1. We say that a Turing machine $M$ accepts a string $w \in \{0,1\}^*$ if on input $w$, $M$ halts and outputs 1. A Turing machine $M$ is said to have property $R$ if whenever $M$ accepts $w$ it accepts $w^R$. (Note: $w^R$ denotes the string obtained by reversing string $w$; e.g., $(011)^R$ is 110.) Define a function $R : \{0,1\}^* \rightarrow \{0,1\}$ as follows: $R(\alpha) = 1$ if $M_\alpha$ has property $R$ and $R(\alpha) = 0$ otherwise. Prove that the function $R$ is uncomputable.

**Hint:** For any binary string $x \in \{0,1\}^*$, consider the function $\text{HALT}_x$ defined as follows: $\text{HALT}_x(\alpha) = 1$ if $M_\alpha$ halts on input $x$ and $\text{HALT}_x(\alpha) = 0$ if $M_\alpha$ does not halt on input $x$.

Reduce the problem of computing $\text{HALT}_x$, for any $x \in \{0,1\}^*$, to the problem of computing $R$. In other words, if there is a TM $M_R$ that computes $R$ then one could use this TM to construct a new TM $M_x$ that could solve $\text{HALT}_x$. This step is the heart of the proof and conceptually most challenging.

Finally, argue that since $\text{HALT}$ is uncomputable, $\text{HALT}_x$ is uncomputable for some $x \in \{0,1\}^*$. This contradicts the existence of $M_R$.

2. Problem 1.12 (Chapter 1, Page 35).

**Hint for Part 3(a):** Fix $x \in \{0,1\}^*$ and again consider $\text{HALT}_x$ and show that Rice’s Theorem implies that $\text{HALT}_x$ is uncomputable. This is fairly straightforward if you understand the statement of Rice’s Theorem. Then argue that if $\text{HALT}$ were computable, then $\text{HALT}_x$ would also be computable (for every $x \in \{0,1\}^*$) and this contradicts the implication about $\text{HALT}_x$ derived from Rice’s Theorem.

**Hint for Part 3(b):** The textbook provides a hint for this on page 531 and again $\text{HALT}_x$ plays an important role. Here I expand on this hint. Consider a non-trivial set $S$ of partial functions and without loss of generality suppose that $\Phi$ (the function that is not defined anywhere) is not in $S$. Since $S$ is non-empty it contains a partial function $f$ that is computed by some TM $M_f$. Also, since $f \neq \Phi$, $f$ is defined for some $x \in \{0,1\}^*$.

Now reduce $\text{HALT}_x$ to $f_S$. In other words, assume that the function $f_S$ is computable by some TM $M_S$ and use the existence of $M_S$ to design a TM $M_x$ for $\text{HALT}_x$. As in the previous problem, this would contradict the uncomputability of $\text{HALT}$.