Lecture Notes CS:5350 SetCover cont. & a randomized 1/2-approx. for MaxSat
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1 Review

1.1 Randomized LP rounding for SetCover

Given our previous notations, there are m elements in the ground set X, $X = \{1, 2, \ldots, m\}$, and there are a collection of subsets of X as $S_1, S_2, \ldots, S_m \subseteq X$, where the union of these subsets covers the the ground set. We want to find the smallest sub-collection of these subsets that covers the entire ground set.

And the LP relaxation we are working with is:

1.1.1 LP relaxation

$$
\begin{aligned}
&min && \sum_{i=1}^{n} x_i && \\
&\text{subject to} && \sum_{i:j \in S_i} x_i \geq 1 && \text{for every } j \in X \\
&\text{and} && x_i \geq 0 && \text{for all } i = 1, 2, \ldots, n
\end{aligned}
$$

And our algorithm is as follows:

1.1.2 Algorithm

1. Solve LP to obtain solution $\{x_i^* | i = 1, 2, \ldots, n\}$
2. **for** each $i = 1, 2, \ldots, n$ **do**

   2.1. $z_i \leftarrow \begin{cases} 1, & \text{with prob. } x_i^* \\ 0, & \text{with prob. } 1 - x_i^* \end{cases}$
3. Output $C = \{i | z_i = 1\}$

Recalling the two lemmas below:

Lemma 1. $E[|C|] \leq OPT$.

Lemma 2. *For any* $j \in X$, *Prob[* C *covers* j *]* $\geq 1 - \frac{1}{e}$.

PROOF:

$$\begin{aligned}
Prob[\ C \text{ covers } j\ ] &= 1 - Prob[\ C \text{ does not cover } j\ ] \\
&= 1 - \prod_{i:j \in S_i} (1 - x_i^*) \\
&\geq 1 - \prod_{i:j \in S_i} exp(-x_i^*) && \text{using } 1 + x \leq e^x \\
&= 1 - exp(-\sum_{i:j \in S_i} x_i^*) \\
&\geq 1 - exp(-1) = 1 - \frac{1}{e} && \square
\end{aligned}$$

2 Improvement to algorithm 1.1.2

Now what can we do to improve the algorithm 1.1.2?

> **We repeat (*i.e.*, amplify) this algorithm $c \ln n$ times and return the union of the "covers".**

Lemma 3. *Let C' denote the "cover" returned by this new algorithm. Then,*

$$E[|C'|] \leq c \ln n . OPT.$$

Hint: This is because of *(i)* $E[|C|] \leq OPT$, *(ii)* being amplified $c \ln n$ times, and *(iii)* the linearity of expectation.

Lemma 4. *For any $j \in X$,*

$$Prob[\ C' \text{ covers } j\ ] \geq 1 - \frac{1}{n^c}$$

Hint: Given that $Prob[\ C \text{ does not cover } j\ ] \leq \frac{1}{e}$ and that we amplify the algorithm 1.1.2 $c \ln n$ times, we get $Prob[\ C' \text{ does not cover } j\ ] \leq (\frac{1}{e})^{c \ln n}$, and thus it gets upper-bounded by $\frac{1}{n^c}$.

Lemma 5. $Prob[\text{ all } j \in X \text{ are covered by } C'\ ] \geq 1 - \frac{m}{n^c}$

Hint: By using union bound.

Now for further improvement, we design our new algorithm as follows:

> **We repeat (*i.e.*, amplify) the above algorithm T times and return the smallest cover.**

3 MaxSat

MaxSat is another example of randomized LP rounding.

Input: $x_1, x_2, \ldots, x_n$– boolean variables
$C_1, C_2, \ldots, C_m$– disjunctive clauses formed using the variables and their negations
Output: A truth assignment to $x_1, x_2, \ldots, x_n$ that satisfies the maximum number of clauses

Example:

$$\begin{aligned} & x_1, x_2, x_3 \\ C_1 = \; & x_1 \vee \overline{x}_2 \\ C_2 = \; & \overline{x}_1 \vee x_2 \vee \overline{x}_3 \\ C_3 = \; & \overline{x}_2 \vee \overline{x}_3 \\ C_4 = \; & \overline{x}_1 \vee \overline{x}_2 \vee x_3 \end{aligned}$$

We can satisfy all above clauses by only assigning $x_1 \leftarrow F$ and $x_2 \leftarrow F$.

Observation:

MaxSat is NP-complete because solving MaxSat in polynomial time implies a polynomial solution to Sat problem.

3.1 Algorithm 1

To each variable x_i independently assign T/F values with prob. $\frac{1}{2}$.

Lemma 6. *Let S be the set of clauses satisfied by this algorithm. Then,*

$$E[|S|] \geq \frac{m}{2}.$$

PROOF:

$$Y_i = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{Otherwise} \end{cases}$$

$$Y = \text{number of satisfied clauses}$$

$$\therefore Y = \sum_{j=1}^{m} Y_j$$

$$\therefore E[Y] = \sum_{j=1}^{m} E[Y_j]$$

$$= \sum_{j=1}^{m} Prob[\ Y_j = 1\ ]$$

Given that

$$Prob[\ Y_j = 1\ ] = 1 - Prob[\text{ no literal in } C_j \text{ is true }]$$

$$= 1 - (\frac{1}{2})^{l_j} \qquad \text{where } l_j \text{ is the number of literals in } C_j$$

$$\geq 1 - \frac{1}{2} = \frac{1}{2}$$

We have

$$\sum_{j=1}^{m} Prob[\ Y_j = 1\ ] \geq \frac{m}{2}$$