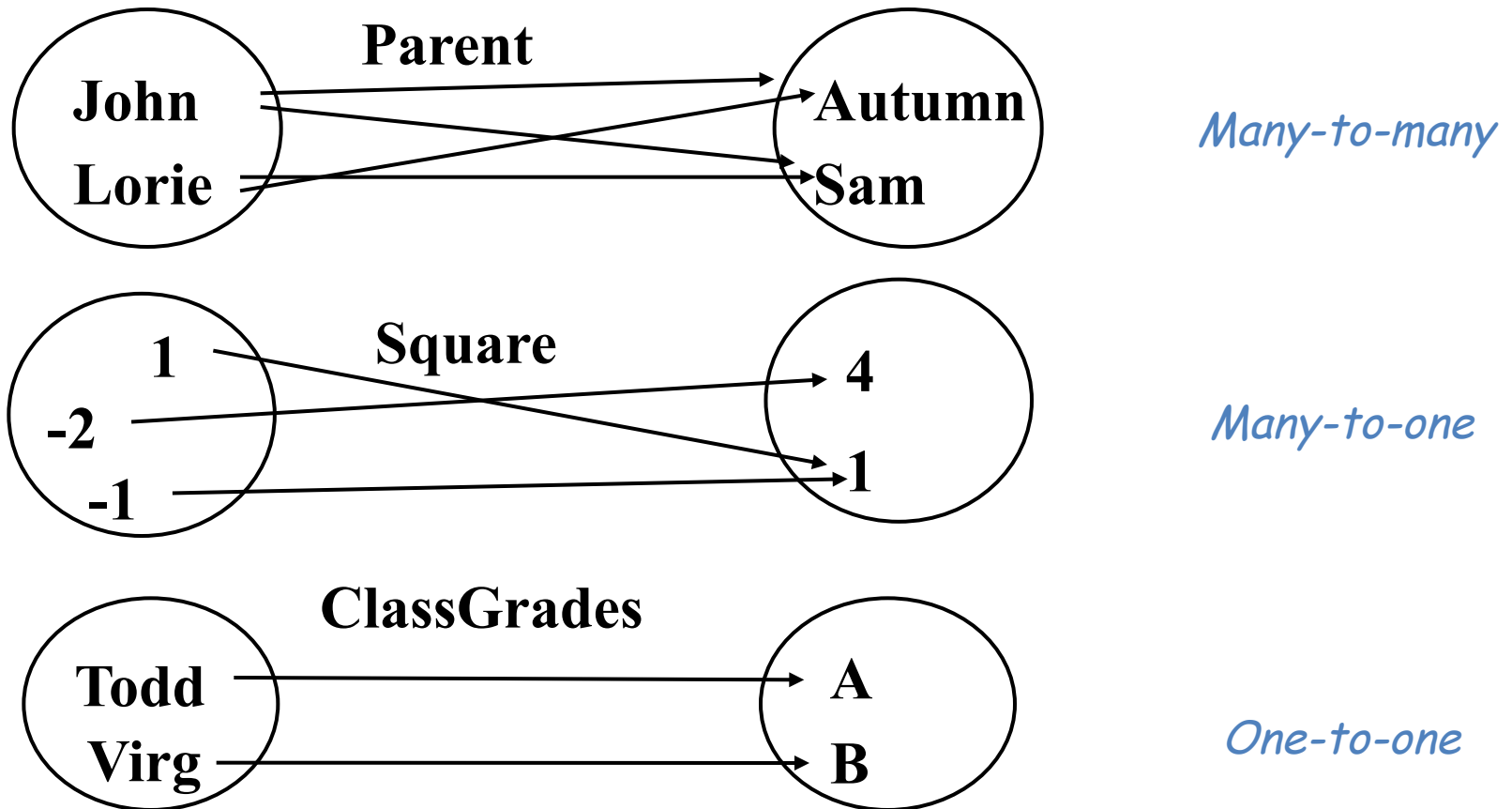
- Instead of $F: A_1 \times A_2 \times \dots \times A_n \times B$

we write $F: A_1 \times A_2 \dots \times A_n \rightarrow B$

Exercises

- Which of the following are functions?
 - Parent == {(John,Autumn), (John,Sam)}
 - Square == {(1,1), (-1,1), (-2,4)}
 - ClassGrades == {(Todd,A), (Virg,B)}

Relations vs. Functions



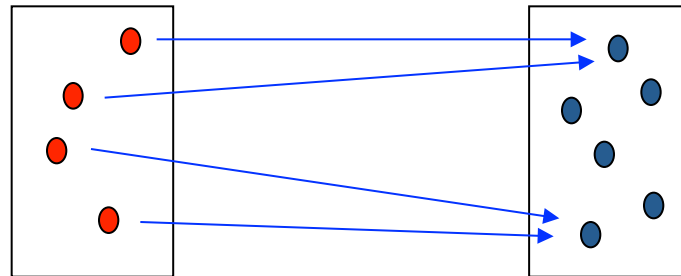
In other words, a function is a relation that is X-to-one.

