Warm-up Analysis of Luby's Distributed Graph Coloring Algorithm

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Assume that $P(v)$ is initially set to $\{1, 2, \ldots, 2\Delta\}$ for all nodes v. This asumption is just for this "warm-up" analysis; in the actual algorithm, $P(v) = \{1, 2, \ldots, \text{degree}(v) + 1\}$ initially for all nodes v. Note that this assumption implies that $|P(v)| \geq \Delta$ for all nodes v, *throughout* the algorithm. Fix a round i and let u be an arbitrary node that has not been colored after the first $i - 1$ rounds. Let X_u denote the event that u is colored in round i. We will show that $Pr(X_u) \geq 1/4$.

Let $W_{c,u}$ denote the event that node u has selected color c, as a "tentative" color. Note that $Pr(W_{c,u}) = 1/|P(u)|$ if $c \in P(u)$ and otherwise $Pr(W_{c,u}) = 0$.

$$
\begin{aligned}
Pr(X_u) &= Pr(\exists c \in P(u) : W_{c,u} \wedge \forall v \in N(u) : \overline{W_{c,v}}) \\
&= \sum_{c \in P(u)} Pr(W_{c,u} \wedge \forall v \in N(u) : \overline{W_{c,v}}) \\
&= \sum_{c \in P(u)} Pr(W_{c,u}) \cdot Pr(\forall v \in N(u) : \overline{W_{c,v}}) \\
&= \frac{1}{|P(u)|} \sum_{c \in P(u)} Pr(\forall v \in N(u) : \overline{W_{c,v}}) \\
&= \frac{1}{|P(u)|} \sum_{c \in P(u)} \prod_{v \in N(u)} Pr(\overline{W_{c,v}}) \qquad (1)
\end{aligned}
$$

Now we focus on finding a lower bound for $Pr(\overline{W_{c,v}})$. If $c \notin P(v)$ then $Pr(\overline{W_{c,v}}) = 1$. If $c \in P(v)$ then $Pr(\overline{W_{c,v}}) = 1 - 1/|P(v)|$. Therefore, independent of whether $c \in P(v)$, we see that $Pr(\overline{W_{c,v}}) \geq (1 - 1/|P(v)|)$. Since $|P(v)| \geq \Delta$, this implies that $Pr(\overline{W_{c,v}}) \geq (1 - 1/\Delta)$. Plugging this lower bound into the right hand side of (1) we get

$$
Pr(X_u) \geq \frac{1}{|P(u)|} \sum_{c \in P(u)} \prod_{v \in N(u)} \left(1 - \frac{1}{\Delta}\right). \qquad (2)
$$

Since $|N(u)| \leq \Delta$,

$$
\prod_{v \in N(u)} \left(1 - \frac{1}{\Delta}\right) \geq \left(1 - \frac{1}{\Delta}\right)^{\Delta}.
$$

Plugging this into (2), we get

$$
Pr(X_u) \geq \frac{1}{|P(u)|} \sum_{c \in P(u)} \left(1 - \frac{1}{\Delta}\right)^{\Delta} = \left(1 - \frac{1}{\Delta}\right)^{\Delta}.
$$

Now we use the fact that $(1 - 1/x)^x \geq 1/4$ for all $x \geq 2$ to obtain the result $Pr(X_u) \geq 1/4$ for $\Delta \geq 2$.