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1 LP example

Let us consider this LP example, we introduced in last class:

$$\begin{aligned} \max_{\{x_1,x_2,x_3,x_4\}} \quad & x_1 + 2x_2 + x_3 + x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 2 \\ & x_2 + x_4 \leq 1 \\ & x_1 + 2x_3 \leq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned} \tag{1}$$

Someone has proposed a solution, $x_1 = 1, x_2 = \frac{1}{2}, x_3 = 0, x_4 = \frac{1}{2}$ which gives us the objective function value = 2.5. They claim that this solution is optimal.

Let us find multipliers one for each of the constraints and consider a linear combination of the constraints and the multiplier.
Using multipliers $(1, 1, 0)$ gives us, $1 \times (1) + 1 \times (2) + 0 \times (3)$. This evaluates to $x_1 + 3x_2 + x_3 + x_4 \leq 3$.

$$\implies \text{Objective function value} \leq x_1 + 3x_2 + x_3 + x_4 \leq 3$$

Is there a better upper bound? Consider, another set of multipliers $(\frac{1}{2}, 1, \frac{1}{2})$ that gives, $\frac{1}{2} \times (1) + 1 \times (2) + \frac{1}{2} \times (3)$. This evaluates to $x_1 + 2x_2 + 1.5x_3 + x_4 \leq 2.5$.

$$\implies \text{Objective function value} \leq x_1 + 2x_2 + 1.5x_3 + x_4 \leq 2.5$$

The second choice of multipliers gives more strong upper bound on the right hand side. We can generalize our goal is to find the scaling factors that give us the best possible upper bound (for maximization problem, lower bound for minimization problem) to the objective function.

2 LP duality

Given a LP problem (referred as a *primal* problem) with a set of constraints, we approach to find the optimal multipliers for each of the constraints so as to obtain the strongest bound possible. We can express the problem of finding the best multipliers as another LP which is called *dual* LP.

Suppose that we have a maximization LP in standard form.

$$\begin{aligned} \max_{x_1,x_2,\ldots,x_n} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned} \tag{2}$$

Goal:
Find non-negative multipliers $y_1, y_2, \ldots, y_m$ such that, $b_1y_1 + b_2y_2 + \ldots b_my_m$ is minimized and following constraints are satisfied.

$$\begin{aligned} & c_1 \leq y_1a_{11} + y_2a_{21} + \ldots + y_ma_{m1} \\ & c_2 \leq y_1a_{12} + y_2a_{22} + \ldots + y_ma_{m2} \\ & \vdots \\ & c_n \leq y_1a_{1n} + y_2a_{2n} + \ldots + y_ma_{mn} \\ & y_1, y_2, \ldots, y_m \geq 0 \end{aligned} \tag{3}$$

In the LP example given in (1), we have already discussed this objective. We chose $y_1 = \frac{1}{2}, y_2 = 1, y_3 = \frac{1}{2}$ such that it satisfied the coefficient constraints given in (3) and also gave the stronger upper bound $b_1y_1 + b_2y_2 + b_3y_3 = 2.5$

What we find that the goal basically gives us another LP problem, this time a minimization problem with following standard form:

$$\begin{aligned} \min_{y_1,y_2,\ldots,y_m} \quad & b_1y_1 + b_2y_2 + \ldots + b_my_m \\ \text{s.t.} \quad & a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1 \\ & a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2 \\ & \vdots \\ & a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \leq c_n \\ & y_1, y_2, \ldots, y_m \geq 0 \end{aligned} \tag{4}$$

So if we have a LP in maximization form (2), which we are going to call the *primal* LP, its *dual* LP (4) is a minimization problem that can be formed by having one variable for each constraint of the primal and having one constraint for each variable of the primal such that for any feasible $(x_1, x_2, \ldots x_n)$ and $(y_1, y_2, \ldots, y_m)$,

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n \leq b_1y_1 + b_2y_2 + \ldots + b_my_m.$$

It is convenient to think of these LPs in matrix form.
Maximization form:

$$\begin{aligned} \max_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{5}$$

Minimization form:

$$\begin{aligned} \min_{y} \quad & b^T y \\ \text{s.t.} \quad & A^T xy \geq c \\ & y \geq 0 \end{aligned} \tag{6}$$