Special Kinds of Functions

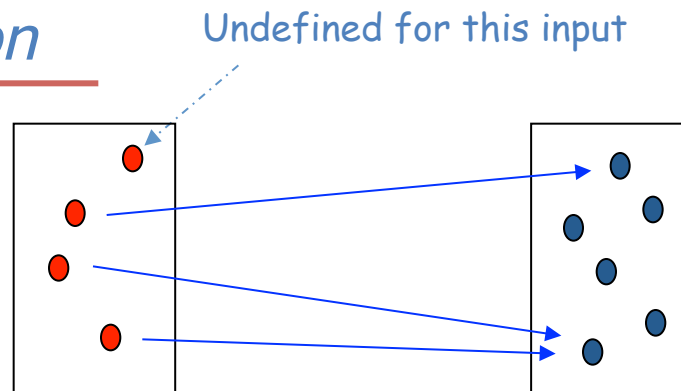
- Consider a function f from S to T
- f is *total* if defined for all values of S
- f is *partial* if defined for some values of S
- Examples
 - Squares : $Z \rightarrow N$, Squares = $\{\dots, (-1,1), (0,0), (1, 1), (2,4), \dots\}$
 - Abs = $\{(x,y) : Z \times N \mid (x < 0 \text{ and } y = -x) \text{ or } (x \geq 0 \text{ and } y = x)\}$

Function Structures

Total Function



Partial Function



Note: the empty relation is a partial function

Special Kinds of Functions

A function $f: S \rightarrow T$ is

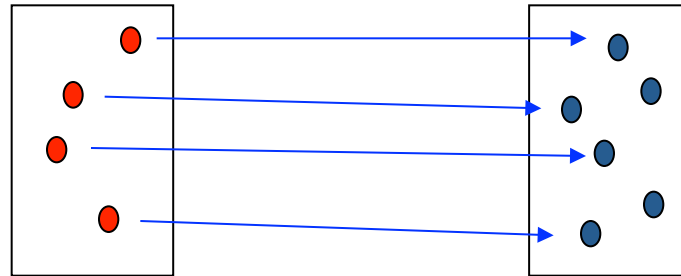
- *injective* (*one-to-one*) if no image element is associated with multiple domain elements
- *surjective* (*onto*) if its image is T
- *Bijjective* if it is both injective and surjective

We'll see that these come up frequently

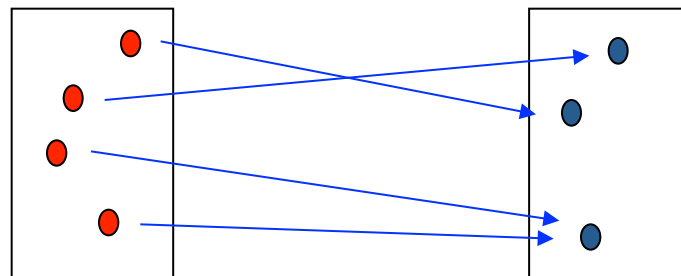
– can be used to define properties concisely

