22C:44 Homework 7 Solution

1. A counterexample with 4 intervals is shown below. In the top portion of the figure I show how the greedy algorithm will produce a partition containing 3 sets, whereas in the bottom portion of the figure I show that a size 2 partition suffices.

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1 2 1
 3
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2. Consider the graph on the left in the above figure. Each of vertices G, H, and I have degree 4 and the each of the remaining vertices have degree 5. Suppose the greedy algorithm picks vertex G first. G and all of its neighbors are deleted from the graph and we get the graph shown on the right. This graph is a clique with 4 vertices \( K_4 \) and therefore only one vertex can be picked from it. So the greedy algorithm produces a solution with 2 vertices. On the other hand, it is clear that \( \{D, E, F\} \) is a larger solution to the problem. Had the greedy algorithm chosen vertex H or vertex I first we would still have obtained a size-2 solution.

3. (a) Suppose the vertices of the graph are arbitrarily labeled 1 through \( n \). Without loss of generality, suppose the algorithm process vertices in the order 1 through \( n \). Also suppose that the \( \Delta + 1 \) colors the algorithm is allowed to use are \( \{1, 2, \ldots, \Delta + 1\} \). We prove the claim by induction on the number of vertices processed. The inductive hypothesis is that after \( i \) vertices have been processed, each of the vertices 1 through \( i \) is assigned a color in the set \( \{1, 2, \ldots, \Delta + 1\} \) such that for any edge \( \{p, q\} \in E, p, q \leq i \), the color assigned to \( p \) is distinct from the color assigned to \( q \). Note that after all \( n \) vertices have been processed this gives us a valid coloring of the graph.

**Base Case:** After 1 vertex has been processed color 1 is used for vertex 1 and the induction hypothesis is trivially true.

**Inductive Case:** Suppose that the inductive hypothesis is true after \( i \) vertices have
been processed. We now show that it is also true after \((i + 1)\) vertices have been processed. Vertex \(i + 1\) has at most \(\Delta\) neighbors. This implies that it has at most \(\Delta\) neighbors that have already been assigned a color. This in turn implies that there is at least one color in the set \(\{1, 2, \ldots, \Delta + 1\}\) that has not been used for any neighbor of \(i + 1\). The greedy algorithm chooses the smallest color in \(\{1, 2, \ldots, \Delta + 1\}\) that has not been used for a neighbor, to color \(i + 1\). Therefore the inductive hypothesis holds after processing \(i + 1\) as well.

(b) Consider a path with 4 vertices, 1, 2, 3, and 4. Suppose the greedy algorithm colors the vertices in the order 1, 4, 2, and 3. Then vertices 1 and 4 will get assigned color 1, vertex 2 will get assigned color 2, and vertex 3 will get assigned color 3 because it has a neighbor colored 1 and a neighbor colored 2.

(c) The path 1, 2, 3, 4 can be colored with 2 colors by assigning color 1 to vertices 1 and 3 and color 2 to vertices 2 and 4.

4. The optimal Huffman tree is completely unbalanced and leads to the following assignment of codes: \(\text{code}(a) = 0^7\), \(\text{code}(b) = 0^61\), \(\text{code}(c) = 0^51\), \(\text{code}(d) = 0^41\), \(\text{code}(e) = 0^31\), \(\text{code}(f) = 0^21\), \(\text{code}(g) = 01\), and \(\text{code}(h) = 1\). Here I am using \(0^i\) to denote a string with \(i\) 0’s.

Let \(c_1, c_2, \ldots, c_n\) be the \(n\) characters with Fibonacci frequencies \(f_1, f_2, \ldots, f_n\), where \(f_1 = f_2 = 1\) and \(f_i = f_{i-1} + f_{i-2}\) for all \(i, 3 \leq i \leq n\). Then the codes assigned to characters \(c_i\) are

\[
\text{code}(c_1) = 0^{n-1} \quad \text{code}(c_i) = 0^{n-i}1 \quad \text{for all} \; i = 2, 3, \ldots, n.
\]

5. See solution to Problem 2 in Homework 9 from Fall 2000. This problem set and its solution is posted in the practice problems section of the course page.