Lecture Notes CS:5350 Design and Analysis of Algorithms
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1 Revisiting Maxflow-Mincut Theorem Weak Version

In this section we will briefly return to the weak version of the Maxflow-Mincut Theorem in order to examine the relation of maximum flows and minnimum cuts in a graph.

Theorem 1 *Let f be an arbitrary feasible flow and (S,T) be an arbitrary cut. Then,*

$$|f| \leq ||S,T||.$$

Moreover, $|f|$=$||S,T||$ iff all edges from S to T are saturated by f and all edges from T to S are avoided by f.

When we stated this theorem in the previous class, we neglected to add the strengthening of the result that starts with the "Moreover". But, it is easy to see this strongrer version of the result follows from the proof we discussed in the previous class. In that proof, we showed that for any feasible flow f and cut (S,T),

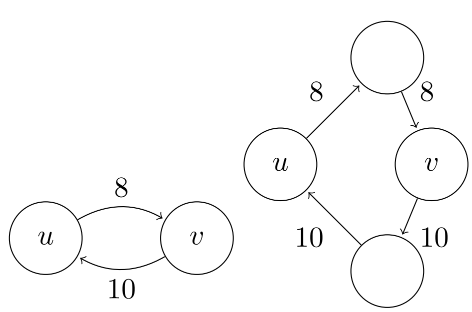
$$|f| = \sum f_{p\to q} - \sum f_{x\to y}.$$

The first summation is over $p \to q \in E$, where $p \in S$ and $q \in T$ and the second summation is over $x \to y \in E$, where $y \in S$, and $x \in T$. Using the fact that f is feasible, we see that $|f| = \sum c_{p\to q}$ iff $f_{x\to y} = 0$ for all $x \to y \in E$, $x \in T$, $y \in S$, and $f_{p\to q} = c_{p\to q}$, $p \to q \in E$, $p \in S$, and $q \in T$. In other words, $|f| = ||S,T||$ iff all edges from S to T are saturated by f and all edges from T to S are avoided by f.

2 Maxflow-Mincut Theorem

Theorem 2 *Let f^* be a flow with maximum value. Let (S^*,T^*) be a cut with minimum capacity. Then $|f^*| = ||S^*,T^*||$.*

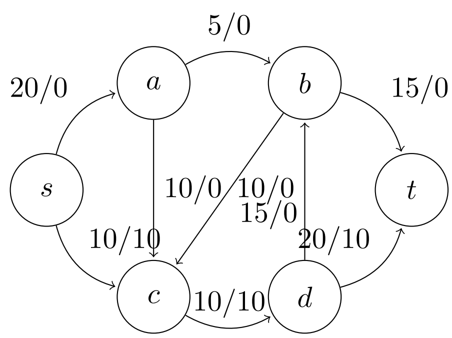
Proof: Without loss of generality we will assume that if $u \to v \in E$ then $v \to u \notin E$. This is because if a graph contains edges $u \to v$ and $v \to u$ as shown in the figure (left) below, then we can add "dummy" nodes as shown in the figure on the right and we no longer have two edges going in opposite directions between a pair of nodes.



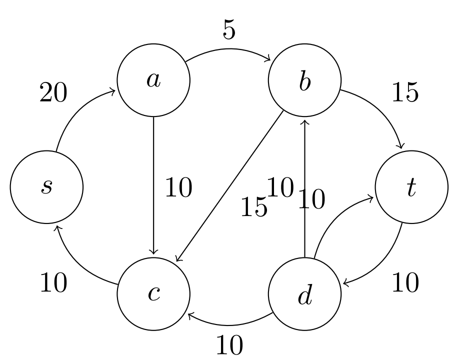
Definition: *Residual Capacity.* Let f be a feasible flow. The residual capacity function $c_f : V \times V \to \mathbf{R}$ is defined as:

$$c_f(u \to v) = \begin{cases} c_(u \to v) - f_{u \to v} & \text{if } u \to v \in E \\ f_{v \to u} & \text{if } v \to u \in E \\ 0 & \text{Otherwise.} \end{cases}$$

Observation: $c_f(u \to v) \geq 0$ for all $u \to v \in V \times V$.
Definition: (**Residual Graph.**) $G_f = (V, E_f)$ is such that $E_f = \{u \to v | c_f(u \to v) > 0\}$.
Definition: (**Augmenting Path**) Any path P from s to t in the residual graph G_f is called an augmenting path.
In the following figure, edges are labeled with the scheme of *capacity*/*flow*. This picture shows a feasible flow.



The figure below is the residual graph for the above graph and flow with an augmenting path $s \to a \to b \to t$. Let $F = \min\{c_f(s \to a), c_f(a \to b), c_f(b \to t)\}$. Then F units of flow can be pushed through in the residual graph from s to t, implying that we can obtain a new flow from f whose value is 5 units higher.



With respect to an arbitrary feasible flow f there are two cases:

(i) there is an augmenting path, or

(ii) there is no augmenting path.

Case (i): Let P be a simple augmenting path in G_f. Let F be the minimum of all residual capacities of the edges in the path, i.e., $F = \min_{u \to v \in P}\{c_f(u \to v)\}$. Now we define a new flow $f' : E \to \mathbf{R}$ as follows:

$$f' = \begin{cases} f(u \to v) + F & \text{if } u \to v \in P \\ f(u \to v) - F & \text{if } v \to u \in P \\ f(v \to v) & \text{Otherwise.} \end{cases}$$

We will show that f' is (a) a flow, (b) feasible, and (c) $|f'| = F + |f|$ Since $F > 0$
$|f'| > |f|$
Therefore f is not a Maxflow. $\square$