1 Karger’s Mincut Algorithm

Algorithm 1: KARGERMINCUT(G):

1. $G_0 \leftarrow G$
2. for $i \leftarrow 1$ to $n-2$ do
   3. Pick an edge $e_i$ uniformly at random from $G_{i-1}$
   4. $G_i \leftarrow CONTRACT(G_{i-1}, e_i)$
3. end
4. return The number of edges between two remaining vertices in $G_{n-2}$

Figure 1: An example of running KARGERMINCUT() on a graph $G$, suppose that the red edge is selected uniformly at random in each step. In this case, the returned cut size is 4, and this is not the size of a minimum cut in $G$. 
Analysis: Let $C$ be the set of edges in a mincut in $G$. Note that $G$ could have several mincuts, and $C$ is one of these chosen arbitrarily. Define following events,

1. $E_i = \text{the edge } e_i \text{ is not } C$.

2. $F_i = E_1 \cap E_2 \cap \cdots \cap E_i$. Thus, $F_i$ is the event that none of the edges $e_1, e_2, \ldots e_i$ belongs to $C$.

Lemma 1 Let $c$ be the size of a mincut in $G$. For any edge $e$ in $G$, $\text{CONTRACT}(G, e)$ has a mincut of size no less than $c$.

Proof: By contradiction, suppose $G' = \text{CONTRACT}(G, e)$. Let $e = \{u, v\}$. Suppose that $G'$ has a mincut $C'$ of size $c' < c$. Now, we undo the $\text{CONTRACT}$ operation, i.e., separate out $u$ and $v$, adding edges that were originally between $u$ and $v$, and also separating out edges going to $uv$ in $G'$ to either $u$ or $v$. Because the only new edges we added are between $u$ and $v$, and they are in the same side of the cut $C'$, the size of this cut remains unchanged. Now, the restored graph is exactly the same as $G$, but it has a cut of size $c' < c$. This contracts our assumption that $c$ is the size of mincut in $G$.

Recall that $F_{n-2}$ is the event that none of $e_1, e_2, \ldots e_{n-2}$ belongs to $C$. If $F_{n-2}$ happens, we know the algorithm $\text{KARGERMINCUT}()$ will return a right answer. Thus, we are interested in $Pr(F_{n-2})$ and in particular showing that $Pr(F_{n-2})$ is large enough.

$$Pr(F_{n-2}) = Pr(E_1 \cap E_2 \cap \cdots \cap E_{n-2})$$
$$= Pr(E_1)Pr(E_2|E_1)Pr(E_3|E_1 \cap E_2) \cdots Pr(E_{n-2}|E_1 \cap E_2 \cap \cdots \cap E_{n-1})$$

So we now calculate $Pr(E_1)$ using the fact that $Pr(E_1) = 1 - Pr(E_1)$. Since $E_1$ is the complement of $E_1$, it is the event that $e_1$ belongs to $C$. Thus,

$$Pr(E_1) = \frac{c}{\text{The number of edges in } G_0^c}.$$ 

Now note that the number of edges in $G_0$ is at least $\frac{n \cdot c}{2}$. This follows from the fact that since the size of a mincut in $G_0$ is $c$, every vertex in $G_0$ has degree at least $c$. Otherwise, we could have separated that vertex with degree less than $c$ from the rest of the graph by deleting fewer than $c$ edges. Thus, we would have a cut of size less than $c$ in $G_0$. Thus,

$$Pr(E_1) \leq \frac{c}{n \cdot c/2} = \frac{2}{n}$$

and therefore

$$Pr(E_1) \geq 1 - \frac{2}{n} = \frac{n - 2}{n}.$$ 

We will next compute $Pr(E_2|E_1)$. We use $Pr(E_2|E_1) = 1 - Pr(E_2|E_1)$. As before,

$$Pr(E_2|E_1) = \frac{c}{\text{The number of edges in } G_1^c}.$$
The number of edges in $G_1$ is no less than $\frac{(n-1)c}{2}$, the argument is similar to the above for $G_0$. Thus,

$$Pr(E_2|E_1) \leq \frac{c}{(n-1)c/2} = \frac{2}{n-1}$$

$$Pr(E_2|E_1) \geq 1 - \frac{2}{n-1}.$$ 

Similarly,

$$Pr(E_3|E_1 \cap E_2) \geq 1 - \frac{2}{n-2}$$

$$\vdash$$

$$Pr(E_{n-2}|E_1 \cap E_2 \cap \cdots \cap E_{n-3}) \geq 1 - \frac{2}{3}$$

Then,

$$Pr(F_{n-2}) \geq \left(1 - \frac{2}{n}\right) \left(1 - \frac{2}{n-1}\right) \left(1 - \frac{2}{n-3}\right) \cdots \left(1 - \frac{2}{3}\right)$$

$$\geq \left(\frac{n-2}{n}\right) \left(\frac{n-3}{n-1}\right) \left(\frac{n-4}{n-2}\right) \cdots \left(\frac{3}{3}\right)$$

$$\geq \frac{2 \cdot 1}{n(n-1)} = \frac{2}{n(n-1)}.$$ 

In other words, Karger’s MinCut algorithm produces a correct answer with probability $\geq \frac{2}{n(n-1)}$. To amplify the correctness probability, we repeat the alogrithm $t$ times. Then, return the smallest cut we found. The following shows the algorithm.