Function Structures

Injective Function



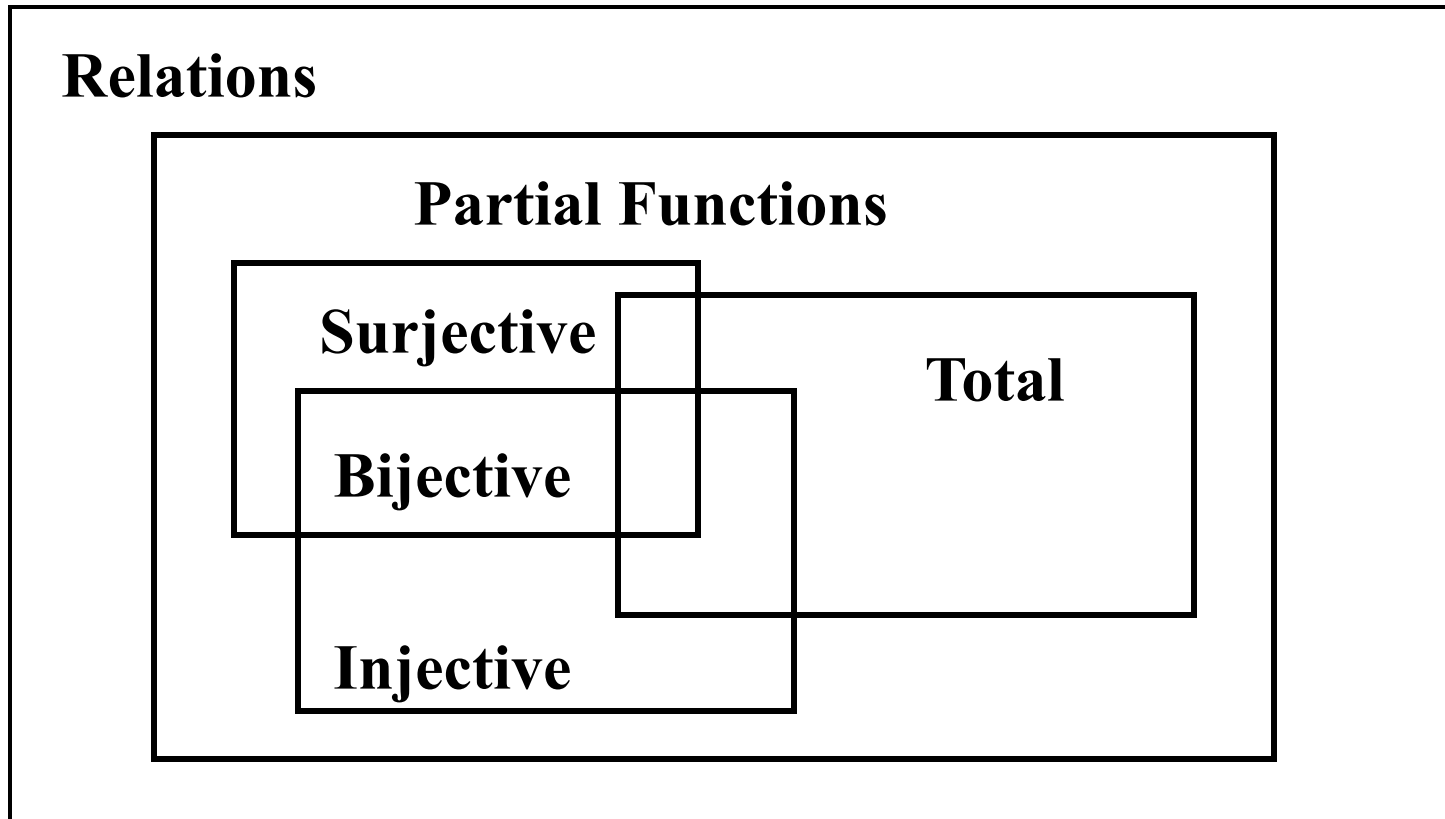
Surjective Function



Exercises

- What kind of function/relation is Abs?
 - $\text{Abs} = \{(x,y) : \mathbb{Z} \times \mathbb{N} \mid (x < 0 \text{ and } y = -x) \text{ or } (x \geq 0 \text{ and } y = x)\}$
- How about Squares?
 - Squares : $\mathbb{Z} \times \mathbb{N}$, $\text{Squares} = \{(x,y) : \mathbb{Z} \times \mathbb{N} \mid y = x^*x\}$

Special Cases



Functions as Sets

- Functions are relations and hence sets
- We can apply all of the usual operators
 - $\text{ClassGrades} == \{(\text{Todd}, A), (\text{Jane}, B)\}$
 - $\#(\text{ClassGrades} \cup \{(\text{Matt}, C)\}) = 3$

Exercises

- In the following if an operator fails to preserve a property give an example
- What operators preserve function-ness?
 - \cap ?
 - \cup ?
 - \setminus ?
- What operators preserve surjectivity?
- What operators preserve injectivity?

Relation Composition

- Use two relations to produce a new one
 - map domain of first to image of second
 - Given $s: A \times B$ and $r: B \times C$ then $s;r : A \times C$

$$s;r \equiv \{(a,c) \mid (a,b) \in s \text{ and } (b,c) \in r\}$$

- For example
 - $s == \{(\text{red},1), (\text{blue},2)\}$
 - $r == \{(1,2), (2,4), (3,6)\}$
 - $s;r = \{(\text{red},2), (\text{blue},4)\}$

Relation Closure

- Intuitively, the **closure** of a relation $r: S \times S$, written r^+ , is what you get when you keep navigating through r until you can't go any farther.

$$r^+ \equiv r \cup (r;r) \cup (r;r;r) \cup \dots$$

- For example
 - GrandParent == Parent;Parent
 - Ancestor == Parent⁺

Relation Transpose

- Intuitively, the **transpose** of a relation $r: S \times T$, written $\sim r$, is what you get when you reverse all the pairs in r .

$$\sim r \equiv \{(b,a) \mid (a,b) \in r\}$$

- For example
 - $\text{ChildOf} == \sim \text{Parent}$
 - $\text{DescendantOf} == (\sim \text{Parent})^+$

Exercises

- In the following if an operator fails to preserve a property give an example
- What properties, i.e., function-ness, onto-ness, 1-1-ness, by the relation operators?
 - composition (;)
 - closure (+)
 - transpose (~)

Acknowledgements

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David Garlan's slides from Lecture 3 of his course of Software Models entitled "Sets, Relations, and Functions"

(<http://www.cs.cmu.edu/afs/cs/academic/class/15671-f97/www/>)