Lecture Notes CS:5350 Introduction to Greedy approximation Algorithms
Lecture 29: Dec. 10, 2019
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1 Derandomization

We have a $\frac{1}{2}$-approximation(in expectation) for MaxSet. (We do, also have better approximation for sure. Say $(1 - \frac{1}{e})$ and $\frac{3}{4}$ approximation. And they can all be derandomized but not we just use this $\frac{1}{2}$-approximation as an example).
Recall how do we get the $\frac{1}{2}$-approximation? We use {"flip coins method. We can de-randomization this algorithm, that means, make it deterministic while it still achieving a $\frac{1}{2}$-approximation.
Remark:

- There are lots of problems we do not know if they can derandomized or not. Even we are allowed to lose runtime, still there are problems we don't know how to derandomized.
- One big open question : Can all problems that can be solved in polynomial time using randomization that also can be solved in polynomial time without randomization? This is a open question. we do not know the answer.

The method we will use us called the method of conditional expectations.
Problem Notations:

- $X_1, X_2, X_3 \ldots X_n$ denote the Boolean variables
- $C_1, C_2 \ldots C_n$ denote the clauses
- Y_j is the indicator variable indicating if the clause C_j is satisfied.
- Y random variable the number of satisfied clauses.

Theorem: $\mathbb{E}[Y] >= m/2$ (showed in previous lecture). Here, $m/2$ is the number of clauses.
Process of the derandomization:
Consider the variables in order, for example, $X_1, X_2, X_3 \ldots X_n$ (the order doesn't matter, just arbitrary). And we assign value to X_i deterministically. First we assign a value to X_1 , we calculate $\mathbb{E}[Y|X_1 = T]$, then calculate $\mathbb{E}[Y|X_1 = F]$ Pick X_1 equal to T or X_1 equal to F depending on which value maximum this conditional expectation. if $\mathbb{E}[Y|X_1 = T]$ is as larger as $\mathbb{E}[Y|X_1 = F]$ then we can set X_1 to True, otherwise we can set it to False.
Remark:

- This is different from other algorithm, here we do not have backtracking. We are doing it in greedy way. We only cares about this one step and pick the larger one to set X_i keep going to setting next X_{i+1}.

$\mathbb{E}[Y|X1 = T]$ is the Expectation of Y condition on setting X_1 to True. Therefore $\mathbb{E}[Y|X_1 = T]$ equals to Number of satisfied Clauses + remain Expectation(problitically, same way to calculate expectation as before).

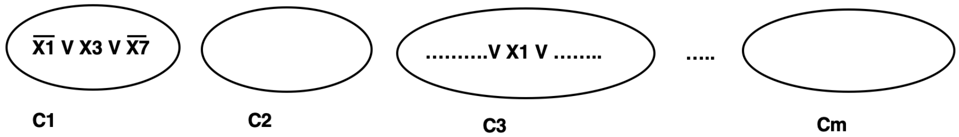

Figure 1: Example of assign value to X_1 = True

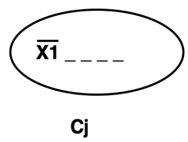

Figure 2: Example of C_j has $\bar{X_1}$

In Figure 1, what if X_1 in C_3, C_3 is satisfied if X1 is setting to True. Now for this C_3, there is no randomize anymore. In Figure 2, let's say C_1 has $\bar{X_1}$,then$C_1$is false. Then we can remove $\bar{X_1}$, since it has no influence to this C_1. So X_1 is gone and some is satisfied,

Remark:

- How do we calculate this $\mathbb{E}[Y|X_1 = T]$? Just as before , Let's consider C_j below. We set X_1 to True. we want to know what is the probability of C_j is satisfied? It is equal to Prob (C_j) = 1 - Prob(not to be satisfied). Prob(satisfied) $= \frac{1}{2} \times \frac{1}{2} \rightarrow$ Prob (C_j not satisfied) $= 1 - \frac{1}{2} \times \frac{1}{2}$.

$\mathbb{E}[Y]$ can be written as : $\mathbb{E}[Y] = \mathbb{E}[Y|X_1 = T] \times \text{Prob}(X_1 = T) + \mathbb{E}[Y|X_1 = F] \times \text{Prob}(X_1 = F) = \frac{1}{2} \times \mathbb{E}[Y|X_1 = T] + \frac{1}{2} \times \mathbb{E}[Y|X_1 = F]$ (as we can see this is just average, but it also can be weighted in other ways)
For example: $\max\{\mathbb{E}[Y|X_1 = T], \mathbb{E}[Y|X_1 = F]\} \geq \mathbb{E}[Y]$ we will take the one as large as $\mathbb{E}[Y]$.
So, when we pick $X_1 = T$ or $X_1 = F$ to maximize the conditional expectation, we are guaranteed that the conditional expectation $\mathbb{E}[Y|X_1] \geq \mathbb{E}[Y] \geq m/2$. so what is left in expectation is sufficiently high.
Now we will continue assign a value to X_2. To make exposition clearer, suppose that X_1 was set to T in the previous step, $\mathbb{E}[Y|X_1 = T, X_2 = T]$, $\mathbb{E}[Y|X_1 = T, X_2 = F]$. Pick a value for X is the larger of these Two. we calculated in the exactly the same way. Observe:

$$\begin{aligned}
\mathbb{E}[Y|X_1 = T] &= \mathbb{E}[Y|X_1 = T, X_2 = T] \times \text{Prob}(X_2 = T) + \mathbb{E}[Y|X_1 = T, X_2 = F] \times Prob(X_2 = F) \\
&= 1/2 \times \mathbb{E}[Y|X_1 = T, X_2 = F] + 1/2 \times \mathbb{E}[Y|X_1 = T, X_2 = F]
\end{aligned}$$