If there is a primal LP problem, it has a corresponding dual problem. The reverse is also true since dual of dual is a primal. We can represent the (6) as below:

$$\begin{aligned} \max_{y} \quad & (-b)^T y \\ \text{s.t.} \quad & (-A)^T y \leq (-c) \\ & y \geq 0 \end{aligned} \tag{7}$$

The problem in (7) is in standard form so we can take its dual to get the LP.

$$\begin{aligned} \min_{x} \quad & (-c)^T x \\ \text{s.t.} \quad & (-A^T)^T x \geq (-b) \\ & x \geq 0 \end{aligned} \tag{8}$$

which is basically,

$$\begin{aligned} \max_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{9}$$

Theorem 1 *Weak LP duality*
If x is a feasible solution of the primal LP and y is a feasible solution of the dual LP then,

$$c^T x \leq b^T y$$

Theorem 2 *Strong LP duality*
If x^* is an optimal solution of the primal LP and y^* is an optimal solution of the dual LP then,

$$c^T x^* = b^T y^*$$

Example:
Given the following primal maximization problem, convert it to a dual LP problem.

$$
\begin{aligned}
\max_{x_1,x_2,x3} \quad & 3x_1 + 2x_2 + x_3 \\
\text{s.t.} \quad & x_1 + 4x_3 \leq 7 \\
& x_2 + 2x_3 \leq 1 \\
& x_1 + x_2 + x_3 \leq 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned} \tag{10}
$$

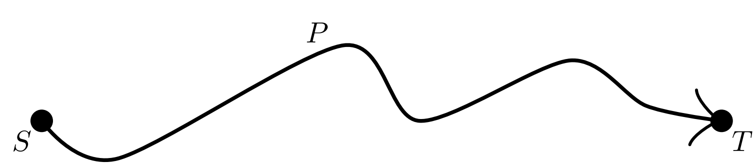
Solution:

$$
\begin{aligned}
\min_{y_1,y_2,y_3} \quad & 7y_1 + y_2 + 3y_3 \\
\text{s.t.} \quad & y_1 + y_3 \geq 3 \\
& y_2 + y_3 \geq 2 \\
& 4y_1 + 2y_2 + y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned} \tag{11}
$$

3 MaxFlow LP

This section introduces an alternate LP for *MAXFLOW* that will make it easier to write and interpret the dual LP.

Choice variables: For each $s \to t$ path P in G lets define a variable X_p denoting the flow along path P.



X_p = flow along P

Notes:

- Following LP formulation depends on the flow decomposition theorem. In flow decomposition, we can decompose any feasible flow into a finite number of paths. Different paths P can include the same edge e in G. So there would be different flows X_p across an edge e.
- The $s \to t$ paths need not to be disjoint in any sense and therefore there can be exponentially many $s \to t$ paths. Therefore, the number of choice variables x_p can be exponential in the size of the graph.

LP formulation:

$$\begin{aligned} \max_{X_P} \quad & \sum_{P} X_P \\ \text{s.t.} \quad & \sum_{P: e \in P} X_P \leq c_e, \text{for every } e \in E \\ & X_P \geq 0, \text{for all paths } P \end{aligned} \tag{12}$$

This LP is in standard form. If we take this to be the primal LP, by comparing with (5) we get,

$$c_{1\times r} = (1, 1, \dots, 1), \text{ where } r = \text{number of } s \to t \text{ paths in } G \text{ and,}$$
$$b_{1\times m} = (c_{e_1}, c_{e_2}, \dots, c_{e_m})^T, \text{ where m is the number of edges.}$$

$$A_{m\times r} = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1 & \dots & 0 & 1 \end{bmatrix} \tag{13}$$

$$A_{ij} = \begin{cases} 1 & \text{if edge } i \text{ is an incident on path } j \\ 0 & \text{otherwise} \end{cases}$$

Matrix A is also called edge-path incidence matrix.

Dual form:
Let consider dual variable y_e for each edge $e \in E$. Then,

$$\begin{aligned} \min_{y_e} \quad & \sum_{e \in E} c_e y_e \\ \text{s.t.} \quad & \sum_{e \in P} y_e \geq 1, \text{ for each } s \to t \text{ path } P \\ & y_e \geq 0, \text{for each } e \in E \end{aligned} \tag{14}$$

Remember, we wanted to illustrate that the MAXFLOW-MINCUT theorem is just a special case of LP duality. But, at first glance this LP seems to have nothing to do with the MINCUT. In the next class, we will come up with an interpretation of this LP that connects it to MINCUT.