**Algorithm 2: ImprovedKargerMinCut(G):**

1. $k \leftarrow \infty$
2. for $i \leftarrow 1$ to $t$ do
3. \hspace{1em} $z \leftarrow$ KargerMinCut(G)
4. \hspace{1em} if $z < k$ then
5. \hspace{2em} $k \leftarrow z$
6. end
7. end
8. return $k$

**Analysis:** Let $c$ = the size of minimal cut in graph $G$, $k$ is the number return by the algorithm. Then,

$$Pr(k \neq c) \leq \left(1 - \frac{2}{n(n-1)}\right)^t$$

Here is an inequality that turns out to be super-useful; $1 + x \leq e^x$ for all reals $x$. Thus, we get

$$Pr(k \neq c) \leq e^{-\frac{2t}{n(n-1)}}$$

- If we pick $t = n(n-1)$, then $Pr(k \neq c) \leq e^{-2} = \frac{1}{e^2}$.
- If we pick $t = n(n-1) \ln n$, then $Pr(k \neq c) \leq e^{-2\ln n} = \frac{1}{n^2}$. 

3
2 Random Variables

Definition: A random variable is a variable that takes on different values with associated probabilities.

Example:

Algorithm 3:

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>while not $\frac{</td>
<td>L_1</td>
<td>}{</td>
<td>L</td>
<td>} \leq \frac{2</td>
<td>L_1</td>
</tr>
<tr>
<td>2</td>
<td>$(L_1, L_2) \leftarrow \text{RANDOMIZEDPARTITION}(L)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>end</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>return $(L_1, L_2)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let $I =$ number of iterations of the Algorithm 3. The following is the probability distribution of $I$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$\cdots$</th>
<th>$i$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Pr(I = i)$</td>
<td>$\frac{1}{3}$</td>
<td>$\left(\frac{2}{3}\right)\frac{1}{3}$</td>
<td>$\left(\frac{2}{3}\right)^2\frac{1}{3}$</td>
<td>$\cdots$</td>
<td>$\left(\frac{2}{3}\right)^{i-1}\frac{1}{3}$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

Table 1: The probability distribution of $I$

This distribution is a geometric distribution with parameter $p = \frac{1}{3}$.

Definition: The expectation of a discrete random variable $X$ denoted $E[X]$ is

$$E[X] = \sum_i i \cdot Pr(X = i).$$

Following the definition, we can compute $E[I]$ by the following formula,

$$E[I] = \sum_{i \geq 1} i \left(\frac{2}{3}\right)^i \frac{1}{3} = \frac{1}{3} \sum_{i \geq 1} i \left(\frac{2}{3}\right)^i.$$

$$S = 1 \cdot \left(\frac{2}{3}\right)^0 + 2 \cdot \left(\frac{2}{3}\right)^1 + 3 \cdot \left(\frac{2}{3}\right)^2 + \cdots + i \cdot \left(\frac{2}{3}\right)^{i-1} + \cdots \quad (1)$$

$$\frac{2}{3} S = 1 \cdot \left(\frac{2}{3}\right)^1 + 2 \cdot \left(\frac{2}{3}\right)^2 + \cdots + (i-1) \cdot \left(\frac{2}{3}\right)^{i-1} + i \cdot \left(\frac{2}{3}\right)^i + \cdots \quad (2)$$

Subtracting (2) from (1), we get

$$\frac{1}{3} S = \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \cdots = \frac{1}{1 - 2/3} = 3.$$

Thus, $S = 9$ and $E[I] = 3$. In general, $E[X] = 1/p$, for a geometric random variable with parameter $p$. 

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3 Geometric Random Variable

Definition: A geometric random variable $X$ with parameter $p$, $0 < p < 1$, has the following probability distribution over $i = 1, 2, 3, \cdots$

$$Pr(X = i) = (1 - p)^{i-1}p.$$ 

It is easy to show the the probabilities indeed add upto 1. In other words,

$$\sum_{i \geq 1} Pr(X = i) = 1.$$

From line 2 to line 3, it is because the fact that $\frac{1}{1-x} = 1 + x + x^2 + \cdots$, for $0 < x < 1$.

Lemma 2 Let $X$ be a discrete random variable that takes on values $i = 1, 2, 3, \cdots$.

$$E[X] = \sum_{i \geq 1} Pr(X \geq i)$$

Example: let $X$ be a geometric random variable with parameter $p$. $Pr(X \geq t) = (1 - p)^{t-1}$, since $Pr(X \geq t)$ means that the first $t - 1$ trails fail. Then, using this fact and Lemma 2, we can compute $E[X]$ as following,

$$E[X] = \sum_{i \geq 1} Pr(X \geq i)$$

$$= \sum_{i \geq 1} (1 - p)^{i-1}$$

$$= \frac{1}{1 - (1 - p)}$$

$$= \frac{1}{p}.$$
4 Linearity of Expectation

Theorem 3 Let $X_1, X_2, X_3, X_4, \ldots, X_n$ be random variables with finite expectations. Then,

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Important note: no independence needed!

