1. Given two feasible \((s, t)\)-flows \(f_1\) and \(f_2\) in a graph \(G = (V, E)\), define the average flow \(\bar{f}\) as
   \[
   \bar{f}(u \to v) = \frac{1}{2} \cdot f_1(u \to v) + \frac{1}{2} \cdot f_2(u \to v)
   \]
   for every edge \(u \to v \in E\). Is \(\bar{f}\) a feasible flow? Support your answer either with a proof or with a counterexample.

2. Consider the following two-source maximum flow problem, which we will call 2SourceMaxFlow. You are given a directed graph \(G = (V, E)\) and three special vertices \(s_1, s_2, t \in V\). Think of \(s_1\) and \(s_2\) as two sources and \(t\) and the single target or sink. A flow is defined as before, except that flow conservation need not hold at \(s_1, s_2, \) or \(t\). The definition of a feasible flow is as before. The value of the flow is the net outflow at \(s_1\) plus the net outflow at \(s_2\). As in the case of the MaxFlow problem, in order to solve the 2SourceMaxFlow problem we want to find a feasible flow with maximum value.

   Describe a polynomial-time algorithm for this problem, by efficiently reducing it to MaxFlow. In other words, given an input \(\langle G = (V, E), s_1, s_2, t, c : E \to \mathbb{R_\geq} \rangle\) to 2SourceMaxFlow, (a) show that it can be modified in polynomial time into an input \(\langle G' = (V', E'), s', t', c' : E' \to \mathbb{R_\geq} \rangle\) to MaxFlow and (b) from the solution to MaxFlow on this input, we can extract, again in polynomial-time a solution to 2SourceMaxFlow on the original input.

3. The input is a directed graph \(G = (V, E)\), special vertices \(s, t \in V\), and two capacity functions \(c_1, c_2 : E \to \mathbb{R_\geq}\). A flow \(f : E \to \mathbb{R}\) is said to be feasible for the input if \(c_1(u \to v) \leq f(u \to v) \leq c_2(u \to v)\) for all edges \(u \to v \in E\). (Recall that a flow, by definition, satisfies flow conservation at all vertices except \(s\) and \(t\).)

   Show that there is a polynomial-time algorithm for determining if there is a feasible flow for the input.

4. Consider the directed graph \(G\) given below, with source \(s\) and target \(t\) identified.
(a) Let $f$ be a flow in $G$ defined by $f(s \to a) = 12$, $f(a \to b) = 12$, $f(b \to t) = 12$ and $f(u \to v) = 0$ for all other edges $u \to v$ in $G$. Draw the residual graph $G_f$.

(b) Identify an augmenting path $P$ in $G_f$ that has the fewest number of edges. Define a new flow $f'$ obtained by augmenting $f$ along path $P$. Draw the residual graph $G_{f'}$.

(c) Is $f'$ a maximum flow? If your answer is “yes,” prove it by showing a cut with capacity equal to the value of $f'$. If your answer is “no,” then identify a flow $f''$ such that $|f''| > |f'|$.

5. Problem 17 from Chapter 10 in Prof. Erickson’s notes.

6. Problem 16(a) from Chapter 11 in Prof. Erickson’s notes. **Hints:** (i) Think about how we reduced the Baseball Elimination problem to a flow problem. (ii) Even if you cannot reduce this matrix rounding problem the standard MAXFLOW problem, consider reducing to one of the MAXFLOW variants described earlier in this handout.