So $\max\{\mathbb{E}[Y|X_1 = T, X_2 = F], \mathbb{E}[Y|X_1 = T, Y_2 = T]\} \geq \mathbb{E}[Y|X_1 = T]$ continue to preserve the lower bound. By the reasoning for step 1,$\mathbb{E}[Y|X1 = T] \geq E[Y] \geq m/2$. So the larger of these two conditional expectations $\geq m/2$.
For the last Node, What it should be?
As shown in Figure 3, $\mathbb{E}[Y|X_1 = b_1, X_2 = b_2 \ldots X_n = b_n] = Y$ (Y is number of clauses satisfied

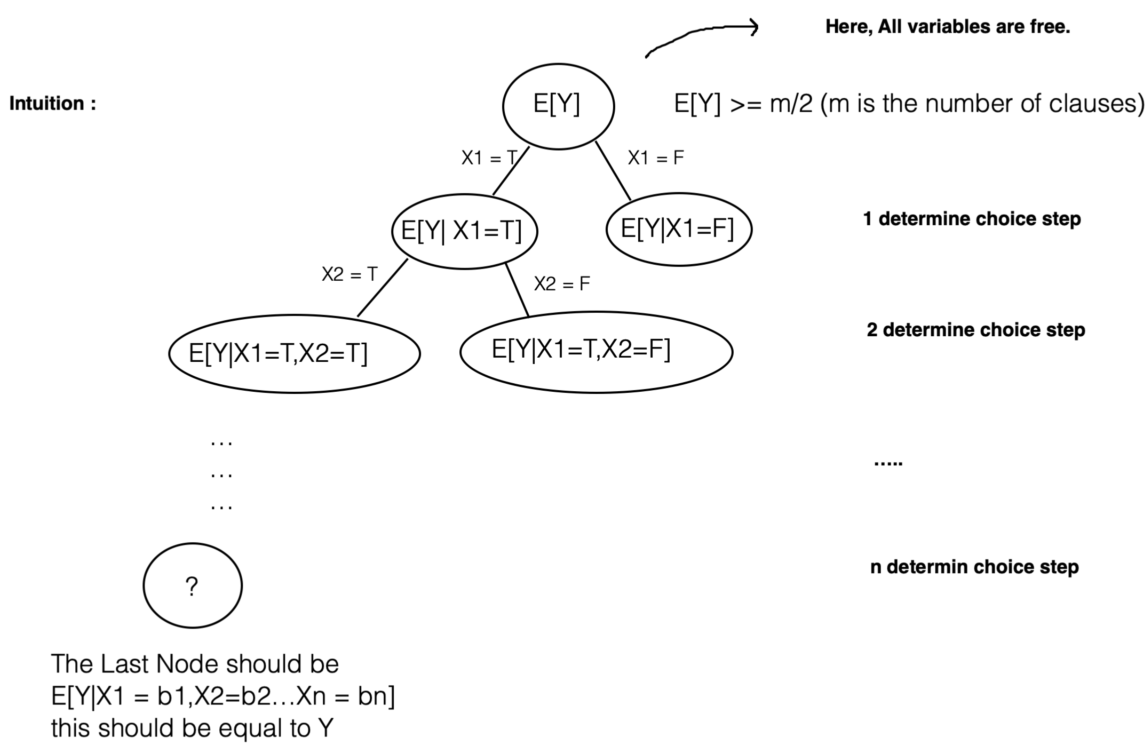

Figure 3: Intuition of derandomization

depending on setting). As a result, the random variable Y is changed from completely random to completely depend on variables node and keep guarantee $\mathbb{E}[Y] \geq m/2$. Therefore we can see from Step1 to Step2, Y is not as randomized as before, it is partial disposed. In each step, the Y is becoming less randomized. And in the last Step Y is completely deterministic and no longer randomized anymore since all the X_i is assigned we can now know what Y is based on the value of X_i.

Remark: If the two $\mathbb{E}[Y|X = T]$ and $\mathbb{E}[Y|X = F]$ are same, then we just arbitrary pick one from them to break the tie. This method is quite popular. We also have $\frac{1}{2}$ approximation in MaxCut problem. And we can also apply the exactly same idea to de-randomize MaxCut.

2 Practice problem:

Now we start to go over some problems as practice problems.

Problem 1 : 1.5 (Might different in pdf file) The problem is about Vertax Cover We will not do the part a here, but just list part a as a fact here.

(a) Fact: The Vertax Cover LP relaxation has the half-integrality property, for example: there is an optimal solution for which $X_i 0, \frac{1}{2}, 1$ for all vertices i, such an half-integral solution can be computed in polynomial-time.

(b) Design a 3/2-approximation (much better) for MVC on planar graphs, use part(a) and use the fact that planar graph can be 4-colored in polynomial time

For example: Here we have a planar graphs (Figure 4 with color and value assigned:

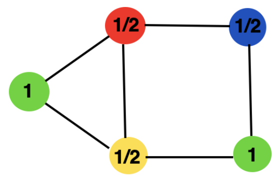

Figure 4: Example of a planar graph

We will cover this problem in next class.