CHAPTER 6:

Some Continuous Probability Distributions

Continuous Uniform Distribution: 6.1

Definition: The density function of the continuous random variable X on the interval $[A, B]$ is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A} & A \leq x \leq B \\ 0 & \text{otherwise.} \end{cases}$$

Application: Some continuous random variables in the physical, management, and biological sciences have approximately uniform probability distributions. For example, suppose we are counting events that have a Poisson distribution, such as telephone calls coming into a switchboard. If it is known that exactly one such event has occurred in a given interval, say $(0, t)$,then the actual time of occurrence is distributed uniformly over this interval.

Example: Arrivals of customers at a certain checkout counter follow a Poisson distribution. It is known that, during a given 30-minute period, one customer arrived at the counter. Find the probability that the customer arrived during the last 5 minutes of the 30-minute period.

Solution: As just mentioned, the actual time of arrival follows a uniform distribution over the interval of $(0, 30)$. If X denotes the arrival time, then

$$P(25 \leq X \leq 30) = \int_{25}^{30} \frac{1}{30} dx = \frac{30 - 25}{30} = \frac{1}{6}$$

Theorem 6.1: The mean and variance of the uniform distribution are

$\mu = \int_A^B x \frac{1}{B-A} = \left[\frac{x^2}{2(B-A)} \right]_A^B = \frac{B^2 - A^2}{2(B-A)} = \frac{A+B}{2}.$

It is easy to show that

$$\sigma^2 = \frac{(B-A)^2}{12}$$

Normal Distribution: 6.2

Definition: The density function of the normal random variable X, with mean μ and variance σ^2, is

$$n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(1/2)[(x-\mu)/\sigma]^2} \quad -\infty < x < \infty,$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$

Example: The SAT aptitude examinations in English and Mathematics were originally designed so that scores would be approximately normal with $\mu = 500$ and $\sigma = 100$.

It can be shown that the parameters of μ and σ^2 are indeed the mean and the variance of the normal distribution. (the proof is not required).

Areas Under the Normal Curve: 6.3

$$P(x_1 < X < x_2) = \int_{x1}^{x_2} n(x;\mu,\sigma)dx =$$
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-(1/2)[(x-\mu)/\sigma]^2}dx$$

Example: Suppose we are looking at a national examination whose scores are approximately normal with $\mu = 500$ and $\sigma = 100$. If we wish to find the probability that a score falls between 600 and 750, we must evaluate the integral

$$P(600 \leq X \leq 750) = \int_{600}^{750} n(x;500,100)dx =$$
$$\frac{1}{\sqrt{2\pi}\cdot 100}\int_{600}^{750} e^{-(1/2)[(x-500)/100]^2}dx$$

This cannot be done in closed form.

The use of tables for evaluating normal probabilities

We can use a single table to compute probabilities for a normal distribution with any mean and variance. The table used for this purpose is that for $\mu = 0$ and $\sigma = 1$.

Definition: The distribution of a normal random variable with mean zero and variance 1 is called a **standard normal distribution**. We denote a standard normal variable by Z

Examples:
$P(Z \leq 2.1) = 0.9821$; $P(Z < -1.34) = 0.0901$; $P(Z < 1.34) = 0.9099$
$P(Z \geq 2) = 1 - P(Z < 2) = 1 - 0.9772 = 0.0228$.
$P(-1 < Z < 1.5) = P(Z < 1.5) - P(Z < -1) = 0.9332 - 0.1587 = 0.7745$.

Transformation to standard normal: We can transform any normal random variable X with mean μ and standard deviation σ into a standard normal random variable Z with mean 0 and standard deviation 1. The linear transformation is

$$Z = \frac{X-\mu}{\sigma}$$

Example 6.4 on p. 150: Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

Example: Suppose we are looking at a national examination whose scores are approximately normal with $\mu = 500$ and $\sigma = 100$. What is the probability that a score falls between 600 and 750?

Example: The grade point average of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation .8. What fraction of the students will posses a grade point average in excess of 3.0?

Using the Normal Curve in Reverse

Example: Given that Z has normal distribution with $\mu = 0$ and $\sigma = 1$, find the value of z that has 80% of the area to the left.

$P(Z < 0.845) = 0.8$. Thus $z = 0.845$.

Ex. 2 on p. 156: Find the value of z if the area under a standard normal curve
(a) to the right of z is 0.3622.

(c) between 0 and z, with $z > 0$, is 0.4838;

(d) between $-z$ and z, with $z > 0$, is 0.9500.

Example: Given that X has normal distribution with $\mu = 100$ and $\sigma = 10$, find the value of x that has 80% of the area to the left.

In general,

$$\frac{x-\mu}{\sigma} = z \quad \text{implies that} \quad x = \sigma z + \mu$$

Ex. 3 on p. 156: Given a standard normal distribution, find the value of k such that
(a) $P(Z < k) = 0.0427$
(b) $P(Z > k) = 0.29946$
(c)$P(-0.93 < Z < k) = 0.7235$

Applications of Normal Distribution: 6.4

Example: Scores on an examination are assumed to be normally distributed with mean 78 and variance 36. Suppose that students scoring in the top 10% of this distribution are to receive an A grade. What is the minimum score a student must achieve to earn an A grade.