Definition: Independence of random variables, random variables $X$ and $Y$ are independent if all values $x, y$,

$$Pr(X = x \cap Y = y) = Pr(X = x)Pr(Y = y).$$

EXAMPLE: let $X =$ sum of two 6-sided dice outcomes. To compute $E[X]$, we have two ways, first directly by definition. Looking at the following probability distribution table. So we can calculate $E[X]$ as follows.

<table>
<thead>
<tr>
<th>$i$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$\cdots$</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Pr(X = i)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\cdots$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

Table 2: The probability distribution of $X$

However, this sum is kind of tedious to deal with. So let us look at this another way, using linearity of expectation. For $i = 1, 2$: $X_i =$ outcome of $i^{th}$ dice. Then,

$$X = X_1 + X_2$$
$$E[X] = E[X_1 + X_2]$$
$$= E[X_1] + E[X_2]$$
$$= 2 \cdot E[X_1]$$
$$= 2 \cdot \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6)$$
$$= 7$$

Theorem 4 Let $X, Y$ be random variables with finite expectation, then $E[X + Y] = E[X] + E[Y]$. 
Proof:

\[ E[X + Y] = \sum_i \sum_j (i + j) Pr(X = i \cap Y = j) \]
\[ = \sum_i \sum_j i \cdot Pr(X = i \cap Y = j) + \sum_i \sum_j j \cdot Pr(X = i \cap Y = j) \]
\[ = \sum_i \sum_j Pr(X = i \cap Y = j) + \sum_j \sum_i Pr(X = i \cap Y = j) \]
\[ = \sum_i i \cdot Pr(X = i) + \sum_j j \cdot Pr(Y = j) \]
\[ = E[X] + E[Y] \]

\[ \Box \]

5 Coupon Collector’s Problem

Suppose each box of cereal contain one of \( n \) distinct coupons, and assume a coupon in a box is chosen uniformly at random from \( n \) distinct coupons. You win once you obtain at least one of every distinct type of coupon. The question is how many boxes of cereal you must buy to win. Let \( X \) = the number of boxes to buy for winning. Apparently, \( X \) is a random variable, you can be luck enough so that only buy \( n \) boxes to win, but this is just high unlikely when \( n \) is large. We are interested in \( E[X] \). Let \( X_i = \) the number of boxes you need to buy while you have \( i - 1 \) distinct coupons.

![Figure 2: The visualization of buying enough cereal boxes to win](image)

By the definition of \( X_i, i = 1, 2, \ldots, n. \)

\[ X = X_1 + X_2 + \cdots + X_n \]
$X_i$ is a geometric random variable with $p = \frac{n-(i-1)}{n}$. Thinking of $X_i$, when you have $i-1$ distinct coupons, the probability to obtain a new distinct coupon is $1 - \frac{i-1}{n} = \frac{n-(i-1)}{n}$, every time you buy one cereal box. Therefore,

$$E[X_i] = \frac{1}{p} = \frac{n}{n - (i - 1)}$$

$$E[X] = E[X_1 + X_2 + X_3 + \cdots + X_n] = E[X_1] + E[X_2] + E[X_3] + \cdots + E[X_n]$$

$$= \sum_{i=1}^{n} \frac{n}{n - i + 1} = n \sum_{i=1}^{n} \frac{1}{n - i + 1} = n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$H_n$ is the $n^{th}$ harmonic number,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + O(1)$$

6 Randomized QuickSort

<table>
<thead>
<tr>
<th>Algorithm 4: QUICKSORT(L1..n):</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 /* Assume L contains distinct elements, which can be removed later</td>
</tr>
<tr>
<td>2 c ← some constant</td>
</tr>
<tr>
<td>3 if $n \leq c$ then</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6 else</td>
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<td>19</td>
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<td>20</td>
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</tbody>
</table>
In the worst case, **QuickSort** form $\Omega(n^2)$ comparison. If the pivot is always the median of current array, then

$$T(n) = 2 \ T(\lceil \frac{n}{2} \rceil) + O(n)$$

$$T(n) = O(n \log n)$$

This is the best case, and we don’t have to be that lucky, if pivot partition $L$ into sublists $L_1, L_2$ such that

$$\frac{|L|}{3} \leq |L_1| \leq \frac{2|L|}{3}$$

we can also have a good running time on expectation. If the above relation fulfilled, then

$$T(n) < T(\frac{n}{3}) + T(\frac{2n}{3}) + O(n)$$

$$\Rightarrow \ T(n) = O(n \log n)$$

$L_1, L_2 \leftarrow \text{RandomizedPartition}(L)$ is a way to achieve it.