Ex. 10 on p. 157:
The finished inside diameter of a piston ring is normally distributed with a mean of 10 centimeters and a standard deviation of 0.03 centimeter.
(a) What proportion of rings will have inside diameters exceeding 10.075 centimeters?

(c) Below what value of inside diameter will 15% of the piston rings fall?

Example: The weekly amount of money spent on maintenance and repairs by a company was observed, over a long period of time, to be approximately normally distributed with mean \$400 and standard deviation \$20. How much should be budgeted for weekly repairs and maintenance to provide that the probability the budgeted amount will be exceeded in a given week is only .1?

Ex 9 on p. 157: A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the amount of drink is normally distributed with a standard deviation equal to 15 milliliters,

(a) what fraction of the cups will contain more than 224 milliliter?

(b) what is the probability that a cup contains between 191 and 209 milliliters?

(c) How many cups will probably overflow if 230 milliliter cups are used for the next 1000 drinks?

(d) below what value do we get the smallest 25% of the drinks?

Normal Approx. to the Binomial: 6.5

Theorem 6.2: If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of $Z = \frac{X-np}{\sqrt{npq}}$ as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

Thus if X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then when n is large, we can approximate the probability that $X = k$ by the area under the normal probability density function between $k - 0.5$ and $k + 0.5$.
That is
$b(k; n, p) \approx P\left(Z < \frac{k+0.5-np}{\sqrt{npq}}\right) - P\left(Z < \frac{k-0.5-np}{\sqrt{npq}}\right)$

Example on p. 163:

$b(0; 100, 0.05) = 0.0059$ (Minitab)

$\mu = 5$ and $\sigma^2 = 4.75$. Thus $\sigma = 2.1794$

$z_1 = \frac{-0.5-5}{2.1794} = -2.5236$

$z_2 = \frac{0.5-5}{2.1794} = -2.0648$

Using Minitab for standard normal,

Output

x	$P(X \leq x)$
-2.0646	0.01195
-2.5236	0.0058

Thus, $P(-2.5236 < Z < -2.0648) = 0.01195 - 0.0058 = 0.0062$.

Example: (Minitab)

$b(8; 36, 0.5) = 0.0004$

$\mu = (36)(0.5) = 18$,
$\sigma^2 = (36)(0.5)(0.5) = 9 \quad \sigma = 3$.
Thus

$z_1 = \frac{7.5-18}{3} = -3.5$

$z_2 = \frac{8.5-18}{3} = -3.1667$

Output from Minitab

x	$P(X \leq x)$
-3.1667	0.0008
-3.5	0.0002

Thus
$P(-3.1667 < Z < -3.5) = P(Z < -3.5) - P(Z < -3.166) = 0.0008 - 0.0002 = 0.0006$

Example: (Minitab)

$B(2; 50, 0.1) = \sum_{x=0}^{2} b(x; 50, 0.1) = 0.1117$

Using normal approximation with $\mu = (50)(0.1) = 5$ and $\sigma = \sqrt{(50)(0.1)(0.9)} = 2.1213$ we have that

$z = \frac{2.5-5}{2.1213} = -1.1785$

$P(Z < -1.1785) = 0.1193$.

Example: An airline finds that 5% of the persons who make reservations on a certain flight do not show up for the flight. If the airline sells 160 tickets for a flight with only 155 seats, what is the probability that a seat will be available for every person holding reservation and planning to fly?

Exercise 4 on p. 164:

A process yields 10% defective items. If 100 items are randomly selected from the process, what is the probability that the number of defectives
(a) exceeds 13?
(b) is less than 8

Exponential Distribution: 6.6

NOTE: Gamma distribution is not required!

The exponential random variable arises in the modeling of the time between occurrence of events.
Examples:

- The time between customer demands for call connections.

- The lifetime of devices and systems.

The continuous random variable X has an **exponential distribution**, with parameter β, if its density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta}e^{-x/\beta} & x > 0 \\ \\ 0 & \text{otherwise.} \end{cases},$$

where $\beta > 0$.

The mean and the variance of exponential distribution are $\mu = \beta$ and $\sigma^2 = \beta^2$.

Example: Assume that X has an exponential distribution with $\beta = 2$. Find $P(1 < X < 4)$.

Relationship to the Poisson Process:

A Poisson random variable with parameter λ, is described by the number of outcomes occurring during a given time. (λ is the mean number of events per unit "time").

Consider now the random variable X described by the time required for the first event to occur. X is a continuous random variable.
It can be shown that X has exponential distribution with $\beta = \frac{1}{\lambda}$.

Recall $f(x) = \begin{cases} \frac{1}{\beta}e^{-x/\beta} & x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$, where $\beta > 0$.

p. Ex. 2 on p. 175

The exponential distribution is frequently applied to the waiting times between successes in a Poisson process. If the number of calls received per hour by a telephone answering service is a Poisson random variable with parameter $\lambda = 6$, we know that the time, in hours, between successive calls has an exponential distribution with parameter $\beta = 1/6$. What is the probability of waiting more than 15 minutes between any two successive calls?